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The complex elements of the skew partition (12,6,3)/(1,0)

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Abstract

Let $\mathscr{F}$ be a free R-module, where R is a commutative ring with 1 and D_nF is the nth degree divided by power. The place polarization technique is a combinatorial approach to computing complex elements. $\partial_{21}^{(k)}$ is a location polarization that occurs from place one to place two and $\partial_{32}^{(k)}$ is a location polarization that occurs from place two to place three.

In this paper, we used the diagrams and divided power, together with their Capelli identities, to find the complex elements of the skew partition (12,6,3)/(1,0).

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1. Introduction

Authors in [1] and [2] study the partition (2,2,2) and (3,3,3) in detail, the complex of Lascaux or complex elements for different partitions modified by many authors, $D(Z_{21})$, $M = \sum D_{p+b} \otimes \mathcal{D}_{q-b} = \sum M_{q-b}$, [3].

In this work we need the following, [4]

$$\partial_{32}^{(1)}\partial_{21}^{(1)} - \partial_{21}^{(1)}\partial_{32}^{(1)} = \partial_{31}^{(1)} \tag{1.1}$$

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$$\partial_{32}^{(k)}\partial_{21}^{(l)} = \sum_{a\geq 0}\partial_{21}^{(l-a)}\ \partial_{32}^{(k-a)}\partial_{31}^{(a)};\ \text{where } \partial_{ij}^{(0)} = \mathrm{I} \qquad (1.2)$$

For k ≠ i

$$\partial_{ij}^{(r)}\partial_{jk}^{(s)} = \sum_{a\geq 0}\partial_{jk}^{(s-a)}\ \partial_{ij}^{(r-a)}\partial_{ik}^{(a)} \qquad (1.3)$$

$$\partial_{jk}^{(s)}\partial_{ij}^{(r)} = \sum_{a\geq 0}(-1)^{a}\partial_{jk}^{(s-a)}\ \partial_{ij}^{(r-a)}\partial_{ik}^{(a)} \qquad (1.4)$$

$$\partial_{21}^{(l)}\ \partial_{31} = \partial_{31}\partial_{21}^{(k)} \qquad (1.5)$$

$$\partial_{32}^{(k)}\partial_{31} = \partial_{31}\partial_{32}^{(k)} \qquad (1.6)$$

We use the same idea in [2] for the skew-partition (12,6,3)/(1,0). Also we can apply that for $\mathcal{PSL}(2,\mathfrak{U})$ and $\mathcal{SL}(2,\mathfrak{U})$, $\mathfrak{U}$ = 31, 37 and weakly essential and uniform module, 2-Visible submodules and modules, [5-8].

2. The Results for (12,6,3)/(1,0)

The complex of Lascoux in this case:

$$D_{14}F\otimes D_6F\otimes D_0F \xrightarrow{\delta_3} \begin{matrix} D_{14}F\otimes D_4F\otimes D_2F \\ \oplus \\ \mathrm{D}_{13}\mathrm{F}\otimes \mathrm{D}_7\mathrm{F}\otimes \mathrm{D}_0\mathrm{F}\end{matrix} \xrightarrow{\delta_2} \begin{matrix} D_{11}F\otimes D_7F\otimes D_2F \\ \oplus \\ \mathrm{D}_{13}\mathrm{F}\otimes \mathrm{D}_4\mathrm{F}\otimes \mathrm{D}_3\mathrm{F}\end{matrix} \xrightarrow{\delta_1} D_{11}F\otimes D_6F\otimes D_3F$$

Consider the following:

$$\begin{tikzcd}
D_{14}F\otimes D_6F\otimes D_0F \arrow[r, "g_1"] \arrow[d, "t_1"'] \arrow[dr, phantom, "\mathrm{S}"] & D_{13}F\otimes D_7F\otimes D_0F \arrow[r, "g_2"] \arrow[d, "t_2"'] \arrow[dr, phantom, "\mathrm{R}"] & D_{11}F\otimes D_7F\otimes D_2F \arrow[d, "t_3"'] \\
D_{14}F\otimes D_4F\otimes D_2F \arrow[r, "f_1"] & D_{13}F\otimes D_4F\otimes D_3F \arrow[r, "f_2"] & D_{11}F\otimes D_6F\otimes D_3F
\end{tikzcd}$$

Diagram 1

So, $g_1(v)= \partial_{21}(v)$; $t_1(v)= \partial_{32}^{(2)}(v)$; $f_2(v)= \partial_{21}^{(2)}(v)$; $t_2(v)= \partial_{32}^{(3)}(v)$; $t_3(v)= \partial_{32}(v)$.

Define $f_1(v): D_{14}F\otimes D_4F\otimes D_2F \to D_{13}F\otimes D_4F\otimes D_3F$; as $f_1(v)=(\frac{1}{3}\partial_{21}\partial_{32} + \partial_{31})(v)$

Proposition 2.1: The diagram S in diagram (1) is commutative.

Proof: $(t_2 \circ g_1)(v) = \partial_{32}^{(3)} \circ \partial_{21}(v) = \partial_{21} \circ \partial_{32}^{(3)} + \partial_{32}^{(2)} \circ \partial_{31}$, and

$$(f_1 \circ t_1)(v) = (\tfrac{1}{3}\partial_{21}\circ\partial_{32} + \partial_{31})\circ\partial_{32}^{(2)} = \partial_{21}\circ\partial_{32}^{(3)} + \partial_{32}^{(2)}\circ\partial_{31}$$

Now we define $g_2(v): D_{14}F\otimes D_7F\otimes D_0F \to D_{11}F\otimes D_4F\otimes D_2F$; by

$$g_2(v) = \tfrac{1}{3}\ \partial_{32}^{(2)}\circ\partial_{21}^{(2)} - \tfrac{1}{2}\ \partial_{32}\circ\partial_{21}\circ\partial_{31} + \partial_{31}^{(2)}$$

Proposition 2.2: The diagram R in diagram (1) is commutative.

Proof: $(f_2 \circ t_2)(v) = \partial_{21}^{(2)} \circ (\frac{1}{3}\ \partial_{32}^{(2)} \circ \partial_{21}^{(2)} - \frac{1}{2}\ \partial_{32} \circ \partial_{21} \circ \partial_{31}\ + \partial_{31}^{(2)})$

$= \partial_{32}^{(3)} \circ \partial_{21}^{(2)} - \partial_{32}^{(2)} \circ \partial_{21} \circ \partial_{31} + \partial_{32} \circ \partial_{31}^{(2)} = \partial_{21}^{(2)}\ \partial_{32}^{(3)}$

$= \partial_{32}^{(3)} \circ \partial_{21}^{(2)} - \partial_{32}^{(2)} \circ \partial_{21} \circ \partial_{31} + \partial_{32} \circ \partial_{31}^{(2)}$ By using Capelli identities

Now,

$$\begin{tikzcd}
D_{14}F \otimes D_6F \otimes D_0F \arrow[r, "g_1"] \arrow[d, "t_1"'] \arrow[dr, phantom, "B"] & D_{13}F \otimes D_7F \otimes D_0F \arrow[r, "g_2"] \arrow[dr, phantom, "C"] & D_{11}F \otimes D_7F \otimes D_2F \arrow[d, "t_3"] \\
D_{14}F \otimes D_4F \otimes D_2F \arrow[r, "f_1"] \arrow[urr, "h"] & D_{13}F \otimes D_4F \otimes D_3F \arrow[r, "f_2"] & D_{11}F \otimes D_6F \otimes D_3F
\end{tikzcd}$$

Diagram 2

h(v): $D_{14}F \otimes D_4F \otimes D_2F \rightarrow\ D_{11}F \otimes D_7F \otimes D_2F$ by h$(v) = \partial_{21}^{(3)}$;where v $\in D_{14}F \otimes D_4F \otimes D_2F$.

Proposition 2,3: The diagram B in diagram (2) is commutative.

Proof: $\left(g_2 \circ g_1\right)(v) = (\frac{1}{3}\ \partial_{32}^{(2)} \circ \partial_{21}^{(2)} - \frac{1}{2}\ \partial_{32} \circ \partial_{21} \circ \partial_{31}\ + \partial_{31}^{(2)}) \circ \partial_{21}$

$= \partial_{32}^{(2)} \circ \partial_{21}^{(3)} -\ \partial_{32} \circ \partial_{21}^{(2)} \circ \partial_{31}\ + \partial_{21} \circ \partial_{31}^{(2)}$

$= \partial_{21}^{(3)} \circ \partial_{32}^{(2)}$

$= (h \circ t_1)(v)$

Proposition 2,4: The diagram C in diagram (2) is commutative.

Proof: $(f_2 \circ f_1)(v) = \partial_{21}^{(2)} \circ (\frac{1}{3} \partial_{21} \circ \partial_{32}\ +\ \partial_{31})$

$= \partial_{21}^{(3)} \circ \partial_{32} + \partial_{21}^{(2)} \circ \partial_{31}$

$= \partial_{32} \circ \partial_{21}^{(3)}$

$= (t_3 \circ h)(v)$

Finally, we can define the maps δ_1 , δ_2 and δ_3 where:

$$\delta_3 : D_{14}F \otimes D_6F \otimes D_0F \longrightarrow \begin{array}{c} D_{14}F \otimes D_4F \otimes D_2F \\ \oplus \\ D_{13}F \otimes D_7F \otimes D_0F \end{array}$$

$$\delta_2 : \begin{array}{c} D_{14}F \otimes D_4F \otimes D_2F \\ \oplus \\ D_{13}F \otimes D_7F \otimes D_0F \end{array} \longrightarrow \begin{array}{c} D_{11}F \otimes D_7F \otimes D_2F \\ \oplus \\ D_{13}F \otimes D_4F \otimes D_3F \end{array}$$

And

$$\delta_1 : \begin{array}{c} D_{11}F \otimes D_7F \otimes D_2F \\ \oplus \\ D_{13}F \otimes D_4F \otimes D_3F \end{array} \longrightarrow D_{11}F \otimes D_6F \otimes D_3F$$

By
$\delta_3(x) = (g_1(x), t_1(x)), \delta_2((x, y)) = (g_2(x) - h(y), f_1(y) - t_2(x))$, $\delta_1((x, y)) = (t_3(x) + f_2(y))$; $\forall (x_1, x_2)$

Proposition 2.5: The sequence

$$0 \longrightarrow D_{14}F \otimes D_6F \otimes D_0F \xrightarrow{\delta_3} \begin{matrix} D_{14}F \otimes D_4F \otimes D_2F \\ \oplus \\ D_{13}F \otimes D_7F \otimes D_0F \end{matrix} \xrightarrow{\delta_2} \begin{matrix} D_{11}F \otimes D_7F \otimes D_2F \\ \oplus \\ D_{13}F \otimes D_4F \otimes D_2F \end{matrix} \xrightarrow{\delta_1} D_{11}F \otimes D_6F \otimes D_3F,$$

complex.

Proof: $\partial_{32}^{(2)}$ *and* ∂_{21} are injective [9], then
$(\delta_2 \circ \delta_3)(x) = \delta_2(g_1(x), t_1(x))$
$(\delta_2 \circ \delta_3)(x) = \delta_2(\partial_{21}(x), \partial_{32}^{(2)}(x)) = (g_2(\partial_{21}(x)) - h(\partial_{32}^{(2)}(x), f_1(\partial_{32}^{(2)}(x)) - t_2(\partial_{21}(x)))$.
So
$g_2(\partial_{21}(x)) - h(\partial_{32}^{(2)}(x)) =$
$(\partial_{32}^{(2)} \circ \partial_{21}^{(3)} - \partial_{32} \circ \partial_{21}^{(2)} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)} - \partial_{32}^{(2)} \circ \partial_{21}^{(3)} + \partial_{32} \circ \partial_{21}^{(2)} \circ \partial_{31} - \partial_{21} \circ \partial_{31}^{(2)}$ $= 0$.
$f_1(\partial_{32}^{(2)}(x)) - t_2(\partial_{21}(x)) = \left(\frac{1}{3}\partial_{21} \circ \partial_{32} + \partial_{31}\right)\partial_{32}^{(2)}(x) - \partial_{32}^{(3)} \circ \partial_{21}(x)$
$= (\partial_{21} \circ \partial_{32}^{(3)} + \partial_{32}^{(2)} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(3)} - \partial_{32}^{(2)} \circ \partial_{31})(x) = 0.$
Finally,
$(\delta_1 \circ \delta_2)(x, y) = \delta_1(g_2(x) - h(y), f_1(y) - t_2(x))$
$= \delta_1((\frac{1}{3}\partial_{32}^{(2)}\partial_{21}^{(2)} - \frac{1}{2}\partial_{32}\partial_{21}\partial_{31} + \partial_{31}^{(2)})(x) - \partial_{21}^{(3)}(y), (\frac{1}{3}\partial_{21}\partial_{32} +$
$\partial_{31})(y) - \partial_{32}^{(3)}(x)$
$=(\partial_{32}^{(3)} \circ \partial_{21}^{(2)} - \partial_{32}^{(2)}\partial_{21}\partial_{31} + \partial_{32}\partial_{31}^{(2)} - \partial_{21}^{(2)}\partial_{32}^{(3)})(x) + (\partial_{21}^{(3)}\partial_{32} + \partial_{21}^{(2)}\partial_{31} -$
$\partial_{32}\partial_{21}^{(3)})(y)$
$= (\partial_{32}^{(3)}\partial_{21}^{(2)} - \partial_{32}^{(2)}\partial_{21}\partial_{31} + \partial_{32}\partial_{31}^{(2)} - \partial_{32}^{(3)} \circ \partial_{21}^{(2)} + \partial_{32}^{(2)} \circ \partial_{21} \circ \partial_{31} -$
$\partial_{32} \circ \partial_{31}^{(2)})(x) + (\partial_{21}^{(3)} \circ \partial_{32} + \partial_{21}^{(2)} \circ \partial_{31} - \partial_{21}^{(3)} \circ \partial_{32} - \partial_{21}^{(2)} \circ \partial_{31})(y) = 0$

Conclusion

From the previous work we concluded that the sequence complex of the elements of the skew partition (12,6,3)/(1,0).

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