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Some properties of the result involution graph of Harada-Norton group HN

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Abstract

Assume that $G \cong HN$ the Harada–Norton group. In this paper, effective investment for the graph $\Gamma_{\mathrm{RI}}^{\mathrm{HN}}$ standard features to acquire meaningful algebraic results for the graph $\Gamma_{\mathrm{RI}}^{\mathrm{HN}}$ and its corresponding group *HN*. For instance, marketing a modern methods to understand the way of create a precise small subgroups in G. Furthermore, performing a full investigation for getting particular $\Gamma_{\mathrm{RI}}^{\mathrm{HN}}$ parameters.

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Keywords: *Harada–norton group, Finite graph, Involutions.*

1. Introduction

Recently, among the most commonly techniques utilized to analyze the structure of finite groups is by using their graphical representation see [1,3,7,8 and 12]. Using the same aforementioned approach, the result involution graph

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on a finite group G was presented by Jund and Salih [5].The symbol ,Γ_G^{RI}, is utilized to represent this sort of graph, also, the symbols V(Γ_G^{RI}), E(Γ_G^{RI}) applied for the set of vertices and edges of Γ_G^{RI} respectively. Thus, for all $\lambda, \mu \in$ V(Γ_G^{RI}), $(\lambda, \mu) \in$ E(Γ_G^{RI}) if and only if $\lambda\mu$ is an involution element of G and $\lambda \neq \mu$. The beginning was with an analysis of Γ_G^{RI} , for some certain finite group such as, S_n , A_n and D_n see [5]. Following then, there was a discernible advancement in the investigation of this graph for various kinds of finite simple groups, such as Mathieu groups [2], also McLaughlin and Higman–Sims groups in [4]. Throughout this paper, we assume that Γ is a simple graph with finite order, and the distance between $\lambda, \mu \in$ V(Γ), is indicted by $d(\lambda, \mu)$. The girth, the clique number, the diameter, the radius and the chromatic number are denoted by ,Gir(Γ),ω(Γ), $Diam(\Gamma)$, R(Γ) and χ(Γ) respectively. For λ ∈ V(Γ), let $\delta_n(\lambda) = \{ \mu \in$ V(Γ): $d(\lambda, \mu) = n\}$ be the n-disc of λ in Γ. Finally, if the V(Γ) $= \Upsilon \cup \Psi$ and $\tau \neq \sigma$ are positive integer such that degree(λ)=τ for each $\lambda \in \Upsilon$ and degree(λ)=σ for each $\lambda \in \Psi$, then Γ is referred to (τ, σ)-biregular graph. For deep details in this regard we refer to [10].This study contributes to developing information associate with the algebraic construction of the group HN by providing a comprehensive analysis of the graph $\Gamma_{HN}^{\mathrm{RI}}$.

2. Preliminary

Throughout this section, some of the pertinent ideas and findings for the graph Γ_G^{RI} are offered when G is a finite group. Assume that $CS(G) = \{CS_1, CS_2, \dots, CS_l\}$ be the set of the G-conjugacy classes, also, let τCS be the union of all G- conjugacy classes with τ representative order.

Proposition 1: [5] Γ_G^{RI} size equitable to 0.5 *($|2CS| * |G|$ - $|4CS|$).

Proposition 2: [2] If $4CS \neq \emptyset$, then degree(λ)= $|2CS| - 1$ for all $\lambda \in 4CS$, and for the rest degree(λ)= |2CS|.

Definition 1: [5] The resize graph ,Γ_G^{RS}, is a simple graph satisfying that V(Γ_G^{RS}) $= CS(G)$ and $(CS_k, CS_l) \in$ E(Γ_G^{RS}) if and only if $(\lambda, \mu) \in$ E(Γ_G^{RI}) for some λ ∈ CS_k and μ ∈ CS_l.

Proposition 3: [2] Assuring the connection of Γ_G^{RS} ensures that Γ_G^{RI} possesses the same quality, and the opposite is also true.

To reinforce the above concepts, we find it useful to equip the following example, which elucidate computationally by Gap [11] and Cedillo [6]. In this paper, we substitute $\Gamma_G^{RS}, \Gamma_G^{RI}$ with Γ^{RS}, Γ^{RI} for simplicity.

Example 1: Consider the general linear group $GL_2(3)$. It is evident by the computational approach that $\Gamma_{GL_2(3)}^{RS}$ and $\Gamma_{GL_2(3)}^{RI}$ are connected. Furthermore, below is a table containing the most frequently applied attributes which appears as a comparison between $\Gamma_{GL_2(3)}^{RS}$ and $\Gamma_{GL_2(3)}^{RI}$:

Table 1
Comparison between $\boldsymbol{\Gamma^{RS}_{GL_2(3)}}$ and $\boldsymbol{\Gamma^{RI}_{GL_2(3)}}$.

|E (Γ^{RS})|	Diam(Γ^{RS})	R(Γ^{RS})	ω (Γ^{RS})	Gir (Γ^{RS})	$\chi(\Gamma^{RS})$
13	3	2	3	3	3
| E (Γ^{RI})|	Diam(Γ^{RI})	Ɽ(Γ^{RI})	ω (Γ^{RI})	Gir (Γ^{RI})	$\chi(\Gamma^{RI})$
309	3	3	4	3	4

3. Main Results

This section promises a thorough analysis of both $\Gamma^{\mathrm{RS}}_{\mathrm{HN}}$ and $\Gamma^{\mathrm{RI}}_{\mathrm{HN}}$. Furthermore, by applying the outcomes of this study to figure out the actions of a certain HN subgroups, within HN. However, it would be feasible to analyze the aforementioned graphs computationally since the HN group is distinguished by its vast size of elements 27303091200000, see the Online Atlas [9]. The techniques greatest practical application is when we calculate the number of edges that connect the vertices of specific HN -conjugacy class together with every vertex in HN-conjugacy classes. For instance, for the class 2CSA the notation 2CSB(79989525000) means $|\{(\lambda,\mu) \in \mathrm{E}(\Gamma^{\mathrm{RI}}_{\mathrm{G}}): \lambda \in$ 2CSA and $\mu \in$ 2CSB $\}|$=79989525000, the same goes for the others.

3.1 *The graph Γ^{RS}_{HN}*

Initially, by utilizing the Online Atlas, we acquire V(Γ^{RS}_{HN}) ={1CSA, 2CS(A,B),3CS(A,B),4CS(A,B,C),5CS(A,B,C,D,E),6CS(A,B,C),7CSA,8CS(A,B), 9CSA,10CS(A,B,C,D,E,F,G,H),11CSA,12CS(A,B,C),14CSA,15CS(A,B,C),19CS (A,B),20CS(A,B,C,D,E),21CSA,22CSA,25CS(A,B),30CS(A,B,C),35CS(A,B),40 CS(A,B)}. Note that, 25CS(A,B) split into two vertices in Γ^{RS}_{HN} which are 25CSA and 25CSA, also, this work for all other vertices in V(Γ^{RS}_{HN}), hence, \|V(Γ^{RS}_{HN})\|=54. Now, by delving into the computational processes of Γ^{RS}_{HN}, we can reach three discs for each vertex in {1CSA,8CSA,19CSA,19CSB,35CSA,35CSB,40CSA}. Otherwise, for the remaining vertices we have only two discs.

Theorem 1: *A detailed set of characteristics for the graph Γ^{RS}_{HN} is presented in the table that identified below:*

Table 2
A detailed set of characteristics for the graph $\boldsymbol{\Gamma^{RS}_{HN}}$

| E (Γ^{RS})|	*Diam*(Γ^{RS})	R(Γ^{RS})	ω (Γ^{RS})	Gir (Γ^{RS})	$\chi(\Gamma^{RS})$
1299	3	2	44	3	44

Proof: For each vertex in V(Γ^{RS}_{HN}), their degree equal respectively: [2, 15, 46, 43, 45, 49, 52, 47, 41, 39, 46, 46, 50, 52, 52, 49, 50, 50, 52, 51, 48, 50, 50, 49, 49, 52, 52, 52, 51, 52, 50, 51, 51, 51, 51, 51, 50, 50, 51, 51, 50, 51, 51, 51, 51, 51, 51, 51, 51, 51, 50, 50, 50, 50], Thus we obtain that \| E(Γ^{RS}_{HN})\|=1299. Moreover, because some vertices in Γ_{HN}^{RS} with two discs and others with three discs. Therefore, R(Γ_{HN}^{RS})=2 and *Diam*(Γ_{HN}^{RS})=3. Since {2CSA,2CSB}∈ δ_1 (1CSA) and δ_1(2CSA) ={1CSA, 2CSB, 3CSA, 4CSA, 4CSB, 5CSA, 5CSE, 6CSA, 6CSB, 8CSB, 10CSB, 10CSF, 10CSG, 10CSH, 12CSA}, then 2CSB ∈ δ_1 (2CSA), thus we get a cycle with three vertex {1CSA,2CSA,2CSB}, and hence we obtain Gir (Γ_{HN}^{RS})=3. According to the discks intersection of Γ_{HN}^{RS}, we can find 31 cliques with sizes {44, 44, 44, 44, 44, 41, 44, 44, 44, 44, 44, 38, 38, 38, 38, 38, 35, 38, 38, 38, 38, 38, 39, 38, 33, 32, 39, 33, 14, 13, 3}, hence, ω(Γ_{HN}^{RS})=44 . Let C be a radom clique with size 44, and let V_t= V(Γ_{HN}^{RS})\C. Then for all X ∈ V_t , there is distinct Y ∈ C such that Y $\notin\delta_1$(X), and so we color X and Y with the same color. Thus, we obtain χ(Γ_{HN}^{RS})= 44.

3.2 *The graph* Γ_{HN}^{RI}

Theorem 2: *Besides that* Γ_{HN}^{RI} *is connected, the salient characteristics of the graph* Γ_{HN}^{RI} *are summarized in the following table:*

Table 3

A detailed set of characteristics for the graph Γ_{HN}^{RI}

| E (Γ_{HN}^{RI})|	Diam(Γ_{HN}^{RI})	Ʀ (Γ_{HN}^{RI})	Gir (Γ_{HN}^{RI})	Regularity
s	3	2	3	(τ,σ) -biregular

Where *s* = 10321029213195860775000, and

$(\tau,\sigma) = (75603375, 75603374)$.

Proof: Applying **Theorem 1**, and **Proposition 3**, one after the other, we prove the graph Γ_{HN}^{RI} is connected. Moreover, since {e,2CSA,2CSB} is a 3-cycle in Γ_{HN}^{RS}, thus the vertices { e,c_1,c_2 } is a 3-cycle in Γ_{HN}^{RI} where c_1,c_2 in 2CSA and 2CSB respectively. Thus, Gir (Γ_{HN}^{RI}) = 3. We have |2CS| = |2CSA| + |2CSB| = 1539000 + 74064375 = 75603375. Likewise, |4CS| = |4CSA| + |4CSB| + |4CSC| = 17775450000 + 47401200000 + 71101800000 = 136278450000. Therefore, by considering the outcomes of the **Proposition 1**, we conclude that |E(Γ_{HN}^{RI})|=*s*, such that *s*=10321029213195860775000. Given that 4CS ≠ ∅, it can be applied **Proposition 2**, then degree(λ) = 75603374 when λ ∈ 4CS, and degree(λ)= 75603375 otherwise, thus Γ_{HN}^{RI} is (75603375, 75603374)-biregular. By using the advantages of the graph Γ_{HN}^{RS}, we find out that the vertices with maximum distance in Γ_{HN}^{RS} are the vertex{e} and the vertices in the second discs of the class 2CSA and 2CSB in Γ_{HN}^{RS} and since $d(e,\lambda) = 1$, for each λ ∈ 2CS, then we have *Diam*(Γ_{HN}^{RI})=3. However, if we remove the vertex {e} form the V(Γ_{HN}^{RS}), we end up with graph has radius 1, thus R(Γ_{HN}^{RS})=2.

Corollary 3: $Dih_\sigma \subset$ HN if σ ∈ {2,3,4,5,6,7,8,9,10,11,12,14,15,20,21,22,25,30}.

Proof: Let λ,μ ∈ HN, then the subgroup <λ,μ> of HN is isomorphic to Dih_σ of size 2σ with the proviso that λ and λμ are elements in certain class of 2CS, and μ be the representative of particular σCS-class. In other words, $Dih_{\sigma} \cong <\lambda,\mu>$ on the condition that the σCS-class of μ in δ_1(2CSA) or δ_1(2CSB). Thus, according to the position of λ in the classes 2CSA or 2CSB, we have either $\sigma \in \{2,3,4,5,6,8,10,12\}$ or $\sigma \in \{2,3,4,5,6,7,8,9,10,11,12,14,15,20,21,22,25,30\}$.

5. Conclusion

Mathematical In conclusion, the inferences concerning the graph Γ_{HN}^{RI} and the group HN which comprising its vertices were obtained by the effective application of the Γ_{HN}^{RI} graph. For instance, advertising a contemporary approach for learning one of the best way to precisely construct specific subgroups throughout HN. Interspersed with this, establish various Γ_{HN}^{RI} and Γ_{HN}^{RS} particularities.

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