*Journal of Discrete Mathematical Sciences & Cryptography*
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 4-A, pp. 1201–1206
DOI : 10.47974/JDMSC-2144

# The property of inverse shadowing in $\mathcal{G}$-group with general properties of sequences

May A. A. Al-Yaseen [†]
*Department of Mathematics*
*College of Education for Pure Sciences*
*University of Babylon*
*Babylon*
*Iraq*

Hayder Kadhim Zghair *
*Department of Software*
*College of Information Technology*
*University of Babylon*
*Babylon*
*Iraq*

Raad Safah Abood [§]
*Directorate of Education of Babylon*
*Ministry of Education*
*Babylon*
*Iraq*

**Abstract**

The property of inverse shadowing is studied in $\mathbb{Z}$– group with the following properties: structurally stable, transversality property, Anosov, expansivity, continuum–wise expansive and expanding. Besides, this property is defined in $\mathcal{G}$– group and studies general properties and relations like transitivity and sensitivity which are the most two important chaotic properties. In order for this property and sequences to be related as in $\mathbb{Z}$– group and the property of shadowing, we define this property of $\mathcal{G}$–inverse shadowing as a sequence

[†] *E-mail:* pure.may.alaa@uobabylon.edu.iq
* *E-mail:* hyderkkk@uobabylon.edu.iq (Corresponding Author)
[§] *E-mail:* raadalhulali@gmail.com

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of mappings on compact metric $\mathcal{G}$–space and study this property in uniformly conjugate with the product.

***Subject Classification:*** *34D20.*

***Keywords:*** *$\mathcal{G}$–inverse shadowing, Uniformly conjugate, Sequence of mappings, $\mathbb{Z}$–group, Transversality property, Anosov.*

## 1. Introduction

There are various reasons why inverse shadowing is important. It plays *a* crucial role in several applications including mathematical modeling, computer science, cryptography, and linear algebra [1]–[5]. The study is supposed to be dealing with general properties of sequences in an attempt to apply these results in more applications but it is found to be so difficult. Hence, maybe in the future other properties and numerical examples are introduced. When it is compared to previous work about $\mathcal{G}-$inverse shadowing, it is found that there are no results like sequences and it just studies the importance of chaos. In addition, this study introduces new types of it named (Double Period and Controlled $\mathcal{G}-$Inverse Shadowing Property) [6], [7]. Then, the study follows the definition of $\mathcal{G}-$Inverse Shadowing Property by Pilyugin who first defined the minor group and space needed. Let $\mathcal{G}$ be a finitely generated group, $\mathcal{M}$ be a $\mathcal{G}$–space and dist be a metric on $\mathcal{M}$, $\Phi$ be a uniformly continuous action of $\mathcal{G}$, and $\mathfrak{f}_{\mathfrak{g}}$ be a homeomorphism mapping on space $\mathcal{M}$ for all $\mathfrak{g} \in \mathcal{G}$, $(\mathfrak{f}_{\mathfrak{g}}(\mathfrak{p}) = \Phi_{\mathfrak{f}}(\mathfrak{g}, \mathfrak{p}))$ , and $\mathcal{S}$ be a finite generating set of $\mathcal{G}$ which is symmetric. A family $\Gamma = \{\gamma_{\mathfrak{g}} \in C(\mathcal{M}, \mathcal{M}) : \mathfrak{g} \in \mathcal{G}\}$ is a method $\mathcal{d}$ for $\Phi$ concerning $\mathcal{S}$ such that where $\mathfrak{s} \in \mathcal{S}$ for all $\mathfrak{g} \in \mathcal{G}$ and $x \in \mathcal{M}$ $dist(\gamma_{\mathfrak{s}\mathfrak{g}}(x), \Phi(\mathfrak{s}, \gamma_{\mathfrak{g}}(x))) < d$ , and $\gamma_e$ is the identity. A trajectory of the method $\Gamma$ is a family $\{x_{\mathfrak{g}} = \gamma_{\mathfrak{g}}(x_{\mathfrak{e}}) \in \mathcal{M} : \mathfrak{g} \in \mathcal{G}\}$. The first definition of inverse shadowing which uses $\mathcal{d}-$ method was obtained by Kloeden and Ombach in dynamical systems [8]. In $\mathcal{G}-$space, Pilyugin [9] is defined as follows: the action $\Phi$,w.r.t. $\mathcal{S}$, has the property of inverse shadowing if there is $\mathcal{d} > 0$ for each $\varepsilon > 0$ such that for each $\Gamma$ method $\mathcal{d}$ and $\mathfrak{p} \in \mathcal{M}$ w.r.t $\mathcal{S}$ there is a $\{x_{\mathfrak{g}}\}$ of $\Gamma$ applying $dist(\mathfrak{f}_{\mathfrak{g}}(\mathfrak{p})x_{\mathfrak{g}}) < \varepsilon$, $\mathfrak{g} \in \mathcal{G}$, where $\mathfrak{f}_{\mathfrak{g}}(\mathfrak{p}) = \Phi_f(\mathfrak{g}, \mathfrak{p})$.

## 2. Definitions by the Researchers

Authors study and prove the conjugate of mappings on $\mathcal{G}-$spaces. See [10]–[13]. The property of $\mathcal{G}-$inverse shadowing in sequence of mappings is defined by the researchers as follows: firstly, definition of methods in [14] is modified to sequences which are needed to define the property of inverse shadowing in group $\mathcal{G}$.

**Definition 2.1:** A family $\Upsilon = \{\gamma_{\mathfrak{g}_{\mathfrak{n}}} : \gamma_{\mathfrak{g}_{\mathfrak{n}}} = \gamma_{\mathfrak{g}_1} \circ \gamma_{\mathfrak{g}_2} \circ \ldots \circ \gamma_{\mathfrak{g}_{\mathfrak{n}}} \in C(\mathcal{M}, \mathcal{M}) : \mathfrak{g} \in \mathcal{G}, \mathfrak{n} = 1,2, \ldots\}$. Is a $\mathcal{d}-$ method for $\Phi$ in a sequence of mappings concerning $\mathcal{S}$,

such that $\mathrm{dist}(\gamma_{s\mathfrak{g}_\pi}(\mathrm{x}), \Phi(s, \gamma_{\mathfrak{g}_\pi}(x))) < d$ , $\mathfrak{g} \in \mathcal{G}$ , $s \in \mathcal{S}$ , $x \in \mathcal{M}$; A trajectory of the method $\Upsilon$ is a family $\{x_{\mathfrak{g}_\pi} = \gamma_{\mathfrak{g}_\pi}(x_e) \in \mathcal{M}: \mathfrak{g} \in \mathcal{G}\}$

**Proposition 2.2:** The action $\Phi$ has the property of inverse shadowing in a sequence of mappings if here is $d > 0$ for all $\varepsilon > 0$ such that for any $\Upsilon$ and all $\mathfrak{p} \in \mathcal{M}$, there is $\{x_{\mathfrak{g}_\pi}\}$ of $\Upsilon$ satisfying $\mathrm{dist}(\mathcal{F}_{\mathfrak{g}_\pi}(\mathfrak{p}), x_{\mathfrak{g}}) < \varepsilon$,$\mathfrak{g} \in \mathcal{G}$, where $\mathcal{F}_{\mathfrak{g}_\pi} = \mathfrak{f}_{\mathfrak{g}_\pi} \circ \mathfrak{f}_{\mathfrak{g}_{\pi-1}} \circ \ldots \circ \mathfrak{f}_{\mathfrak{g}}$..

## 3. Main Results

Let $(\mathcal{M}, dist_1)$ and $(\mathcal{K}, dist_2)$ be $\mathcal{G}$ −metric spaces. Let $\mathcal{I} = \{\mathcal{F}_{\mathfrak{g}_\pi}: \mathcal{F}_{\mathfrak{g}_\pi} = \mathfrak{f}_{\mathfrak{g}_\pi} \circ \mathfrak{f}_{\mathfrak{g}_{\pi-1}} \circ \ldots \circ \mathfrak{f}_{\mathfrak{g}}, \mathfrak{g} \in \mathcal{G}, \pi = 0,1,2, \ldots\}$, And $\mathcal{J} = \{\mathcal{H}_{\mathfrak{g}_\pi}: \mathcal{H}_{\mathfrak{g}_\pi} = \mathfrak{h}_{\mathfrak{g}_\pi} \circ \mathfrak{h}_{\mathfrak{g}_{\pi-1}} \circ \ldots \circ \mathfrak{h}_{\mathfrak{g}}, \mathfrak{g} \in \mathcal{G}, \pi = 0,1,2, \ldots\}$ be sequences of continuous mappings on $\mathcal{M}$ and $\mathcal{K}$ respectively.

**Proposition 3.1:** Let $\mathcal{I}$ be uniformly conjugate to $\mathcal{J}$. Then $\mathcal{I}$ has the property of $\mathcal{G}$ −inverse shadowing if and only if $\mathcal{J}$ has this property.

***Proof:*** Let $\varepsilon > 0$ be given. Since $\mathcal{I}$ is uniformly conjugate to $\mathcal{J}$, there exists a uniform homeomorphism. $\mathcal{Q}_{\mathfrak{g}}: \mathcal{M} \to \mathcal{K}$ such that $\mathcal{Q}_{\mathfrak{g}} \circ \mathcal{F}_{\mathfrak{g}_\pi} = \mathcal{H}_{\mathfrak{g}_\pi} \circ \mathcal{Q}_{\mathfrak{g}}$.., i.e. $\mathcal{F}_{\mathfrak{g}_\pi} \circ \mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{\pi-1}} = \mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{\pi-1}} \circ \mathcal{H}_{\mathfrak{g}_\pi}$, or all $\pi \geq 0$. Now, $\mathcal{Q}_{\mathfrak{g}}$ is uniformly continuous being a uniform homeomorphism. Therefore, there exists on $\varepsilon_0 > 0$ such that

$dist_1(x, y) < \varepsilon_0$ implies $dist_2(\mathcal{Q}_{\mathfrak{g}}(x), \mathcal{Q}_{\mathfrak{g}}(y)) < \varepsilon$. Let $\mathcal{I}$ have $\mathcal{G}$ −inverse shadowing. Since $\mathcal{Q}_{\mathfrak{g}}$ being a uniform homeomorphism, $\mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{-1}}$ is uniformly continuous mapping, therefore for $d_0 > 0$ there exists a $d > 0$ such that for any $x, y \in \mathcal{K}$ ,$dist_2(x, y) < d$ implies $dist_1(\mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{\pi-1}}(x), \mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{\pi-1}}(y)) < d_0$. Let $\{\gamma_{\mathfrak{g}_\pi}\}$ be a trajectory of $d$ −method for $\mathcal{J}$ such that $dist_2((\gamma_{\mathcal{H}_{\mathfrak{g}_\pi}})_{s\mathfrak{g}}(y), \Phi(s, (\gamma_{\mathcal{H}_{\mathfrak{g}_\pi}})(y))) < d$, i.e.

$$dist_1((\gamma_{\mathcal{F}_{\mathfrak{g}_\pi}})_{s\mathfrak{g}}(\gamma_{\mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{-1}}})(y), \Phi)(s, \gamma_{\mathcal{F}_{\mathfrak{g}_\pi}}(\gamma_{\mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{-1}}})(y)) < d_0$$

which implies $\{\mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{-1}}(y)_{\mathfrak{g}\pi}\}$ is a trajectory of $d_0$ −method for $\mathcal{I}$. Thus, there exists $x \in \mathcal{M}$ such that $dist_1(\mathcal{F}_{\mathfrak{g}_\pi}(x), \mathcal{Q}_{\mathfrak{g}\mathfrak{g}_{-1}}(y)_{\mathfrak{g}_\pi}) < \varepsilon_0$, and hence $dist_2(\mathcal{Q}_{\mathfrak{g}}(\mathcal{F}_{\mathfrak{g}_\pi}(x)), (y)_{\mathfrak{g}_\pi}) < \varepsilon$. Now, for all $\pi \geq 0$, $\mathcal{Q}_{\mathfrak{g}} \circ \mathcal{F}_{\mathfrak{g}_\pi} = \mathcal{Q}_{\mathfrak{g}} \circ \mathfrak{f}_{\mathfrak{g}_\pi} \circ \mathfrak{f}_{\mathfrak{g}_{\pi-1}} \circ \ldots \circ \mathfrak{f}_{\mathfrak{g}}$ , … Generally speaking, $\mathfrak{h}_{\mathfrak{g}_\pi} \circ \mathfrak{h}_{\mathfrak{g}_{\pi-1}} \circ \ldots \circ \mathfrak{h}_{\mathfrak{g}} \circ \mathcal{Q}_{\mathfrak{g}} = \mathcal{H}_{\mathfrak{g}_\pi} \circ \mathcal{Q}_{\mathfrak{g}}$

Implies $dist_2(\mathcal{H}_{\mathfrak{g}_\pi}(\mathcal{Q}_{\mathfrak{g}}(x)), (y)_{\mathfrak{g}_\pi}) < \varepsilon$, i.e. there is $\{\gamma_{\mathfrak{g}_\pi}\}$ of $d$ −method for any point $\mathcal{Q}_{\mathfrak{g}}(x) \in \mathcal{K}$. Thus $\mathcal{J}$ has $\mathcal{G}$ −inverse shadowing. In the same way, we can prove the converse.

**Proposition 3.2:** Both $\mathcal{I}$ and $\mathcal{J}$ have the property of $\mathcal{G}$ −inverse shadowing if and only if $\mathcal{I} \times \mathcal{J} = \{\mathcal{F}_{\mathfrak{g}_\pi} \times \mathcal{H}_{\mathfrak{g}_\pi}: \mathfrak{g} \in \mathcal{G}, \pi = 0,1,2, \ldots\}$ has this property.

***Proof:*** Define metric $dist$ on $\mathcal{M} \times \mathcal{K}$ by $dist((x_1, y_1), (x_2, y_2)) = max\{dist_1(x_1, x_2), dist_2(y_1, y_2)\}$, $(x_1, y_1), (x_2, y_2) \in \mathcal{M} \times \mathcal{K}$. Note that, for any $\mathfrak{g} \in \mathcal{G}$, $(\mathcal{F}_{\mathfrak{g}_\pi} \times \mathcal{H}_{\mathfrak{g}_\pi})(x, y) = (\mathcal{F}_{\mathfrak{g}_\pi}(x), \mathcal{H}_{\mathfrak{g}_\pi}(y))$, $(x, y) \in \mathcal{M} \times \mathcal{K}$. Let

$\varepsilon > 0$ be given and $\mathcal{J}, \mathcal{J}$have $\mathcal{G}$ −inverse shadowing, then there exists a $d_1 > 0$ and $d_2 > 0$ such that: for any $\mathfrak{p} \in \mathcal{M}$ and any $d_1$ −method there is $\{\mathfrak{x}_{\mathfrak{g}\mathfrak{n}}\}$ satisfying

$dist_1(\mathcal{F}_{\mathfrak{g}_\mathfrak{n}}(\mathfrak{p}), x_{\mathfrak{g}\mathfrak{n}}) < \varepsilon$, and, for any $d_2$ −method and any $\mathfrak{q} \in \mathcal{M}$ there is $\{\gamma_{\mathfrak{g}\mathfrak{n}}\}$ satisfying $dist_2(\mathcal{H}_{\mathfrak{g}_\mathfrak{n}}(\mathfrak{q}), \mathcal{y}_{\mathfrak{g}\mathfrak{n}}) < \varepsilon$ .Let $d = min\{d_1, d_2\}$ and $\{(x_\mathfrak{g}, \mathcal{y}_\mathfrak{g})\}$ be a trajectory for $\mathcal{J} \times \mathcal{J}$. Then

$$dist((\gamma_{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}} \times \mathcal{H}_{\mathfrak{g}_\mathfrak{n}}})_{\mathcal{s}\mathfrak{g}}(x, \mathcal{y}), \qquad \Phi(\mathcal{s}, (\gamma_{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}} \times \mathcal{H}_{\mathfrak{g}_\mathfrak{n}}})_g(x, \mathcal{y}))) < d,$$

and by definition of $dist$ , implies

$$dist_1((\gamma_{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}}})_{\mathcal{s}\mathfrak{g}}(x), \Phi(\mathcal{s}, (\gamma_{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}}})_g(x))) < d_1,$$
$$dist_2((\gamma_{\mathcal{H}_{\mathfrak{g}_\mathfrak{n}}})_{\mathcal{s}\mathfrak{g}}(\mathcal{y}), \Phi(\mathcal{s}, (\gamma_{\mathcal{H}_{\mathfrak{g}_\mathfrak{n}}})_g(\mathcal{y}))) < d_2.$$

Hence $dist(\mathcal{F}_{\mathfrak{g}_\mathfrak{n}}(\mathfrak{p}), \mathcal{H}_{\mathfrak{g}_\mathfrak{n}}(\mathfrak{q}), (x_{\mathfrak{g}\mathfrak{n}}, \mathcal{y}_{\mathfrak{g}\mathfrak{n}})) < \varepsilon$. So, for any method $d$ and any $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{M} \times \mathcal{K}$ there exists a trajectory $\{\mathfrak{x}_{\mathfrak{g}\mathfrak{n}}, \gamma_{\mathfrak{g}\mathfrak{n}}\}$ satisfying $dist((\mathcal{F}_{\mathfrak{g}_\mathfrak{n}} \times \mathcal{H}_{\mathfrak{g}_\mathfrak{n}})(\mathfrak{p}, \mathfrak{q}), (x_{\mathfrak{g}\mathfrak{n}}, \mathcal{y}_{\mathfrak{g}\mathfrak{n}})) < \varepsilon$. Then $\mathcal{J} \times \mathcal{J}$ has $\mathcal{G}$ −inverse shadowing. The converse can be proved by similar arguments.

**Proposition 3.3:** If $\mathcal{J}$ has the property of $\mathcal{G}$ −inverse shadowing then $\mathcal{J}^\mathfrak{k} = \{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_\mathfrak{k}} : \mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_\mathfrak{k}} = \mathfrak{f}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_\mathfrak{k}} \circ \mathfrak{f}_{\mathfrak{g}_{\mathfrak{n}-1}\mathfrak{g}_\mathfrak{k}} \circ \ldots \circ \mathfrak{f}_{\mathfrak{g}\mathfrak{g}_\mathfrak{k}}, \mathfrak{g} \in \mathcal{G}, \mathfrak{n} = 0,1,2, \ldots\}$ has this property for all $\mathfrak{k} > 1$.

***Proof:*** Since $\mathcal{J}$ has $\mathcal{G}$ −inverse shadowing, therefore there exists $\{\mathfrak{x}_{\mathfrak{g}\mathfrak{n}}\}$ satisfying $dist(\mathcal{F}_{\mathfrak{g}_\mathfrak{n}}(\mathfrak{p}), x_{\mathfrak{g}\mathfrak{n}}) < \varepsilon$. Let $\{\gamma_{\mathfrak{g}\mathfrak{n}}\}$ be a trajectory of $d$ −method for $\mathcal{J}^\mathfrak{k}$. Then

$dist((\gamma_{\mathcal{H}_{\mathfrak{g}_{\mathfrak{n}+1}}})_{\mathcal{s}\mathfrak{g}}(\mathcal{y}), \Phi(\mathcal{s}, (\gamma_{\mathcal{H}_{\mathfrak{g}_{\mathfrak{n}+1}}})_\mathfrak{g}(\mathcal{y}))) < d$, for all $\mathfrak{n} \geq 0$, where $\mathcal{H}_{\mathfrak{g}_\mathfrak{n}} = \mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_{\mathfrak{k}-\mathfrak{k}+1,\mathfrak{n}\mathfrak{k}-k+2,\ldots,\mathfrak{n}\mathfrak{k}}}$, which means,

$$dist((\gamma_{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_{\mathfrak{k}-\mathfrak{k}+1,\mathfrak{n}\mathfrak{k}-k+2,\ldots,\mathfrak{n}\mathfrak{k}}}})_{\mathcal{s}\mathfrak{g}}(\mathcal{y}), \Phi(\mathcal{s}, (\gamma_{\mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_{\mathfrak{k}-\mathfrak{k}+1,\mathfrak{n}\mathfrak{k}-k+2,\ldots,\mathfrak{n}\mathfrak{k}}}})_\mathfrak{g}(\mathcal{y}))) < d.$$

For $0 \leq \mathfrak{j} < \mathfrak{k}$ and $\mathfrak{n} \geq 0$, put $x_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}}} = \mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_{\mathfrak{k}-\mathfrak{k}+1,\mathfrak{n}\mathfrak{k}-k+2,\ldots,\mathfrak{n}\mathfrak{k}}}(\mathcal{y})$. Claim $\{\mathfrak{x}_{\mathfrak{g}\mathfrak{n}}\}$ is a trajectory for $d$ −method of $\mathcal{J}$ such that $dist((\gamma_{\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}})_{\mathcal{s}\mathfrak{g}}(x), \Phi(\mathcal{s}, (\gamma_{\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}})_\mathfrak{g}(x))) < d$, for all $\mathfrak{n} \geq 0$, and for all $\mathfrak{j}$, $0 \leq \mathfrak{j} < \mathfrak{k}$. Choose any $\mathfrak{n} \geq 0$. Now, for any $j$, $0 \leq \mathfrak{j} < \mathfrak{k} - 2$, $\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}(x_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}) = \mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}(\mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_{\mathfrak{k}-\mathfrak{k}+1,\mathfrak{n}\mathfrak{k}-k+2,\ldots,\mathfrak{n}\mathfrak{k}}}(\mathcal{y}))$. Thus,

$$dist((\gamma_{\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}})_{\mathcal{s}\mathfrak{g}}(x_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}), \Phi(\mathcal{s}, (\gamma_{\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}})_\mathfrak{g}(x_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{j}+1}}))) < d,$$

for all $j$, $0 \leq \mathfrak{j} < \mathfrak{k} - 2$. Now, if we put $\mathfrak{j} = \mathfrak{k} - 1$, then we get

$$dist((\gamma_{\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{k}}}})_{\mathcal{s}\mathfrak{g}}(x_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{k}}}), \Phi(\mathcal{s}, (\gamma_{\mathfrak{f}_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{k}}}})_\mathfrak{g}(x_{\mathfrak{n}\mathfrak{g}_{\mathfrak{k}+\mathfrak{k}}}))) < d.$$

Hence the claim. By $\mathcal{G}$ −inverse shadowing of $\mathcal{F}_{\mathfrak{g}\mathfrak{i}}$, for any method $d$ and any $\mathfrak{p} \in \mathcal{M}$ there is $\{\mathfrak{x}_{\mathfrak{g}\mathfrak{i}}\}$ satisfying $dist(\mathcal{F}_{\mathfrak{g}\mathfrak{i}}(p), x_{\mathfrak{g}_\mathfrak{i}}) < \varepsilon$, for all $\mathfrak{i} \geq 0$. In particular, for $\mathfrak{i} = \mathfrak{k}\mathfrak{n}$, $dist(\mathcal{F}_{\mathfrak{g}\mathfrak{k}\mathfrak{n}}(p), x_{\mathfrak{g}_{\mathfrak{k}\mathfrak{n}}}) < \varepsilon$. Thus, $dist(\mathcal{F}_{\mathfrak{g}_\mathfrak{n}\mathfrak{g}_\mathfrak{k}}(\mathfrak{p}), \mathcal{y}_{\mathfrak{g}_\mathfrak{n}}) < \varepsilon$, Thus, $\mathcal{J}^\mathfrak{k}$ has $\mathcal{G}$ −inverse shadowing.

In the next proposition, it is noticed that the inverse mapping of the same property also is preserved in sequences.

**Proposition 3.4:** If $\mathcal{J}$ has the property of $\mathcal{G}$ −inverse shadowing then

$$\mathcal{J}^{-1} = \{\mathcal{F}_{\mathfrak{g}_{\mathfrak{n}}\mathfrak{g}_{-1}} : \mathcal{F}_{\mathfrak{g}_{\mathfrak{n}}\mathfrak{g}_{-1}} = \mathfrak{f}_{\mathfrak{g}_{\mathfrak{n}}\mathfrak{g}_{-1}} \circ \mathfrak{f}_{\mathfrak{g}_{\mathfrak{n}-1}\mathfrak{g}_{-1}} \circ \dots \circ \mathfrak{f}_{\mathfrak{g}\mathfrak{g}_{-1}}, \mathfrak{g} \in \mathcal{G}, \mathfrak{n} = 0,1,2, \dots\}$$

has this property.

***Proof:*** For $d > 0$, $\{\mathfrak{x}_{\mathfrak{g}\mathfrak{n}}\}$ is a trajectory of $d$ −method for $\mathcal{J}_{\mathfrak{g}}$ if

$$dist((\gamma_{\mathcal{F}_{\mathfrak{n}+1}})_{s\mathfrak{g}}x, \Phi(s,)\gamma_{\mathcal{F}_{\mathfrak{n}+1-\mathfrak{g}}}(x))) < d, \mathfrak{n} \geq 0 \text{ , and}$$

$$dist((\gamma_{\mathcal{F}_{\mathfrak{n}}})_{s\mathfrak{g}}x, \Phi(s, (\gamma_{\mathcal{F}_{\mathfrak{n}}})_{\mathfrak{g}}(x))) < d, \mathfrak{n} \leq -1,$$

i.e. $dist((\gamma_{\mathcal{F}_{\mathfrak{n}}})_{s\mathfrak{g}}x, \Phi(s, (\gamma_{\mathcal{F}_{\mathfrak{n}}})_{\mathfrak{g}}(x))) < d$, $\mathfrak{n} \geq 1$, $dist((\gamma_{\mathcal{F}_{\mathfrak{n}+1}})_{s\mathfrak{g}}x, \Phi(s, (\gamma_{\mathcal{F}_{\mathfrak{n}+1}})_{\mathfrak{g}}(x))) < d, \mathfrak{n} \leq 0$. Putting $\mathfrak{n} = -\mathfrak{m}$ we have

$$dist((\gamma_{\mathcal{F}_{\mathfrak{m}\mathfrak{g}_{-1}}})_{s\mathfrak{g}}(x), \Phi(s, (\gamma_{\mathcal{F}_{\mathfrak{m}\mathfrak{g}_{-1}}})_{\mathfrak{g}}(x))) < d, \mathfrak{m} \leq -1,$$

$$dist((\gamma_{\mathcal{F}_{\mathfrak{m}+1\mathfrak{g}_{-1}}})_{s\mathfrak{g}}(x), \Phi(s, (\gamma_{\mathcal{F}_{\mathfrak{m}+1\mathfrak{g}_{-1}}})_{\mathfrak{g}}(x))) < d, \mathfrak{m} \geq 0.$$

Thus $\{y_{g_n}\}$ is a $d$ −method's trajectory for $\mathcal{J}^{-1}$. If $\mathfrak{p} \in \mathcal{M}$, there is $\{\mathfrak{x}_{\mathfrak{g}-1}\} = \{\gamma_{\mathfrak{g}\mathfrak{n}}\}$ a trajectory applying $dist(\mathcal{F}_{\mathfrak{g}\mathfrak{g}_{-1}}(\mathfrak{p}), x_{\mathfrak{g}-1}) < \varepsilon$, for all $\mathfrak{g} \in \mathcal{G}$, i.e. $dist(\mathcal{F}_{\mathfrak{g}\mathfrak{g}_{-1}}(\mathfrak{p}), y_{\mathfrak{g}\mathfrak{n}}) < \varepsilon$, Thus, $\mathcal{J}^{-1}$ has $\mathcal{G}$ −inverse shadowing. The converse will be proved by using the same steps.

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*Received December, 2024*