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The special case for resolution of two rows : Weyl module for skewpartitions (9,6)/(2,0) & (9,6)/(3,0)

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Abstract

The present article is to study a certain cases for resolutions two rows for Weyl module, where F & $D_r\mathrm{F}$ are free modules over a commutative ring R & divided power algebra for degree r, respectively in the cases for the skew-partition (9,6)/(2,0) & (9,6)/(3,0). Two cases for resolution. Weyl modules are associated with the two-row skew-partition when the second row comes forward to the left for the first, creeping two & three in different cases. To determine the limits for that characteristic-free element & f& the exactness for the Weyl resolution, simply describe contracting homotopy for the skew-partition using a homological approach, which means the letter-place techniques, contracting homotopy, "place polarization".

Subject Classification: *18G85, 55N10.*

Keywords: *Resolution, Resolution for Weyl module, Place polarization, Skew-partition, Box map.*

1. Introduction

The homological properties for algebras are useful in many branches for modern analysis, including operator theory, harmonic & complex analysis. The method divides a region Ω into connected parts & creates loops Υ_i in Ω. Each loop Υ in Ω is homologous to an integer linear combination for the Υ_i, creating a suitable domain for analysis. [1][2]

Assume F a free module on the commutative ring $\mathcal{R}$ with identity, $D_r\mathrm{F}$ be a divided power algebra for degree r that underlies F.

A partition for length $n = \ell(\lambda)$ be sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ for non - negative integers in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.The pair (λ, μ) is presented in a relative sequence with $\mu \leq \lambda$ implying that $\mu_i \leq \lambda_i$ for all

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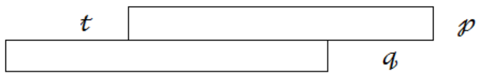
$i \geq 1$. The symbol λ/μ represents relative sequences. If λ & μ are both partitions, a relative sequence λ/μ is known as a skew partition**.** Researchers in [3] began using "letter place- algebra" & "differential bar- complex". In [3] [4], researchers depicted the Weyl module K_(λ/μ) Ƒ, which is related to the skew shape.



For $\mathcal{K}_{\lambda/\mu}F = \mathrm{Im}(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu}: DF \to \wedge F$ (Weyl map), the Weyl module is presented **b**y the **box** map [5] so we have

$$\sum D_{p+\kappa} \otimes D_{q-\kappa} \xrightarrow{\square} D_p \otimes D_q \xrightarrow{d'_{\lambda/\mu}} \mathcal{K}_{\lambda/\mu} \longrightarrow 0 \qquad \text{.....(1)}$$

& use "letter place-algebra" A map is

$$\left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(p+k)} \\ 2^{(q-k)}\end{matrix}\right) \xrightarrow{\partial_{21}^{(k)}} \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(p)}\ 2^{(k)} \\ 2^{(q-k)}\end{matrix}\right) \to \sum\nolimits_{\omega} \left(\begin{matrix}\omega_{(1)} \\ \omega'\omega_{(2)}\end{matrix}\middle| \begin{matrix}(t+1)'(t+2)' \dots (p+t)' \\ 1'2'3' \dots q'\end{matrix}\right);$$

Where $\omega \otimes \omega' \in D_{p+k} \otimes D_{q-k}$, $\square = \sum_{k=t+1}^{q} \partial_{21}^{(k)}$, & $d'_{\lambda/\mu} = \partial_{q'2} \dots \partial_{1'2}$ $\partial_{(p+t)'1} \dots \partial_{(t+1)'1}$, sequence for the place polarization, from positive place {1, 2} to negative place {1', 2'… $(p+t)$'}. In specific [6] the box map □ sends element $x \otimes y$ for $D_{p+k} \otimes D_{q-k}$ to$\sum x_p \otimes x'_k\, y$; when $\sum x_p \otimes x'_k$ component for diagonal for x of $D_p \otimes D_k$[5]. The researchers in [6] [7-13] dealt with the following connotations:

Let Z_{21} be the st& for the generator for a "divided power- algebra" $D(Z_{21})$ in one free generator. "A divide power- element" $Z_{21}^{(k)}$ for degrees k for a free generator Z_{21} acts over $D_{p+k} \otimes D_{q-k}$ by" place-polarizations "for degree k from places 1 to places 2.

"A graded- algebra" with identity $\mathcal{A} = D(Z_{21})$ act over a graded modules $M = D_{p+k} \otimes D_{q-k} = \sum M_{q-k}$.Hence, M be graded left $\mathcal{A}$-modules; when $\omega = Z_{21}^{(k)} \in \mathcal{A}$ & $v \in D_{\beta_1} \otimes D_{\beta_2}$, thus:

$$\omega(v) = Z_{21}^{(k)}(v) = \partial_{21}^{(k)}(v)$$

If we pick (t^{+})- graded str& for degree q

$$M_{\bullet}: 0 \to M_{q-t} \xrightarrow{\partial_S} \dots \to M_e \xrightarrow{\partial_S} M_1 \xrightarrow{\partial_S} M_0,$$

For the normalized" Bar- complex" $\mathrm{Bar}(M, \mathcal{A}; S, \bullet)$; where $S = \{x\}$.At this point should be shown some fundamental st&ard concepts that are needed for the work

The authors for [6] [8] define the maps $\{S_i\}$ as follows:

$$S_0: D_p \otimes D_q \to \sum\nolimits_{k>0} Z^{(t+k)} x D_{p+t+k} \otimes D_{q-t-k} \qquad \dots (2).$$

$$\left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(p)} & 2^{(k)} \\ 2^{(q-k)} & \end{matrix}\right) \to \begin{cases} 0\,;\ \text{if } k \leq t \\ Z_{21}^{(k)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(p+k)} \\ 2^{(q-k)}\end{matrix}\right);\ \text{if } k > t \end{cases}$$ As for the higher dimensions

$$\mathcal{S}_{\ell-1}: \sum_{k_i>0} Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \dots Z_{21}^{(k_{\ell-1})} x\, D_{p+t+|k|} \otimes D_{q-t-|k|} \qquad \dots (3)$$

$$\to Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \dots Z_{21}^{(k_{\ell-1})} x\, Z_{21}^{(k_\ell)} x\, D_{p+t+|k|} \otimes D_{q-t-|k|}$$

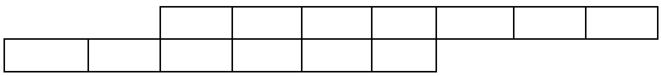
$$Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \dots Z_{21}^{(k_{\ell-1})} x \begin{pmatrix} \omega \\ \omega' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p+t+|k|)} & 2^{(m)} \\ 2^{(q-t-|k|-m)} & \end{matrix}\right)$$

$$\to \begin{cases} 0\ ;\text{if } m = 0 \\ Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \dots Z_{21}^{(k_{\ell-1})} x\, Z_{21}^{(m)} x \begin{pmatrix} \omega \\ \omega' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p+t+|k|+m)} \\ 2^{(q-t-|k|-m)} \end{matrix}\right) ;\text{if } m > 0 \end{cases}$$

While the authors for [9] write the resolution for modules as M_i , $i = 0,1,\dots,\ q-t,\ M_0 = D_p \otimes D_q,\ \&\ M_i = Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \dots Z_{21}^{(k_i)} x\, D_{p+t+|k|} \otimes D_{q-t-|k|}$, for $i \geq 1$. The authors in[10] [11] , demonstrate a term & an exactness for a resolutions Weyl module, this work shows the limits for the terms & the exactness for the resolution Weyl module in a different case for skew-partitions (9,6)/(2,0) & (9,6)/(3,0).

2. The out Come for the (9,6)/(2,0)

This part, the terms, & the exactness should be found in the two-row resolution for Weyl modules of a skew-partitions (9,6)/(2,0) with t = 2 .



A terms for characteristics free resolutions two rows is:

$M_0 = D_7 \otimes D_6$

$M_1 = Z_{21}^{(3)} x\, D_{10} \otimes D_3 \oplus Z_{21}^{(4)} x\, D_{11} \otimes D_2 \oplus Z_{21}^{(5)} x\, D_{12} \otimes D_1 \oplus Z_{21}^{(6)} x\, D_{13} \otimes D_0$

$M_2 = Z_{21}^{(3)} x Z_{21}^{(1)} x\, D_{11} \otimes D_2 \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x\, D_{12} \otimes D_1 \oplus Z_{21}^{(3)} x Z_{21}^{(2)} x\, D_{12} \otimes D_1 \oplus$
$Z_{21}^{(5)} x Z_{21}^{(1)} x\, D_{13} \otimes D_0 \oplus Z_{21}^{(4)} x Z_{21}^{(2)} x\, D_{13} \otimes D_0 \oplus Z_{21}^{(3)} x Z_{21}^{(3)} x\, D_{13} \otimes D_0$

$M_3 = Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{12} \otimes D_1 \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{13} \otimes D_0 \oplus$
$Z_{21}^{(3)} x Z_{21}^{(2)} x Z_{21}^{(1)} x\, D_{13} \otimes D_0 \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(2)} x\, D_{13} \otimes D_0$

$M_4 = Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{13} \otimes D_0$.

The complex for this terms is

$$\begin{tikzcd}
\mathcal{M}_4 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] & \mathcal{M}_3 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_3"'] & \mathcal{M}_2 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_2"'] & \mathcal{M}_1 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_1"'] & \mathcal{M}_0 \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_0"'] \\
\mathcal{M}_4 \arrow[r, "\partial_x"] & \mathcal{M}_3 \arrow[r, "\partial_x"] & \mathcal{M}_2 \arrow[r, "\partial_x"] & \mathcal{M}_1 \arrow[r, "\partial_x"] & \mathcal{M}_0
\end{tikzcd}$$

The construction for a “contracting homotopies” $\{s_i\}$,i = 1,2,3 are:

$$\mathcal{S}_0: D_7 \otimes D_6 \to \sum_{k>0} Z_{21}^{(k+2)} x\, D_{7+k} \otimes D_{6-k} \text{ Such that} \qquad \dots(4)$$

$$\mathcal{S}_0\left(\left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(7)} & 2^{(k)} \\ 2^{(6-k)} & \end{matrix}\right)\right)=\begin{cases}0\ ;\ \text{if } k \le 2 \\ Z_{21}^{(k)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(7+k)} \\ 2^{(6-k)}\end{matrix}\right)\ ;\ \text{if } k = 3,4,5,6\end{cases}$$

$$\mathcal{S}_1 : \sum_{k>0} Z_{21}^{(k+2)} x\, D_{8+k} \otimes D_{4-k} \to Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x\, D_{9+k} \otimes D_{4-k} \qquad \dots(5)$$

Such that $\mathcal{S}_1\left(Z_{21}^{(k+2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)\right)=$

$$\begin{cases}0\ ;\ \text{if } m = 0 \\ Z_{21}^{(k+2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k+m)} \\ 2^{(4-k-m)}\end{matrix}\right)\ ;\ \text{if } m = 1,2,3\end{cases}$$

$$\mathcal{S}_2 : \sum_{k_i>0} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x\, D_{9+|k|} \otimes D_{4-|k|} \to$$
$$Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, D_{9+|k|} \otimes D_{4-|k|}$$

Such that

$$\mathcal{S}_2\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$

$$=\begin{cases}0\ ;\ \text{if } m = 0 \\ Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)} \\ 2^{(4-|k|-m)}\end{matrix}\right)\ ;\ \text{if } m = 1,2,\end{cases} \text{; Where } |k| = k_1 + k_2.$$

$$\mathcal{S}_3 : \sum_{k_i>0} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, D_{9+|k|} \otimes D_{4-|k|}$$
$$\to Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x\, D_{9+|k|} \otimes D_{4-|k|} \qquad \dots.(6)$$

$$\mathcal{S}_3\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$

$$=\begin{cases}0\ ;\ \text{if } m = 0 \\ Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)} \\ 2^{(4-|k|-m)}\end{matrix}\right)\ ;\ \text{if } m = 1\end{cases};$$

Where $|k| = k_1 + k_2 + k_3$. Now, we have

$$\mathcal{S}_0 \partial_x\left(Z_{21}^{(k+2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)\right) = \mathcal{S}_0 \partial_{12}^{(k+2)} \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)$$

$$= \binom{k+2+m}{m} Z_{21}^{(k+2+m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k+m)} \\ 2^{(4-k-m)}\end{matrix}\right), \&$$

$$\partial_x \mathcal{S}_1\left(Z_{21}^{(k+2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)\right) = \partial_x\left(Z_{21}^{(k+2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k+m)} \\ 2^{(4-k-m)}\end{matrix}\right)\right)$$

$$= -\binom{k+2+m}{m} Z_{21}^{(k+2+m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k+m)} \\ 2^{(4-k-m)}\end{matrix}\right) + Z_{21}^{(k+2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)$$

Then $\mathcal{S}_0 \partial_x + \partial_x \mathcal{S}_1 = \mathrm{id}_{M_1}$.

$$\mathcal{S}_1 \partial_x\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$

$$= \mathcal{S}_1\left[-\binom{|k|+2}{k_2} Z_{21}^{(|k|+2)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|)} & 2^{(m)}\\ 2^{(4-|k|-m)} & \end{matrix}\right) + Z_{21}^{(k_1+2)} x \partial_{21}^{(k_2)} \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(8+|k|)} & 2^{(m)}\\ 2^{(4-|k|-m)} & \end{matrix}\right)\right]$$

$$= -\binom{|k|+2}{k_2} Z_{21}^{(|k|+2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)}\\ 2^{(4-|k|-m)}\end{matrix}\right) + \binom{k_2+m}{m} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2+m)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)}\\ 2^{(4-|k|-m)}\end{matrix}\right), \&$$

$$\partial_x \mathcal{S}_2\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|)} & 2^{(m)}\\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$

$$= \partial_x\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)}\\ 2^{(4-|k|-m)}\end{matrix}\right)\right)$$

$$= \binom{|k|+2}{k_2} Z_{21}^{(|k|+2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)}\\ 2^{(4-|k|-m)}\end{matrix}\right) - \binom{k_2+m}{m} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2+m)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|+m)}\\ 2^{(4-|k|-m)}\end{matrix}\right) +$$

$$Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x \left(\begin{matrix}\omega\\ \omega'\end{matrix}\middle| \begin{matrix}1^{(9+|k|)} & 2^{(m)}\\ 2^{(4-|k|-m)} & \end{matrix}\right); \text{ Where} |k| = k_1 + k_2.$$

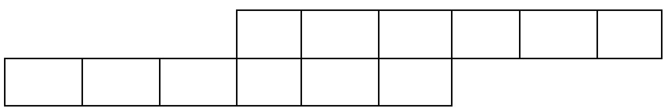
Then $\mathcal{S}_1\partial_x + \partial_x\mathcal{S}_2 = \mathrm{id}_{M_2}$.

By the same way we can prove $\mathcal{S}_2\partial_x + \partial_x\mathcal{S}_3 = \mathrm{id}_{M_3}$.

From overhead we conclude $\{\mathcal{S}_0, \mathcal{S}_i\}, i = 1,2,3$ a"contracting- homotopy" [11] [12] that means the complex be exact.

3. The outcome for the case (9, 6)/ (3, 0)

This part, the resolution two row for Weyl module should be found for the skew-partition (9,6)/(3,0) where t = 3 .



Aterms for characteristics free resolutions Weyl module:

$M_0 = D_6 \otimes D_6$

$M_1 = Z_{21}^{(4)} x\, D_{10} \otimes D_2 \oplus Z_{21}^{(5)} x\, D_{11} \otimes D_1 \oplus Z_{21}^{(6)} x\, D_{12} \otimes D_0$

$M_2 = Z_{21}^{(4)} x Z_{21}^{(1)} x\, D_{11} \otimes D_1 \oplus Z_{21}^{(4)} x Z_{21}^{(2)} x\, D_{12} \otimes D_0 \oplus Z_{21}^{(5)} x Z_{21}^{(1)} x\, D_{12} \otimes D_0$

$M_3 = Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{12} \otimes D_0$

. Then the complex is

$$\begin{tikzcd}
\mathcal{M}_3 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"'] & \mathcal{M}_2 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"'] \arrow[dl, "\mathcal{S}_2"'] & \mathcal{M}_1 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"'] \arrow[dl, "\mathcal{S}_1"'] & \mathcal{M}_0 \arrow[d, "\mathrm{id}"'] \arrow[dl, "\mathcal{S}_0"'] \\
\mathcal{M}_3 \arrow[r, "\partial_x"] & \mathcal{M}_2 \arrow[r, "\partial_x"] & \mathcal{M}_1 \arrow[r, "\partial_x"] & \mathcal{M}_0
\end{tikzcd}$$

The construction for a "contracting homotopiies" $\{s_i\}, i = 1, 2$ are:

$$\mathcal{S}_0: \mathrm{D}_6 \otimes \mathrm{D}_6 \rightarrow \sum_{k>0} Z_{21}^{(k+3)} x\, \mathrm{D}_{6+k} \otimes \mathrm{D}_{6-k} \quad \ldots(7),$$

Such that

$$\mathcal{S}_0\left(\left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(6)} & 2^{(k)} \\ 2^{(6-k)} & \end{matrix}\right)\right) = \begin{cases} 0\,;\ \text{if } k \leq 3 \\ Z_{21}^{(k)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(6+k)} \\ 2^{(6-k)}\end{matrix}\right);\ \text{if } k = 4,5,6 \end{cases}$$

$\mathcal{S}_1: \sum_{k>0} Z_{21}^{(k+3)} x\, \mathrm{D}_{9+k} \otimes \mathrm{D}_{3-k} \rightarrow Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x\, \mathrm{D}_{9+k} \otimes \mathrm{D}_{3-k} \ldots(8)$ Such that

$$\mathcal{S}_1\left(Z_{21}^{(k+3)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(3-k-m)} & \end{matrix}\right)\right) = \begin{cases} 0\,;\ \text{if } m = 0 \\ Z_{21}^{(k+3)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k+m)} \\ 2^{(3-k-m)}\end{matrix}\right);\ \text{if } m = 1,2 \end{cases}$$

$$\mathcal{S}_2: \sum_{k_i>0} Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x\, \mathrm{D}_{9+|k|} \otimes \mathrm{D}_{3-|k|} \rightarrow Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, \mathrm{D}_{9+|k|} \otimes \mathrm{D}_{3-|k|} \quad ..(8),$$

Such that

$$\mathcal{S}_2\left(Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(3-|k|-m)} & \end{matrix}\right)\right)$$

$$= \begin{cases} 0\,;\ \text{if } m = 0 \\ Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+|k|+m)} \\ 2^{(3-|k|-m)}\end{matrix}\right);\ \text{if } m = 1 \end{cases}; \text{ Where} |k| = k_1 + k_2.$$

Now, we have

$$\mathcal{S}_0 \partial_x \left(Z_{21}^{(k+3)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(3-k-m)} & \end{matrix}\right)\right) = \mathcal{S}_0 \partial_{12}^{(k+3)} \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(3-k-m)} & \end{matrix}\right)$$

$$= \binom{k+3+m}{m} Z_{21}^{(k+3+m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k+m)} \\ 2^{(3-k-m)}\end{matrix}\right), \&$$

$$\partial_x \mathcal{S}_1 \left(Z_{21}^{(k+3)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(3-k-m)} & \end{matrix}\right)\right) = \partial_x \left(Z_{21}^{(k+3)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k+m)} \\ 2^{(3-k-m)}\end{matrix}\right)\right)$$

$$= -\binom{k+3+m}{m} Z_{21}^{(k+3+m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k+m)} \\ 2^{(3-k-m)}\end{matrix}\right) + Z_{21}^{(k+3)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+k)} & 2^{(m)} \\ 2^{(3-k-m)} & \end{matrix}\right)$$

Then $\mathcal{S}_0 \partial_x + \partial_x \mathcal{S}_1 = \mathrm{id}_{M_1}$.

$$\mathcal{S}_1 \partial_x \left(Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(3-|k|-m)} & \end{matrix}\right)\right)$$

$$= \mathcal{S}_1 \Bigg[-\binom{|k|+3}{k_2} Z_{21}^{(|k|+3)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(3-|k|-m)} & \end{matrix}\right) + Z_{21}^{(k_1+3)} x \partial_{21}^{(k_2)} \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+|k|)} & 2^{(m)} \\ 2^{(3-|k|-m)} & \end{matrix}\right)\Bigg]$$

$$= -\binom{|k|+3}{k_2} Z_{21}^{(|k|+3)} x Z_{21}^{(m)} x \left(\begin{matrix}\omega \\ \omega'\end{matrix}\middle|\begin{matrix}1^{(9+|k|+m)} \\ 2^{(3-|k|-m)}\end{matrix}\right)$$

$$+\binom{k_2+m}{m} Z_{21}^{(k_1+3)} x Z_{21}^{(k_2+m)} x \left(\begin{array}{c|c} \omega & 1^{(9+|k|+m)} \\ \omega' & 2^{(3-|k|-m)} \end{array}\right) \&,$$

$$\partial_x \mathcal{S}_2 \left(Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x \left(\begin{array}{c|cc} \omega & 1^{(9+|k|)} & 2^{(m)} \\ \omega' & 2^{(3-|k|-m)} & \end{array}\right)\right)$$

$$= \partial_x \left(Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} \omega & 1^{(9+|k|+m)} \\ \omega' & 2^{(3-|k|-m)} \end{array}\right)\right)$$

$$= \binom{|k|+2}{k_2} Z_{21}^{(|k|+3)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} \omega & 1^{(9+|k|+m)} \\ \omega' & 2^{(3-|k|-m)} \end{array}\right)$$

$$-\binom{k_2+m}{m} Z_{21}^{(k_1+3)} x Z_{21}^{(k_2+m)} x \left(\begin{array}{c|c} \omega & 1^{(9+|k|+m)} \\ \omega' & 2^{(3-|k|-m)} \end{array}\right)$$

$$+Z_{21}^{(k_1+3)} x Z_{21}^{(k_2)} x \left(\begin{array}{c|cc} \omega & 1^{(9+|k|)} & 2^{(m)} \\ \omega' & 2^{(3-|k|-m)} & \end{array}\right); \text{ Where} |k| = k_1 + k_2.$$

Then $\mathcal{S}_1 \partial_x + \partial_x \mathcal{S}_2 = \mathrm{id}_{M_2}$.

From overhead we conclude $\{\mathcal{S}_0, \mathcal{S}_i\}, i = 1,2$ be "contracting homotopy" [11] [12] that means the complex be exact.

4. Discussion & Conclusion

On close analysis, the summary for all the results which came out for the research for this paper is as follows:

It has been discovered that in partition (9,6), the greater the shift, the smaller the number for characteristic free elements belonging to it, as happened that the number for characteristic free term for resolution for a skew- partition (9,6)/(2,0) is five, & the first term for these terms began with exponent 3, whereas the number for characteristic free term for resolutions for a skew-partitions (9,6)/(3,0) is four, & the first term for these terms began with exponent 4, & the first term for these terms began with an exponent for four, & the maps' complexes are exact in all cases**.**

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