

On planar domination in graphs

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Abstract

Let G be a finite, undirected, simple graph with no isolated vertices. A new model of graph domination is introduced in this paper, and named “planar domination”. The properties of planar domination number are determined. Also, the relation between the domination number and planar domination number is discussed. the size of graph having planar domination number is calculated.

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1. Introduction

In this paper, a finite, simple, and undirected graph $G = (V, E)$ is dealt with. If $X \subseteq V(G)$ and $u \in X$ then the private neighbor of u with respect to X , is defined by $pn[u, X] = \{v : N[v] \cap X = \{u\}\}$. For theoretical terms and the fundamental concepts of graph theory, see ([5,14]). One of the most rapidly developing areas in graph theory is the analysis of related subset issues of dominating sets. Graph theory is used in many fields and has many applications ([2,13,7]). A subset D of $V(G)$ is a dominating set in G if $N(v) \cap D \neq \emptyset$; $\forall v \in V - D$, so that $N[D] = V$, then that set is said to be a dominating set. The cardinality of the lowest dominating set D of G is known as the domination number, and denoted by $\gamma(G)$. The ideas of dominating sets and domination numbers were first presented by Ore [12].

The objective used and the concepts of different types of domination; to obtain new concepts for the domination number by introducing certain conditions, which are put on the dominating set [1,4] and some of which are put outside the dominating set [3,11], or are put on together sets [5,6]. The domination number is of great importance especially mathematics, as it is included in the calculations of most well-known mathematical fields as fuzzy graphs [9], labeled graphs [10]. Here, planar domination with certain constraints is introduced with its characteristics.

Corollary 1.1: [8] (i) If G is a connected simple planar graph with $n (\geq 3)$ vertices and m edges, then $m \leq 3n - 6$. (ii) If, in addition, G has no triangles, then $m \leq 2n - 4$.

2. Planar Dominating set

Definition 2.1: Let $G = (V, E)$ be a graph without isolated vertex, a subset $D \subseteq V(G)$ is a planar dominating set if D dominating set, and induced subgraph $\langle D \rangle$ is planar graph such that $deg(v) \geq 2$ for all $v \in D$. (for example, see Fig.1 (a) and (b)).

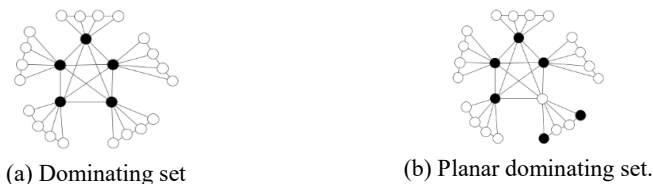


Figure 1

Definition 2.2. Let $G = (V, E)$ be a graph, a subset $D \subseteq V(G)$ is called a minimal planar dominating set if it has no proper planar dominating set. A minimum planar dominating set is a planar dominating set of the smallest cardinality in a graph G .

Definition 2.3: The planar domination number denotes $\gamma_{pl}(G)$ is a minimum cardinality over all planar dominating sets in G .

Properties 2.4: Let G be a graph of order n having the planar domination number, then we have:

1. The order of G is $n \geq 3$ and $\gamma_{pl}(G) \geq 1$.
2. $\delta(G) \geq 1$ and $\Delta(G) \geq 2$.
3. For each vertex $v \in D$, $\deg(v) \geq 2$.
4. If v is a support vertex then $v \in D$.
5. Every pendent vertex $v, v \in V - D$.

Theorem 2.5: Let G be a graph, if $\gamma_{pl}(G) > \gamma(G)$, then $|G| \geq 13$.

Proof: By hypotheses, G has the planar domination number and it is greater than the domination number, so G has no isolated vertex. In addition, for all vertices there is a vertex of degree greater than two adjacent to each vertex of degree one, so, we will rule out the idea of taking a vertex with one degree in the dominating set. Furthermore, since $\gamma_{pl}(G) > \gamma(G)$, then each dominating set contains an induced subgraph which is isomorphic to K_5 at least if not then $\gamma_{pl}(G) = \gamma(G)$ and this is a contradiction to the hypotheses. As a result, from the above, the graph is isomorphic to $K_5 \odot P_1$. Each pendant vertex cannot belong to a dominating set, since G has a planar dominating set. Thus, in this case, G has no planar dominating set and so, one of the vertices must be excluded from the induced subgraph which is isomorphic to K_5 which belongs to the dominating set. Now, to dominate this vertex there must be more than one vertex that is adjacent to it and not belong to the induced subgraph which is isomorphic to K_5 . If the number of vertices is two as we mentioned above these vertices have no pendants, so these vertices with the vertex that belongs to the induced subgraph which is isomorphic to K_5 constitute a triangle. Therefore, it is possible to replace the existing vertex in the induced subgraph which is isomorphic to K_5 with one of the two vertices mentioned above, so $\gamma_{pl}(G) = \gamma(G)$ and this is a contradiction with the hypotheses. If the number of vertices is three as we mentioned above these vertices have no pendants, so these vertices with the vertex that belongs to the induced subgraph which is isomorphic to K_5 constitute a fan graph it is possible to replace the existing vertex in the induced subgraph which is isomorphic to K_5 with one of the two vertices mentioned above, so $\gamma_{pl}(G) = \gamma(G)$ and this is a contradiction with the hypotheses. If the number of vertices is four as we mentioned above each of these vertices have no pendants, so these vertices with the vertex that belongs to the induced subgraph which is isomorphic to K_5 constitute a fan graph. In this case, there is no vertex from the four vertices that can dominate the other three vertices with a vertex that belongs to the induced subgraph which is isomorphic to K_5 , so $\gamma_{pl}(G) > \gamma(G)$, and the number of vertices is 13 at least.

Proposition 2.6: If G have a planar domination number γ_{pl} then the size of G is $(n - \gamma_{pl}) \leq m \leq \binom{n}{2} + \frac{1}{2} (-\gamma_{pl}^2 + 5\gamma_{pl}) - 4$.

Proof: Let D be a γ_{pl} -set of a graph G , then we have two cases:

Case 1: To prove the lower bound, suppose that $G[D]$ and $G[V - D]$ are two null graphs to be G has as few edges as possible. Now by the definition of the planar domination, every vertex in $V - D$ must be adjacent to at least one vertex in D to be dominated and the degree of every vertex in D is 2. Thus, each vertex in $V - D$ contributes at least one edge to m . Therefore, $|V - D| = (n - \gamma_{pl}) \leq m$.

Case 2: To prove the upper bound, suppose that $G[D]$ and $G[V - D]$ are two induced complete subgraphs of G and $G[D]$ is planar graph that means $|G[D]| < 5$ where the number of edges of D and $V - D$ equal to m_1 and m_2 respectively, then: $m_1 < 2|D| - 4 = (2\gamma_{pl} - 4)$ by Corollary (1.1). $m_2 = \frac{(|V-D|)(|V-D|-1)}{2} = \frac{(n-\gamma_{pl})(n-\gamma_{pl}-1)}{2}$. Now by the definition of planar domination, it is clear that every vertex of D may be adjacent to all vertices in $V - D$ and are not adjacent to any vertices in D , then the number of edges from D to $V - D$ is equal to $m_3 = |D||V - D| = (\gamma_{pl})(n - \gamma_{pl}) = (n\gamma_{pl} - \gamma_{pl}^2)$. Therefore, the number of edges of G is equal to $m = m_1 + m_2 + m_3 = (2\gamma_{pl} - 4) + \frac{1}{2} (n^2 - n\gamma_{pl} - n - n\gamma_{pl} + \gamma_{pl}^2 + \gamma_{pl}) + (n\gamma_{pl} - \gamma_{pl}^2) = \frac{1}{2} (4\gamma_{pl} - 8 + n^2 - n\gamma_{pl} - n - n\gamma_{pl} + \gamma_{pl}^2 + \gamma_{pl} + 2n\gamma_{pl} - 2\gamma_{pl}^2) = \frac{(n^2-n)}{2} + \frac{1}{2} (-\gamma_{pl}^2 + 5\gamma_{pl}) - 4 = \binom{n}{2} + \frac{1}{2} (-\gamma_{pl}^2 + 5\gamma_{pl}) - 4$.

Proposition 2.7: Every planar dominating set D in a graph G is a minimal planar dominating set if:

1. $v \in D$ such that $N(v) \cap D = \emptyset$.
2. $v \in D$, $pn[v, V - D] \neq \emptyset$.
3. $v \in D$, $N(v) \cap D = \{u\}$, such that $|N(u) \cap V - D| = 1$ and $|pn[u, V - D]| = 1$.

Proof: In a graph G , let D be any planar dominant set. Suppose it isn't a minimal planar dominating set then there is at least one vertex say $v \in D$ such that $D - \{v\}$ is a planar dominating set. It is clear that $deg(v) \geq 2$ for all $v \in D$ by Definition 2.1. Here are some possible uses of such removal cases.

Case 1: If $N(v) \cap D = \emptyset$, $v \in D$, then the vertices in set $V - D \cup \{v\}$ are not dominated by set $D - \{v\}$.

Case 2: If $pn[v, V - D] \neq \emptyset$, then when vertex v has at least one private neighbour in set $V - D$ say u , so if v in $V - D$ then D is not a dominating set.

Case 3: If $N(v) \cap D = \{v\}$, such that $|N(u) \cap V - D| = 1$ and $|pn[u, V - D]| = 1$. Then the degree of vertex u is 2, so when $v \notin D$ then the D is not a dominating set.

Proposition 2.8: If G is a planar graph such that $\deg(v) \geq 2$ for all $v \in G$ then G has planar dominating set D .

Proof: Since G is a planar graph that has order n ($n \geq 3$) and $\deg(v) \geq 2$ for all $v \in G$ then G has no isolated vertex or pendant. There are two cases:

Case 1: If G connected then:

Subcase 1: If $n = 3$, Since $\deg(v) \geq 2$ for all $v \in G$, then G is cycle and planar. Therefore, G has a planar dominating set.

Subcase 2: If $n > 3$ Let $D = \{v_1, v_2, v_3, \dots, v_k\}$ be a dominating set in G where $k < n$, and every vertex in G is either in D or adjacent to a vertex in D and D is a subgraph of G , then D is a planar dominating set in G .

Case 2: If G disconnected since $\deg(v) \geq 2$ for all $v \in G$, and every component is a planar graph, therefore G has a planar dominating set as the above case.

3. Discussion and Conclusion

From above, the new type of domination in graphs which is planar domination can be applied in any planar and a non-planar graph, connected and disconnected graph. The planar domination is calculated for some known graphs.

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