

Solving pantograph type through shifted Vieta-Pell polynomials

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Abstract

One of the basic mathematical models in the study of differential equations is the Pantograph delay differential equation. This article effectively solves the nonlinear pantograph delay equations through Vieta-Pell polynomials using the sub-interval collocation method. When the solution score is less than or equal to n , the approach discussed in this article ensures an exact solution. The method that can be directed reduces the problem that is related to solving a system of algebraic equations. A number of completed test instances are provided to demonstrate the accuracy and potency of the novel approach. Additionally, many calculations of the residual errors are used to determine the accuracy of the numerical results. Such residuals are demonstrated to tend to zero.

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1. Introduction

Many real-world applications use nonlinear differential equations for modeling the issues under study. Numerical or approximate solutions are calculated since such equations frequently have difficult or impossible exact solutions. One type of delay differential equation used in various branches of pure and applied mathematics, including number theory, is the pantograph equation [1-6].

The ability of Pantograph delay differential equations (PDDEs) to handle a variety of real-world issues has led to their increased popularity in recent years.

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It has shown to be an effective tool in a number of fields, including chemistry, physics, engineering, and applied sciences, where thermal modeling serves as an application. Problems with unique initial values of pantograph delay equations. It is difficult to find an analytical and accurate solution to these problems, hence approximation and numerical techniques should be used. Numerous researchers investigated the numerical solution. We are interested to discuss a few of the suggested methods. Galerkin method [7], successive method [1], Runge-Kutta method [8], Maclaurin series expansion (MSE) [9], tau method [10], pseudo spectral method [11], perturbation method [12], Variational iteration method VIM [13], backward Euler discretization [14], existence and uniqueness [15] and some applications of PDDEs can be found in [16, 17, 18, 19].

Many researchers have analyzed and approximately solved the pantograph-type delay differential equation. In [20,21] Bernoulli polynomials were used to solve PDDEs. Sara et al. [22] employ a numerical method through first Boubaker polynomials for a PDDEs. Also, [23] adopted Legendre and Chebyshev polynomials method for PDDEs. In [24], an orthogonal exponential polynomial-based matrix approach is suggested as a solution to the high-order PDDEs. Applied Haar wavelets, was proposed for solving PDDEs by J. P. Jaiswal et al [25]. The authors of [26] gave two different canonical forms to solve PDDEs via homotopy perturbation method. Also, the differential transform method (DTM) and the Adomian decomposition method (ADM) were proposed in [27] to obtain estimates of the generalized pantograph equations. In [28], the stability of the numerical solution using the trapezoidal rule to solve generalized pantograph equation. Taylor polynomials were used to estimate the solution of the PDDE in [29]. Additionally, present the numerical solution is based on shifted Orthonormal Bernstein Polynomials (SOBPs) to solve PDDE. In this article, we suggest a subinterval collocation technique for numerically solving the pantograph delay equation (PDDE) based on shifted vieta polynomials of equation (1).

$$u''(t) = \sigma(t)u\left(\frac{t}{p}\right) + \delta(t)\dot{u}\left(\frac{t}{p}\right) + \gamma(t)u(t) + h(t), t \in [0,1], \quad (1)$$

with initial conditions : $u(0) = a, u'(0) = \beta$, where $0 < p < 1$ and a, β are given scalar, $h, \sigma, \delta, \gamma$, are known function $\in C[0,1]$, with initial condition. the function $u : R^n \rightarrow R$ is continuously differentiable and unknown. Considered to be the unique solution $u''(t)$ of equation (1).

2. Shifted Vieta Polynomials Over The Interval[0, 1]

The basis of the Vieta-pell polynomials, represented by $V_m^{0,1}(t)$, of degree m , is of interest when the independent variable's range is $[0,1]$. They can be recursively produced as follows and are a sequence of orthogonal polynomials:

$$V_m^{0,1}(t) = (4t - 2)V_{m-1}^{0,1}(t) - V_{m-2}^{0,1}(t) \quad (2)$$

with $V_0^{0,1}(t) = 2, V_1^{0,1}(t) = 4t - 2$

Furthermore, using the following phrase, the explicit analytical formula for $V_m^{0,1}(t)$ may be found:

$$V_m^{0,1}(t) = m \sum_{i=0}^m \binom{j-i}{i} t^{m-2i}, \quad m = 1, 2, 3, \dots \quad (3)$$

The first seven $V_m^{0,1}(t)$ are given as below:

$$V_0^{0,1}(t) = 2, \quad V_1^{0,1}(t) = 4t - 2, \quad V_2^{0,1}(t) = 16t^2 - 16t + 2,$$

$$V_3^{0,1}(t) = 64t^3 - 96t^2 + 36t - 2,$$

$$V_4^{0,1}(t) = 256t^4 - 512t^3 + 320t^2 - 64t + 2,$$

$$V_5^{0,1}(t) = 1024t^5 - 2560t^4 + 2240t^3 - 800t^2 + 100t - 2$$

$$V_6^{0,1}(t) = 4096t^6 - 12288t^5 + 13824t^4 - 7168t^3 + 1680t^2 - 144t + 2.$$

2.1 Some of the Properties of Vieta Shifted Polynomials Over the Interval $[0,1]$

The result in the present section presents an exact computational formula, which represents the power function as a linear combination of the polynomials $V_m^{0,1}(t)$.

$$1 = \frac{1}{2} V_0^{0,1}(t),$$

$$t = \frac{1}{4} (V_1^{0,1}(t) + V_0^{0,1}(t)),$$

$$t^2 = \frac{1}{16} (V_2^{0,1}(t) + 4V_1^{0,1}(t) + 3V_0^{0,1}(t)),$$

$$t^3 = \frac{1}{64} (V_3^{0,1}(t) + 6V_2^{0,1}(t) + 15V_1^{0,1}(t) + 10V_0^{0,1}(t)),$$

$$t^4 = \frac{1}{256} (V_4^{0,1}(t) + 8V_3^{0,1}(t) + 28V_2^{0,1}(t) + 56V_1^{0,1}(t) + 35V_0^{0,1}(t)),$$

$$t^5 = \frac{1}{1024} (V_5^{0,1}(t) + 10V_4^{0,1}(t) + 45V_3^{0,1}(t) + 120V_2^{0,1}(t) + 210V_1^{0,1}(t) + 126V_0^{0,1}(t))$$

$$t^6 = \frac{1}{4096} (V_6^{0,1}(t) + 12V_5^{0,1}(t) + 66V_4^{0,1}(t) + 220V_3^{0,1}(t) + 495V_2^{0,1}(t) + 792V_1^{0,1}(t) + 462V_0^{0,1}(t)),$$

Theorem 2.2: (Vieta pell Polynomials truncation theorem)

The error in approximating $(u(t))$ by the sum of its first (m) terms is bounded by the absolute value of the next omitted term in the series.

$$\text{If} \quad u_m(t) = \sum_{i=0}^m c_i V_i(t)$$

$$\text{Then } E_v(m) = |u(t) - u_m(t)| \leq \sum_{k=m}^{\infty} |c_k|, \quad \text{for all } u(t) \text{ and all } t \in [0,1]$$

Proof: For any $t \in [0,1]$ and for all k truncated series from the infinite series, the Vieta pell Polynomials are bounded by one, that is, $|u_m(t)| \leq 1$. The theorem is obtained by the difference bounding each phrase and adding up the limits. The bound of a Vieta pell polynomial is one, thus, $|u_m(t)| \leq 1$ for all

$t \in [0,1]$ and for all k The theorem is obtained by taking truncated series from the infinite series, bounding each term in the difference, and then adding the bounds.

3. Description The Technique of Solution Pantograph Delay Differential Equation (PDDE):

We recall eq. (1)

$$u''(t) = \sigma(t)u\left(\frac{t}{p}\right) + \delta(t)\dot{u}\left(\frac{t}{p}\right) + \gamma(t)u(t) + h(t)$$

Using operator this can be written as

$$L[u] = h(t) \quad (4)$$

where the definition of the operator L is

$$L[u(t)] = u''(t) - \sigma(t)u\left(\frac{t}{p}\right) - \delta(t)\dot{u}\left(\frac{t}{p}\right) - \gamma(t)u(t) \quad (5)$$

$u(t)$ and $h(t)$ were functions of vector values of t . An approximation of the unknown function $u(t)$ is the form

$$u_N(t) = \sum_{i=0}^N c_i V_i(t) \quad (6)$$

Where $V_i(t)$ is given in equation (3). When we replace eq. (6) in eq. (4), we obtain

$$L[u_N(t)] = h(t) + R_N$$

$$\text{Where } L[u_N(t)] = u_N''(t) - \sigma(t)u_N\left(\frac{t}{p}\right) - \delta(t)u_N'\left(\frac{t}{p}\right) - \gamma(t)u_N(t) \quad (7)$$

which we possess the error R_N

$$R_N = L\left[\sum_{i=0}^N c_i V_i(t)\right] - h(t) = \sum_{i=0}^N c_i L[V_i(t)] - h(t) \quad (8)$$

Where

$$L[V_i(t)] = V_i''(t) - \sigma(t)V_i\left(\frac{t}{p}\right) - \delta(t)V_i'\left(\frac{t}{p}\right) - \gamma(t)V_i(t) \quad i=0,1,2,\dots,N \quad (9)$$

This Lead to

$$\begin{aligned} a_{ij} &= L[V_j(t_i)] & j &= 0,1,2,\dots,N \\ H_i &= h(t_i) & i &= 0,1,2,\dots,N \end{aligned} \quad (10)$$

Utilizing the collocation point t_i which is defined as $t_i = a + \frac{b-a}{N}i$, $i = 0,1,2,\dots$, in eq.(10), The foundational matrix is obtained:

$$Ac = H$$

$$A = \begin{pmatrix} a_{00} & \cdots & a_{0N} \\ \vdots & \ddots & \vdots \\ a_{N0} & \cdots & a_{NN} \end{pmatrix}, \quad c = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_N \end{pmatrix}$$

$$a_{ji} = a_j(t_i), H_i = h(t_i) \quad i, j = 0,1,\dots,N$$

4. Test Problems

This section presents two experiments using delay pantograph differential equations (PDDEs) to illustrate the efficacy of the technique. To present light on the conclusions of the previous part, the calculations related to the experiments covered above. The method's errors and the suggested method's errors are compared, as shown in Table 1.

Example 1: The differential equation for the pantograph is discussed $u''(t) - \frac{2}{t}u'(\frac{t}{2}) = 1$, $u(0) = 1, u'(0) = 0$. The exact of problem $u(t) = 1 - t^2$, where $t \in [0,1]$, as shown in Figure 1.

Table 1
Vieta-pell polynomials with varying times are used the difference between the exact and approximation solutions In

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
n = 4 Approximate solution VP (t)	0.9999	0.9907	0.9661	0.9318	0.8939	0.8592	0.83520	0.83020	0.85290	0.9128	1.0202
n = 4 Exact solution	1.000	0.9991	0.9936	0.9811	0.9616	0.9375	0.9136	0.8971	0.8979	0.89271	1.000

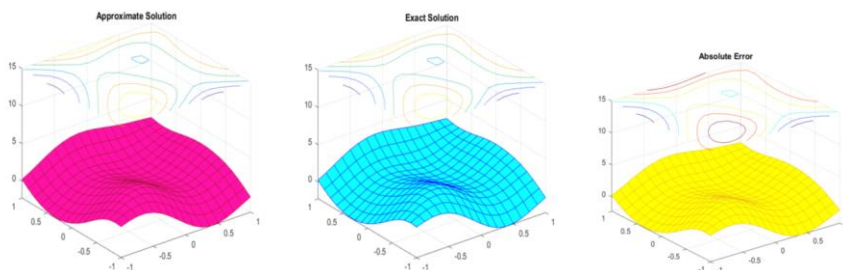


Figure 1
The difference between the exact and approximation solutions, in example 1, vieta-pell polynomials with $M = 4$ and $k = 1$ are applied with absolute error.

Example 2: The differential equation for the pantograph is discussed $u''(t) + \frac{8}{t}u'(\frac{t}{2}) = 14t^2 - 9t$, $u(0) = 1, u'(0) = 0$. The exact problem $u(t) = t^4 - t^3 + 1$, where $x \in [0,1]$, explained in Table 2, and illustrated in Figure 2.

Table 2
Vieta-Pell polynomials with varying parameters are used to analyze the difference between the exact and approximate solutions in example 2.

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
n = 4 Approximate solution VP (t)	0.9999	0.9907	0.9661	0.9318	0.8939	0.8592	0.83520	0.83020	0.85290	0.9128	1.0202
n = 4 Exact solution	1.000	0.9991	0.9936	0.9811	0.9616	0.9375	0.9136	0.8971	0.8979	0.89271	1.000

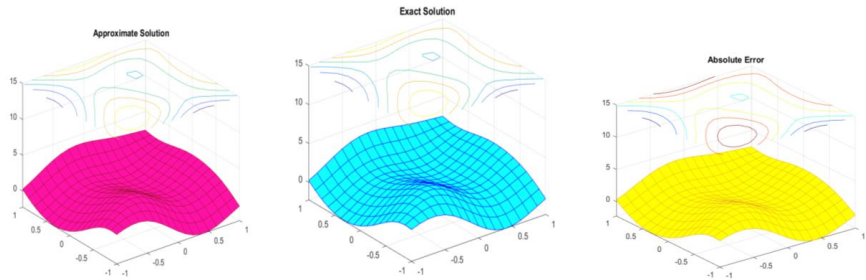


Figure 2
The difference between the exact and approximation solutions In example 2, veita-pell polynomials with $M = 4$ and $k = 1$ are applied with absolute error.

5. Conclusions

A numerical technique based on shifted veita-Pell polynomials is proposed in this paper to estimate the pantograph delay differential equations (PDDEs) through the subintervals collocation approach, the unknown function is approximated in the proposed method, and its derivatives are approximated using shifted veita-pell polynomials. Absolute error comparisons have been made between the solutions to two test problems. For the given scenario, the graph clearly shows that the suggested approach performs better than the established numerical techniques.

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