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On multiplicatively closed sets and prime subact over monoid

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Abstract

In this work the concept of multiplicatively closed set of S-act have been introduced. The relation between multiplicatively closed subset of S-act and compactly packed of S-act

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have been studied and proved some properties of this concepts. Let U be a M. C. set of a monoid S and let U* be a U-closed subset of M. Ш is a subact of M which is maximal in M-U*. If [Ш:M] is maximal in S, then Ш is a prime subact of M

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1. Introduction

Let $\dot{M}$ be a unitary module defined on commutative ring with 1. I is $\boldsymbol{c.P}$ ideal for any collection $\{\boldsymbol{P_\alpha}\}_{\boldsymbol{\alpha\in\lambda}}$ for prime ideal with $\boldsymbol{I} \subseteq \bigcup_{\boldsymbol{\alpha\in\lambda}} \boldsymbol{P_\alpha}$, $\exists \boldsymbol{\beta} \in \boldsymbol{\lambda}$, s.t $\boldsymbol{I} \subseteq \boldsymbol{P_\beta}$. In [1] Reis and Viswanathan called rings is $\boldsymbol{c.P}$ in which every ideal is $\boldsymbol{c.P}$. The generalization of S-Acts has been introduced. It is well known that a subact $\dot{N}$ which is proper is designate prime, if $\boldsymbol{rx} \in \dot{N}$, $\boldsymbol{r} \in$ R also $\boldsymbol{x} \in \dot{M}$ behold either $\boldsymbol{r} \in [\dot{N}:\dot{M}]$ or $\boldsymbol{x} \in \dot{N}$, [2]. Following [3], put a commutative ring with identity R. A subset V of R have been said a multiplicatively closed (M.C) subset of R if: $1 \in V$ and for any v_1,v_2 in V, $v_1v_2 \in V$. Let V be (M.C) of R and M is an R –module, (1) V^* a nonempty subset of M have been said V-closed if $vm \in \mathbf{V}^*$ for every $v \in V$ and $m \in \mathbf{V}^*$. (2) An V-closed subset $\mathbf{V}^*$ have been said saturated if the next provision is hold: where $dm \in \mathbf{S}^*$ for $d \in R$ also $m \in M$, then $d \in \mathbf{V}$ and $m \in \mathbf{V}^*$

2. Main Results

Definition 2.1: If S is monoid, A subset U in R have been said a multiplicatively closed (M.C) sub-set of S if : 1 belongs to U and for any $u_1,u_2 \in U$. Let V be a (M.C) of a monoid S and M is S-act (1) the non-empty sub-set U^* for M have been said U-closed if um belongs U^* for every $v \in U$ and $m \in U^*$. (2) A U-closed subset U^* have been said saturated if the next provisions are hold: where $km \in U^*$ for $k \in S$ and $m \in M$, then $k \in U$ and $m \in U^*$.

Example 2.2: Let Q be act over integer number Z. If U^* is saturated U-closed, then U = Z-{0} and U^* = Q-{0}.

Proof: U^* is non-empty set, since it is saturated U-closed, then $0 \notin U^*$. Put $\frac{p}{q} \in U^* \ni p, q \in Z\text{-}\{0\}$, thus $p\frac{1}{q} \in U^*$, therefore $\frac{1}{q} \in U^*$, hence $q\frac{1}{q^2} \in U^*$, thus $q \in U$, therefore $1 = q\,\frac{1}{q} \in U^*$, hence $1 = p\,\frac{1}{p} \in U^*$, for each $p \in Z\text{-}\{0\}$, then $p \in U$ and $\frac{1}{p} \in U^*$, thus U = Z- {0} and $\frac{p}{q} \in U^*$ for all $p, q \in Z\text{-}\{0\}$, Therefore {0} = Q-U^*, thus U^* = Q-{0}.

Theorem 2.3: *Let U be a M. C. set of a monoid S and let U^* be an U-closed sub-set for M. Ш be subact for M which is maximal in M-U^*. If [Ш:M] is maximal in S , then Ш prime subact for M.*

Proof: Plainly, Ɯ≠M. Let $s.m \in N$ for $s\ in\ S$ and $m \in M$. Suppose that $s\ in$ [Ɯ:M] and $m \notin N$. The subact (Sm+N) $\cap SU^* \neq \varphi$ and the ideal $(< s > +[N:M]) \cap U \neq \varphi$ i.e. $\exists\ x \in$ (Sm+ Ɯ) $\cap$ U and $t \in U \cap (< s > +[\omega: M])$. Thus $n + a_1 m = x,\ a_2 r + h = t$, for $a_1, a_2 in\ S$, n in N and h in [N:M]. Hens tx $\in$ U^* ∩ Ɯ, so we get a contradiction.

Proposition 2.4: Let M, U and U^* be as in (2.3), and a subact Ɯ ⊆ M-U^* then there exists a subact H including Ɯ and maximal in M-U^*.

Proposition 2.5: if M is S-Act & U is m. c. set of a monoid S. U^* is a saturated U-closed, also K is an ideal of S such that $K \cap S = \varphi$. If one of the next conditions satisfy: (a) every f.g ideal in S is cyclic (b) every f.g subact is cyclic. The $KM \cap U^* = \varphi$.

Proof: Suppose that there exists $x \in KM \cap U^*$, then for each i $x = \sum_{i=1}^{n} r_i m_i$, $r_i \in K$ also $m_i \in M$. Now: If (i) every f.g ideal is cyclic, let $L =< r_1, r_2, \dots, r_n >$ and hence $\exists s \in L$ such that $L =< s >$ i.e. $x = sm$ where $s \in K, m \in M$, since U^* is saturated the $s \in U$ which contradicts the fact $K \cap U = \varphi$. By the same way we can prove part (b).

An S-Act M have been said a multiplication (mlt-S-act), if all subact N have a form IM = N for the ideal I for S, i.e, N = [N:M]M. [4].Recall where $\dot{M}$ be mlt-act & $\dot{N}$ subact with $\dot{N} \subseteq \bigcup_{(\alpha\in\lambda)} \dot{N}_\alpha$, wherever is a prime subact and λ finite set, thus $\dot{N} \subseteq \dot{N}_\beta$ where β∈λ [5].

Theorem 2.6: *Let ₩ be a mlt-S-act and U^* be a saturated U-closed relative to M.C. set U of S. Presume that K₩ ⊆ ₩-U^* for all ideal K include in S-U. If a subact Ɯ is maximal in ₩-U^* thus Ɯ is prime subact.*

Proof: Via Theorem 2.3 we can offer that [Ɯ: ₩] is maximal in S-U. Presume that [Ɯ: ₩] is not maximal in S-U therefore ∃ L in S-U ∋ [Ɯ: ₩] ⊊ L, thus L₩⊆ ₩-U^*. However, Ɯ= [Ɯ: ₩] ₩ ⊆ L₩, so that L₩= Ɯ, because Ɯ is maximal in ₩-U^*. i.e.L ⊆ [Ɯ: ₩]. Thus [Ɯ: ₩] is maximal in S-U.

Corollary 2.7: *Let ₩, U, U^* be as in theorem 2.6. If Ɯ in ₩ also maximal in ₩-U^* and one of the next provisions is hold (1) every f.g ideal is cyclic (2) every f.g subact is cyclic (3) ₩ is cyclic S-Act. Then Ɯ is prime subact.*

Proposition 2.8: Put S is a moniod and put U is a M.C. set of S. Put ₩ is a mult. S-U-act and $U^* \neq \emptyset$ in ₩. Suppose that φ = $U^* \cap$ K₩ for all saturated U-closed subset of ₩ also for all ideal K in S-U. Then U^* is saturated U-closed in ₩ if and only if ₩-U^* is a union of prime subacts P_i (i∈I) of ₩ and the set S-U is a union of [P_i: ₩], (i∈I) [6, 7].

Proof: Suppose U^* is saturated U-closed in ₩, then K₩ ∩ U^*= φ, for all K in S-U. Because U^* is saturated thus, $0 \in$ ₩-U^*, i.e. ₩-$U^* \neq \varphi$. Put e is any element in ₩-S^*, then Se ⊆₩-U^*, also by proposition 2.3 there is a subact P include Se and maximal in ₩-U*. The subact P must be prime subact by proposition 2.5. Hence ₩-U*= $\bigcup_{i\in I} P_i$ where P_i is prime subact, (i∈I), put $U_0 = R - \bigcup_{i\in I}[P_i : M]$. Let u∈U, m∈U*, then um $\in$ U*= $\bigcup_{i\in I} P_i$. So that um $\notin P_i$ for each i. Because ach P_i is prime subact then $u \notin [P_i : ₩]\ \forall i \in I$ i.e. $u \in U_0$. Therefore U⊆ U_0. On the other hand, let $s' \in U_0$ then $u' \notin [P_i : M]\ \forall i \in I$. If $m' \in U$, we have $U_0 \subseteq U$ and it follows that $U_0 = U$.

Corollary 2.9: *Put ₩ is a mult- S-U-act, U is a M.C. set of S and U*≠ ∅. Whether one of the next provisions holds: i) every f.g. ideal is cyclic, ii) every f.g. subact is cyclic. Then U* is saturated U-closed subset of ₩ iff the set ₩-U* is a union of prime subacts P_i (i∈I) of ₩ also the set S-U is a union of $[P_i : ₩]$, (i∈I). Recall that N is a proper (appropriate) subact which is c.P, if N have been includ in a union for a class for prime subact, so $\dot{N}$ is contained in one for a member for a class, $\dot{M}$ is compactly packed (c.P) S-Act if all proper subact be c.P [5].*

Proposition 2.10: Let M be a mult- S-U-act. If one of the next provisions are hold: (a) M is cyclic S-act. (b) every f.g ideal is cyclic. (c) every f.g subact is cyclic. Then M is c.p S-act if and only if prime subact is c.p.

Proof: Requirement is clear. Prove the adequacy, presume all prime subact is c.p . Put Ɐ̰ is a proper(appropriate) subact in ₩ together with Ɐ̰⊆ $\bigcup_{\alpha\in\lambda} P_\alpha$, where P_α is a prime subact of ₩ for all $\alpha \in \lambda$. We have two cases: (1) $\bigcup_{\alpha\in\lambda} P_\alpha =$ M. Since Ɐ̰ is proper subact of a mult-S-act then by [2], there is a prime subact P which includes Ɐ̰. By assumption P is c.p and Ɐ̰⊆P⊆ ₩ = $\bigcup_{\alpha\in\lambda} P_\alpha$ therefore P⊆ P_β for $\beta \in \lambda$ that is Ɐ̰⊆ P_β. (2) $\bigcup_{\alpha\in\lambda} P_\alpha \subsetneq$ M. Let U*= ₩− $\bigcup_{\alpha\in\lambda} P_\alpha$. Thus U* is saturated U-closed subset of ₩ where $U = S - \bigcup_{\alpha\in\lambda}[P_\alpha : ₩]$. Since Ɐ̰ ⊆ $\bigcup_{\alpha\in\lambda} P_\alpha$ then Ɐ̰⊆₩-U*and by proposition 2.3 and theorem 2.5 there exists prime subact K of ₩ that contains Ɐ̰. But K ⊆ $\bigcup_{\alpha\in\lambda} P_\alpha$ therefore, $\exists \beta \in \lambda\ \ni$L ⊆ P_β and thus Ɐ̰ ⊆ P_β.

Proposition 2.11: Let M be a faithful finitely generated compactly packed R-S-act. Whether one of the next provisions holds: (a) M is cyclic S-act. (b) every f.g ideal in S is cyclic. (c) every f.g subact is cyclic, then S is compactly packed.

Proof: Put I is an ideal in S with $I \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$ where P_α is prime ideal for each $\alpha \in \lambda$There exists a prime subact L_α in M such that $[L_\alpha : M] = P_\alpha$. When $\bigcup_{\alpha\in\lambda} L_\alpha = M$ thus IM $\subseteq \bigcup_{\alpha\in\lambda} L_\alpha$, also if $\bigcup_{\alpha\in\lambda} L_\alpha \subsetneq M$, put U*=M− $\bigcup_{\alpha\in\lambda} L_\alpha$ then U* is a U-closed subset in M where$U = S - \bigcup_{\alpha\in\lambda} P_\alpha$. Since I⋂ U = φ then by proposition 2.3 IM ⋂ U^*= φ that is IM⊆ $\bigcup_{\alpha\in\lambda} L_\alpha$. But I is a proper (appropriate) ideal in S then IM is a proper subact in M. But M is compactly packed then $\exists \beta \in \lambda\ \ni$IM $\subseteq L_\beta$ and hence $I \subseteq [L_\beta : M] = P_\beta$ and we get I is compactly packed ideal and S is a compactly packed monoid.

Proposition 2.12: If M is S-Act & U is M.C in S. If M is c.p then so is M. In particular, M_P is c.p for each prime subact P of M.

Proof: Assume $H \subseteq \bigcup_{\alpha\in\lambda} W_\alpha$ when H be proper (appropriate) subact for M_S & W_α be prime subact of M_S for any $\alpha \in \lambda$. Define $\varphi: M \to M_S$ as follows: $\varphi(m) = \frac{m}{1}\ \forall m \in M$. So that $\varphi^{-1}(H) \subseteq \varphi^{-1}(\bigcup_{\alpha\in\lambda} W_\alpha)$ and hence $\varphi^{-1}(H) \subseteq \bigcup_{\alpha\in\lambda} \varphi^{-1}(W_\alpha)$. $\varphi^{-1}(W_\alpha)$ is prime subact for each $\alpha \in \lambda$. But M is compactly packed then $\varphi^{-1}(H) \subseteq \varphi^{-1}(W_\beta)$ where $\beta \in \lambda$. Theus $\left(\varphi^{-1}(H)\right)_S \subseteq (\varphi^{-1}(W_\beta))_S$ and hence $H \subseteq W_\beta$, that is M is c.p

3. Discussion and Conclusion

In this work some properties have been proved, assume M is S-Act and U is a m. c. set of a monoid S. U^* is a saturated U-closed, also K is an ideal of S such that $K \cap S = \varphi$. If one of the next conditions satisfy: (a) every f.g ideal in S is cyclic (b) every f.g subact is cyclic. The $KM \cap U^* = \varphi$, Let M be a faithful finitely generated compactly packed R-S-act. Whether one of the next provisions are hold: (a) M is cyclic S-act. (b) every f.g ideal in S is cyclic. (c) every f.g subact is cyclic, then S is compactly packed.

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