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Nearly small semi-prime sub-modules

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Abstract

Suppose that D is an unitary module over a ring that is commutative F, with identity. A non-zero sub-module N of D called as nearly small semi-prime, if for every $t \in F$, $m \in D$,(m) be small of M and t^2 m $\in$ N, which implies that tm $\in$ N + Rad(D) and an F-module M is a nearly small semi-prime module, if (0) is a nearly small semi-prime sub-module. In this work, We investigate the characteristics and properties of these types of sub modules (modules).

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1. Introduction

A sub-module that was nearly semi prime was introduced by Al-Mothafar and Alhakeem in 2015, a proper-submodule N of D is called a nearly semi prime-sub module, if $t^k x \in N$ where $t \in F$, also $x \in D$ then $tx \in N + Rad(D)$, and $Rad(D)$ be a intersection for every $maximal$-submodule in D. At

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(0) is a nearly sem-iprime-submodule for D, the D is called a nearly semi prime-module, see [1]. $small$ semi prime-sub modules and $small$ semi prime-modules were generalized by Ramadhan and in 2021 AL- Mothafar, claimed a proper-sub module N of D $small$ semi prime-sub module if and only if $t^2m \in N$ where $t \in F, m \in D$ and (m) is $small$ in D, then $tm \in N$, [2]. And if (0) is a $small$ semi prime-sub module of D, D is a $small$ semi prime-module of D, [3]. Where " a sub module N in D named small (notationally, $N \ll D$) where$N + K = D$ whenever l sub-modules K in D ,then $K = D$, [4]. Also, the F-$module$ D named hollow module if each proper submodule in D be $small$ in D [5]. Selman [6]. proper submodule N in F-module D named $small$ prime if and only if $r \in F$ & $x \in D$ with $(x) \ll D$ & $rx \in N$ implies $x \in N$ or $r \in [N: D]$" . From the notions $small$ semi-prime submodules and $small$ semi-prime modules, we present the notions nearly $small$ semi-prime sub-modules and nearly $small$ semiprime modules.

2. Nearly Small Semi prime Sub-modules

This part presents a study on a nearly small semi prime-submodules.

Definition 2.1: The proper-submodule N for module D is nearly small semi prime-submodule for D, if $\mathrm{t^2d} \in \mathrm{N}$ where $\mathrm{t} \in \mathrm{F}$, $\mathrm{d} \in \mathrm{D}$ & $(\mathrm{d}) \ll \mathrm{D}$ so, $\mathrm{td} \in \mathrm{N} + \mathrm{Rad(D)}$.

Of the ring F , an ideal I nearly small semi prime-ideal for F , if I nearly small semi prime-submodule of F-module

Examples and Remarks 2.2:

1. It's evident that each $small$ semi prime-sub module is nearly $small$ semi prime- sub module, however, generally speaking, the reverse not true. For instance: if $D = Z_8$ as Z-$module$, $N = (\bar{0})$ and $Rad(Z_8) = (\bar{2})$. N is a nearly $small$ semi-prime, since if $z^2y \in N$ where $z \in Z$ also $(m) \ll Z$. In which $(\bar{0})$, $(\bar{2})$, $(\bar{4})$ are $small$ sub-modules in Z_8, implies $zm \in N + (\bar{2})$. But $N.$ isn't $small$ semi-prime, since $2^2\bar{2} \in N$ but $2.\bar{2} \notin N$.
2. Evident that, all semi prime-sub module is nearly $small$ semi prime-submodule, but conversly in general may not be true. For instance: Consider $D = \mathbb{Z}$ as a $\mathbb{Z}$-$module$, hence every non-zero sub-module N of $\mathbb{Z}$ is nearly $small$ semi-prime. Since, if $z^2m \in N$ with $z \in \mathbb{Z}$ and $(m) \ll \mathbb{Z}$. Since (0) is the only one of a small sub-module in $\mathbb{Z}$, thus $m = 0$. Thus $zm = 0 \in N$. But, if we put $N = 4\mathbb{Z}$, hence it is evident that N isn't semi prime.
3. If D is a hollow-module and $Rad(D) = 0$, then each nearly $small$ semi-prime sub module of D is semi prime-sub module.
4. If $D = Z_{24}$ as a Z –$module$, The sub-module $S = (\bar{8})$ isn't a nearly $small$ semi-prime sub-module of D, since $(\bar{6}) << Z_{24}$ and $\bar{0} = 2^2.\ \bar{6} \in S$, but $2.\bar{6} \notin S + Rad(D)$.

5. In the ring Z every non-zero ideal is a $small$ semi prime. Since (0) is the only small and Z is a integral domain.

Theorem 2.3: *If N a proper submodule for D. A subsequent equivalent:*

1. *N a nearly small semi-prime.*
2. *For every $t \in F$ of D and for every $B \leq D$such that $t^2B \subsetneq N$, implies $t\,B \subsetneq N + Rad(D)$.*

Proof: (1)⟹(2) Assume that $a^2B \subsetneq N$, hence $\exists$ b∈ B as $a^2b \in N$.Since b∈ B and B<< D, hence (b) $\ll$ D by [7, Lemma 4.2]. But N is nearly small semi-prime sub-module of D, so ab ∈ N. Therefore, aB $\subsetneq$ N.

(2)⟸(1) Let $a \in F, y \in D$ and (y) $\ll$ D with $a^2y \in$N. Then $a^2\,(y) \subsetneq N$. Also $a\,(y) \subsetneq N$ by (2). Hence $ay \in N + Rad(D)$. Thus,N is nearly small semi-prime.

We can give the following result.

Theorem 2.4: *If N proper-sub module for D F-module with $Rad(D) \subseteq N$. Then N is nearly small semi prime iff $[N:_D I]$ is a nearly small semi prime-sub module of $D, \forall\ I \subsetneq F$ such that $ID \nsubseteq N$.*

Proof: (⟹): Since ID $\nsubseteq$ N, hence $D \neq [N:_D I]$ Let $t^2y \in [N:_D A]$ (y) << D ; that is $t^2(y) \subsetneq [N:_D I]$, so $t^2Iy \subsetneq N$. By assumption (y) << D implies (Iy)<< D. But N is nearly small semi-prime, so $t(Iy) \subsetneq N + Rad(D)$. by theorem (2.3). But $Rad(D) \subseteq N$, consequently, $Rad(D) + N = N$, so $t(Iy) \subsetneq N$ that's $ty \in [N:_D I]$ ⊆[N:_D I]+Rad(D). so, $[N:_D I]$ nearly small semi-prime.

(⟸): Assume $[N:_D I]$ nearly small semi-prime for D, $\forall$ I $\subsetneq$ F. Put $I = F$. Therefore $[N:_D I] = N$ is a nearly small semi-prime.

Proposition 2.5: If D the F-module& N small sub-module of D. If Rad(D) is a small semi prime-sub module for D, soN be nearly small semi prime sub-module of D.

Proof: Assume $t^2y \in N$ where $t \in F, y \in D$ such that N << D. Then N ⊆ Rad(D) that is $t^2D \in$ Rad(D). But Rad(D) small semi prime for D, so ty ∈ Rad(D) ⊆ N + Rad(D). Thus N nearly small semi prime for D.

Collorary 2.6: *Let D hollow F-module & $Rad(D)$ small semi prime-sub module of D. so N be nearly small semi prime-submodule for D.*

Proposition 2.7: If N nearly small semi prime-sub module for D′, so $f^{-1}(N)$ is a nearly small semi prime-sub module of D, where f: D → D′ be an epimorphism for Kerf<< D.

Proof: Initially, Our should demonstrate $f^{-1}(N)$ is a proper sub-module of D. Let $f^{-1}(N) = D$, hence $f(D) \subseteq N$, it runs counter with presumption. Let

$t \in F, m \in D$ and $(m) << D$ such that $t^2m \in f^{-1}(N)$. Then $t^2f(m) \in N$. But $(m) << D$, hence $f(D) << D'$ by [6], now N nearly small semi pime-submodule of D', so $tf(D) \in N + Rad(D')$. hence $f(tm) \in N + Rad(D')$, then $tm \in f^{-1}(N + Rad(D'))$ this implies $tm \in f^{-1}(N) + f^{-1}(Rad(D'))$. But Kerf<< D, so by [8] $tm \in f^{-1}(N) + Rad(D)$. Thus $f^{-1}(N)$is nearly small semi prime-sub module of D.

Corollary 2.8: *Let D be an F-module, $C \subseteq S \leq D$ and $C << D$. If $\frac{S}{C}$ is a nearly small semi prime-sub module of $\frac{D}{C}$, then S nearly small semi prime-submodule for D.*

Proof: Assume $\pi: D \to D/C$ is canonical epimorphism. Since $\frac{S}{C}$ nearly small semi prime, and Kerπ<< D. Consequently, by prior propoistion $\pi^{-1}\left(\frac{S}{C}\right) = S$ is a nearly small semi prime.

Remark 2.9: A homomorphic image for the nearly small semi prime must not nearly small semi prime, in case if $\varphi: Z_{24} \to Z_{24}$ defined as follows: $\varphi(\bar{z}) = 4\bar{z}$, $\forall\ \bar{z} \in Z_{24}$. If $N = (\bar{2})$, hence N is nearly small semi prime, but $\varphi(N) = (\bar{8})$ isn't nearly small semi prime, see (2 .2 ,4).

Remark 2.10: If $[N: D]$ nearly small semi prime-ideal for F, thus it's not necessitated N be nearly small semi prime-sub module of D. For instance: $D = \mathbb{Z}_{24}$ as the $\mathbb{Z}$ -modules, $N = (\bar{8})$ isn't small semi prime view (2.2,4). But $[N: D] = 8\mathbb{Z}$ be nearly small semi prime-ideals in Z.

Recall, the F-module D be multiplication for any submodule S in D there is ideal I for F s.t $S = ID$, [9].

Proposition 2.11: if D finitely generated faithful multiplication F-module& N proper-submodule for D. suppose $[N: D]$ nearly small semi prime-ideals for F, then N is nearly small semi prime-submodule of D

Proof: Suppose that $t^2y \in N$ when $t \in F$, and $y \in D$ with $(y) \ll D$. However, D be a finitely generated faithful multiplication, hence $(y) = ID$ and $I \ll F$ [9]. Therefore $t^2AD \subseteq N$, then $t^2A \subseteq [N: D]$. But $[N: D]$ is a nearly small semi prime, so $tI \subseteq [N: D] + Rad(F)$ by Theorem (2.3). Consequently, $tID \subseteq N$, then $t(y) \subseteq N$. Thus $ty \in N$. Thus, N nearly small semi prime.

Proposition 2.12: Let C and D be two nearly small semi prime-submodules for module D. so $C \cap D$ nearly small semi prime-sub module for D.

Proof: Let $t \in F, y \in D$ & $(y) \ll D$ with $t^2y \in C \cap D$. Then $t^2y \in C$ and $t^2y \in D$. But C and D are nearly small semi prime-sub modules. Therefore $ty \in C + Rad(D)$ & $ty \in D + Rad(D)$. Hence $ty \in C \cap D + Rad(D)$. Therefore $C \cap D$ is nearly small semi-prime sub-module of Y.

3. **Nearly Small Semi prime-Modules**

We introduce in this part the idea of small semi prime-module and establish some characteristics of small semi prime-modules.

Definition 3.1: A module D is said to be nearly small semi prime if (0) the small semi prime-sub module for D; i,e if $t^2y \in (0)$ and (y) << D, then $ty \in \mathrm{Rad(D)}$. A ring F namely small semi prime iff F be small semi prime F-module.

Remarks & Examples 3.2:

1. For any semiprime F-$module$ is nearly $small$ semi prime, but the convers may not be correct, for instance: Z_4 as $\mathbb{Z}$ -$module$ is $small$ semiprime, but not semi prime.
2. Each hollow nearly $small$ semi prime F-$module$ is semi prime.
3. For any integral domain is a nearly $small$ semi prime ring.
4. $(\bar{0})$ is n't nearly $small$ semi prime of Z_{32} , since 2^2. $(\bar{8})$ = $(\bar{0}$) and $(\bar{8})$ << Z_{32}. But $2\overline{(8)} \notin (\bar{0})$, so Z_{32} as a $\mathbb{Z}$-$module$ is not nearlysmall semi prime.
5. For any $small$ prime F-$module$ is nearly small semi prime , but the conversely may not be correct, for instance : Z_4 as a $\mathbb{Z}$-$module$ is $small$ semi prime but n't small prime by [5].
6. Z_{24} as a $\mathbb{Z}$-module isn't nearly small semi prime, since $(\bar{0})$ is not nearly small semi prime-submodule becuase if 2^2. $(\bar{6})$ = $(\bar{0})$ and $(\bar{6})$ << Z_{24}. But $2(\bar{6}) \notin (\bar{0})$.
7. For any semi simple F-module is nearly small semi prime, however, in general the opposite may n't be correct, for instance: Z as a $\mathbb{Z}$-$module$ is $small$ semi prime but n't semi simple.

Theorem 3.3: *D nearly small semi prime-module if and only if $ann(\ t^2y) = ann(ty)$ for each $(y) \ll D$ and $t \in F$, $t^2y\ \neq 0$.*

Proof: ⇒) It is clear that $\mathrm{ann(ty)} \subseteq \mathrm{ann(\ t^2y)}$. Let $\mathrm{a}\ \in\ \mathrm{ann(\ t^2y)}\ ,\mathrm{t^2y}\ \neq 0$ then $\mathrm{t^2ay = 0, (y)}$ << D, so $\mathrm{(ay) \ll D}$ [6]. D nearly small semi prime, so $\mathrm{iay = 0}$ and hence $\mathrm{a \in ann(ty)}$. Thus ann($\mathrm{t^2y}$) $\subseteq$ $\mathrm{ann(rm)}$. Thus, ann($\mathrm{t^2y}$) $=\ \mathrm{ann(ty)}$.

⇐) It is clear.

Proposition 3.4: Suppose that $\mathrm{D_1}$, $\mathrm{D_2}$ be two F-modules and let $\mathrm{D = D_1 \oplus D_2}$. assume D nearly small semi prime-module, so $\mathrm{D_1}$ and $\mathrm{D_2}$ are nearly small semi prime-module.

Proof: Let $\mathrm{t \in F,\ y \in D_1, (x) \ll D_1}$ such that $\mathrm{t^2y = 0}$, then $\mathrm{i^2(y, 0) = (0,0)}$. But $\mathrm{(y) \ll D_1}$ and $\mathrm{(0) \ll D_2}$, so $\mathrm{(y, 0) \ll D_1 \oplus D_2}$ by [6]. But D is a nearly small semiprime-module of D, so $t(y, 0) = (0,0)$ .Thus, $ty = 0$. Therefore D_1 small semi prime-module.

In the same proof, D_2 nearly small semi prime module.

Theorem 3.5: *Assume $D_1 \cong D_2$. so D_1 is nearly small semi prime iff D_2 is small semi prime.*

Proof: $\Longrightarrow$): Let D_1 is nearly small semi prime. Since $D_1 \cong D_2$, so there exists $\alpha: D_1 \rightarrow D_2$ be a F-isomorphism. Assume that $(y) \ll D_2$. Hence $\alpha^{-1}(y) \ll D_1$ by [6]. Let $t \in F$ such that $t^2(y) = 0$. So $r^2\alpha^{-1}(y) = 0$. Since D_1 small semi prime, so $t\,\alpha^{-1}(y) = 0$ and hence $ty = 0$. Thus, D_2 is small semi prime-module.

$\Longleftarrow$): By similar proof , D_1 is a nearly small semi prime-module.

Proposition 3.6: Assume D nearly small semi prime F-module, so ann C small semi prime ideal for F for every proper small sub-module C of D.

Proof: Suppose that $t \in F$ such that $t^2 \in \text{ann}C$ and $(t) \ll F, 0 \neq C \ll D$. So $t^2C = 0$. On otherwise, D nearly small semi prime, so (0) be small semi prime for D. so $tC = 0$ and so $t \in \text{ann}C$. Thus, ann C nearly small semi-prime ideal for F.

Theorem 3.7: *Assume that D is a f.g F-module, and let S be a multiplicatively closed subset of F. so D nearly small semi prime F-module iff D_s nearly small semi prime F_s-module*

Proof: It follows directly by theorem (2.10) and (2.11).

Now, if F be a small semi-prime ring, then it's unnecessary that D be nearly small semi-prime F-module, for instance: the Z_{24} as the Z-module isn't small semi-prime, but the ring Z be small semi-prime ring.

Theorem 3.8: *Let D be a multiplication finietlygenerated and faithfull F-module.Then D is a nearly small semi prime F-module if and only if F is a nearly small semi prime ring.*

Proof: $\Rightarrow$) Sppose that $t \in F$, $(t) \ll F$ such that $t^2 = 0$. But $(t^2) \subsetneq (t) \ll F$ implies that $(t)^2 \ll F$ by [6]. Since D multiplication finietly generated faithful, thus $(t^2)D = t^2D \ll D$, so $t^2D = 0_D$. now D nearly small semi prime, Hthen $tD = 0_D$ and so $t \in \text{ann}D = 0$. Implies $t = 0$. Hence, F is a nearly small semi prime ring.

$\Leftarrow$) Assume that $t \in F$, $y \in D, (y) \ll D$ such that $t^2y = 0$. Since D is a finitely generated faithful multiplication , then $(y) = ID$ where I ideal in F and $A \ll F$. So $t^2AD = 0_D$, consequently, $t^2A \subsetneq \text{ann}D = (0)$, thus, $t^2A = 0$. Since (0) is a nearly small semi prime ideal, so tA =(0) also tID =(0). Hence $t(y) = 0$, so $ty = 0$. Therefore D is a nearly small semi-prime.

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