

Small annihilator maximal sub-modules

Mahdi Saleh Nayef [†]
 Duaa Hussein Wareed ^{*}
Department of Mathematics
College of Education
Mustansiriyah University
Baghdad
Iraq

Abstract

The main objective of this work is to present and study a new generalization of annihilator maximal sub modules that is small annihilator maximal sub modules. We are investigated the relationship between Maximal Annihilator Maximal Sub modules with Small Annihilator Maximal Sub modules. In addition, we are introduced the concept of Small Annihilator Semi-Simple Modules which serve as generalization of Annihilator Semi-Simple Modules. Many of the basic properties and descriptions of these ideas are discussed. In addition, their relationship to other related concepts is studied.

Subject Classification: Primary 16D10, 16D60. Secondary 16C50.

Keywords: Annihilator maximal sub modules, Small annihilator maximal sub modules, Annihilator semi-simple modules, Small annihilator semi-simple modules.

1. Introduction

In this research we shall all $\mathcal{N}$ be an R -module and all modules are considered to be unitary left R -modules and let R be a commutative ring with identity. A proper sub module $\mathcal{D}$ of an R -module $\mathcal{N}$ (denoted $\mathcal{D} \subsetneq \mathcal{N}$) is referred to be essential and denoted by $(\mathcal{D} \subseteq^e \mathcal{N})$ if $\forall \mathcal{H} \subseteq \mathcal{N}$ where $\mathcal{H} \neq (0)$ we have $\mathcal{D} \cap \mathcal{H} \neq (0)$ [1]. While, a sub module $\mathcal{D} \neq (0)$ is referred to be annihilator essential sub module (briefly AE-sub module) of $\mathcal{N}$ if $\mathcal{D} \cap \mathcal{H} = (0)$ then $\text{ann}(\mathcal{H}) \subseteq^e R, \forall \mathcal{H} \subseteq \mathcal{N}$. Equivalently: Let $(0) \neq \mathcal{D} \subseteq \mathcal{N}$. $\mathcal{D}$ is referred to be annihilator essential of $\mathcal{N}$ if $\mathcal{D} \cap \mathcal{H} \neq (0), \forall \mathcal{H} \subseteq \mathcal{N}$.s.t $\text{ann}(\mathcal{H}) \not\subseteq^e R$ [2]. Let $(0) \neq \mathcal{D} \subseteq \mathcal{N}$, $\mathcal{D}$ is referred to be small annihilator essential sub module of (briefly SAE-sub module) denoted by $\mathcal{D} \subseteq^{s.a.e} \mathcal{N}$ if $\mathcal{D} \cap \mathcal{H} = (0)$ implies

[†] E-mail: mahdisaleh773@uomustansiriyah.edu.iq

^{*} E-mail: duaahusseiniwd@uomustansiriyah.edu.iq (corresponding Author)

$\text{ann}(\mathcal{H}) \subseteq^e R$, $\forall \mathcal{H} \subseteq^s \mathcal{N}$. Equivalently, Let $(0) \neq \mathcal{D} \subseteq \mathcal{N}$. $\mathcal{D}$ is SAE-sub module if $\mathcal{D} \cap \mathcal{H} \neq 0$, $\forall \mathcal{H} \subseteq^s \mathcal{N}$ s.t $\text{ann}(\mathcal{H}) \not\subseteq^e R$ [3]. A sub module $\mathcal{D}$ of $\mathcal{N}$ is referred to be small and denoted by $\mathcal{D} \subseteq^s \mathcal{N}$ if for some $\mathcal{H} \subseteq \mathcal{N}$, $\mathcal{D} + \mathcal{H} = \mathcal{N}$ then $\mathcal{H} = \mathcal{N}$ [1]. Referred a module $\mathcal{N}$ to be uniform modules if $\forall \mathcal{D} \subseteq \mathcal{N}$ where $\mathcal{D} \neq (0)$, $\mathcal{D} \subseteq^e \mathcal{N}$ [4],[5],[12]. While, $\mathcal{N}$ is said to be multiplication module if every sub module of $\mathcal{N}$ is write of the form $I\mathcal{N}$ for some ideal I of R [6],[7]. Let $\mathcal{D} \subsetneq \mathcal{N}$, $\mathcal{D}$ is referred to be maximal sub module (denoted by $\mathcal{D} \subseteq^m \mathcal{N}$) if whenever $\mathcal{H} \subseteq \mathcal{N}$ with $\mathcal{D} \subset \mathcal{H} \subseteq \mathcal{N}$ then $\mathcal{H} = \mathcal{N}$ [1]. Let $\mathcal{D} \subsetneq \mathcal{N}$, $\mathcal{D}$ is referred to be annihilator maximal (briefly AM-sub module) denoted by $\mathcal{D} \subseteq^{a.m} \mathcal{N}$ if whenever $\mathcal{H} \subseteq^{a.e} \mathcal{N}$ with $\mathcal{D} \subset \mathcal{H} \subseteq \mathcal{N}$ implies that $\mathcal{H} = \mathcal{N}$ [2]. Referred a module $\mathcal{N}$ to be annihilator semi-simple modules (briefly A-semi-simple) if $\nexists \mathcal{D} \subseteq^{a.e} \mathcal{N}$ [2]. Let $\mathcal{D} \subsetneq \mathcal{N}$, $\mathcal{D}$ is referred to be Almost Maximal sub module if whenever $\mathcal{H} \subseteq^e \mathcal{N}$ s.t $\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{N}$ then $\mathcal{H} = \mathcal{N}$ [8],[9]. Referred a module $\mathcal{N}$ to be annihilator local maximal module (briefly A-local) if $\mathcal{N}$ has only one AM-sub module [2].

2. Small Annihilator Maximal Sub modules

In this section we will present the concept of Small Annihilator Maximal Sub modules, which is generalizes Annihilator Maximal Sub modules.

Definition 2.1: A proper sub module $\mathcal{D}$ of an R -module $\mathcal{N}$ is referred to be small annihilator maximal sub module (briefly SAM-sub module) denoted by $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ if whenever $\mathcal{H} \subseteq^{s.a.e} \mathcal{N}$ with $\mathcal{D} \subset \mathcal{H} \subseteq \mathcal{N}$ then $\mathcal{H} = \mathcal{N}$.

Remarks and Examples 2.2:

1. Every maximal sub module is SAM-sub module. The contrast cannot be considered valid. As, $\mathcal{N} = 2\mathbb{Z} \oplus 2\mathbb{Z}$ as $\mathbb{Z}$ -module and let $\mathcal{D} = 4\mathbb{Z} \oplus (0) \subset 2\mathbb{Z} \oplus (0) \subseteq 2\mathbb{Z} \oplus 2\mathbb{Z}$. $\mathcal{D}$ is SAM-sub module since $\nexists \mathcal{H} \subsetneq^{s.a.e} \mathcal{N}$ s.t $\mathcal{D} \subset \mathcal{H} \subseteq \mathcal{N}$ but $\mathcal{D} \not\subseteq^m \mathcal{N}$.
2. Every AM- sub module is SAM-sub module, see Ex(2-14)
3. $\mathcal{N} = \mathbb{Z}_p^\infty$ as $\mathbb{Z}$ -module, $\forall \mathcal{D} \not\subseteq^{s.a.m} \mathcal{N}$ since $\exists \mathcal{H} \subseteq \mathcal{N}$ s.t $\mathcal{D} \subset \mathcal{H} \subseteq \mathcal{N}$ where $\mathcal{H} \subseteq^{s.a.e} \mathcal{N}$ but $\mathcal{H} \neq \mathcal{N}$.

Proposition 2.3: Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be an R -modules and $\mathcal{D} \subseteq \mathcal{N}_2$. And $\alpha: \mathcal{N}_1 \rightarrow \mathcal{N}_2$ be an isomorphism. If $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}_2$ then $\alpha^{-1}(\mathcal{D}) \subseteq^{s.a.m} \mathcal{N}_1$.

Proof: To prove that $\alpha^{-1}(\mathcal{D}) \neq \mathcal{N}_1$. Suppose that $\alpha^{-1}(\mathcal{D}) = \mathcal{N}_1$ where $\mathcal{D} \neq \mathcal{N}_2$ so $\exists m \in \mathcal{N}_2$ and $m \notin \mathcal{D}$ so $\alpha^{-1}(m) \in \alpha^{-1}(\mathcal{N}_2) = \mathcal{N}_1 = \alpha^{-1}(\mathcal{D})$ where α is epimorphism. $\exists d \in \mathcal{D}$ s.t $\alpha^{-1}(m) = \alpha^{-1}(d)$. So $\alpha^{-1}(m - d) = (0)$ so $\alpha(\alpha^{-1}(m - d)) = \alpha(0)$ so $m - d = (0)$. Where α is epimorphism we have $m = d$ then $m \in \mathcal{D}$ C!. Then $\alpha^{-1}(\mathcal{D}) \neq \mathcal{N}_1$.

Now, to show that $\alpha^{-1}(\mathcal{D}) \subseteq^{s.a.m} \mathcal{N}_1$. Suppose that $\alpha^{-1}(\mathcal{D}) \not\subseteq^{s.a.m} \mathcal{N}_1$ so $\exists \mathcal{H} \subseteq^{s.a.e} \mathcal{N}_1$ s.t $\alpha^{-1}(\mathcal{D}) \subseteq \mathcal{H} \subseteq \mathcal{N}_1$ so $\mathcal{D} \subseteq \alpha(\mathcal{H}) \subseteq \alpha(\mathcal{N}_1)$ where $\alpha(\mathcal{H}) \subseteq^{s.a.e} \alpha(\mathcal{N}_1)$ by [3, prop(1.5)] since α is epimorphism so $\mathcal{D} \subseteq \alpha(\mathcal{H}) \subseteq \mathcal{N}_2$. Then we get $\mathcal{D} \not\subseteq^{s.a.m} \mathcal{N}_2$ C!. Hence $\alpha^{-1}(\mathcal{D}) \subseteq^{s.a.m} \mathcal{N}_1$.

Remark 2.4: Let $\mathcal{N}$ be an R-module and $\mathcal{D}, \mathcal{H}$ are proper sub modules of $\mathcal{N}$ s.t $\mathcal{D} \subseteq \mathcal{H}$. If $\mathcal{D} \subseteq^{s.a.m} \mathcal{H}$ and $\mathcal{H} \subseteq^{s.a.m} \mathcal{N}$ so $\mathcal{D}$ is not necessary SAM-sub module of $\mathcal{N}$. As: $\mathcal{N} = \mathbb{Z}_{24}$ as $\mathbb{Z}$ -module s.t $\mathcal{D} = (\bar{4})$ and $\mathcal{H} = (\bar{2})$, $\mathcal{D} \subseteq \mathcal{H}$. $\mathcal{D} \subseteq^{s.a.m} \mathcal{H}$ and $\mathcal{H} \subseteq^{s.a.m} \mathcal{N}$ but $\mathcal{D} \not\subseteq^{s.a.m} \mathcal{N}$.

Remark 2.5: Let $\mathcal{H} \subseteq^{s.a.m} \mathcal{N}$ if $\mathcal{D} \subseteq \mathcal{N}$ s.t $\mathcal{D} \simeq \mathcal{H}$ so its not necessary that $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$. As: $\mathcal{N} = \mathbb{Z}$ as $\mathbb{Z}$ -module the sub module $2\mathbb{Z} \subseteq^{s.a.m} \mathcal{N}$ and $2\mathbb{Z} \simeq \mathbb{Z}$. But $\mathbb{Z} \not\subseteq^{s.a.m} \mathbb{Z}$.

Proposition 2.6: If $\mathcal{D}, \mathcal{H} \subsetneq \mathcal{N}$ such that $\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{N}$. If $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ then $\mathcal{H} \subseteq^{s.a.m} \mathcal{N}$.

Proof: Assume that $\mathcal{H} \not\subseteq^{s.a.m} \mathcal{N}$ then $\exists \mathcal{W} \subseteq^{s.a.e} \mathcal{N}$ s.t $\mathcal{H} \subseteq \mathcal{W} \subseteq \mathcal{N}$ then $\mathcal{D} \subseteq \mathcal{W} \subseteq \mathcal{N}$. Then $\mathcal{D} \not\subseteq^{s.a.m} \mathcal{N}$ C!.

Corollary 2.7: If $\mathcal{D}, \mathcal{H} \subsetneq \mathcal{N}$ s.t if $\mathcal{D}, \mathcal{H} \subseteq^{s.a.m} \mathcal{N}$ then $\mathcal{D} + \mathcal{H} \subseteq^{s.a.m} \mathcal{N}$.

Proof: Since $\mathcal{D} \subseteq \mathcal{D} \oplus \mathcal{H} \subseteq \mathcal{N}$ and $\mathcal{H} \subseteq \mathcal{D} \oplus \mathcal{H} \subseteq \mathcal{N}$ then by proposition (2.6) we are done.

Corollary 2.8: If $\mathcal{D}, \mathcal{H} \subsetneq \mathcal{N}$ s.t $\mathcal{D} \cap \mathcal{H} \subseteq^{s.a.m} \mathcal{N}$. Then both of $\mathcal{D}$ and $\mathcal{H}$ are SAM-sub modules of $\mathcal{N}$.

Proof: Since $\mathcal{D} \cap \mathcal{H} \subseteq \mathcal{D} \subseteq \mathcal{N}$ and $\mathcal{D} \cap \mathcal{H} \subseteq \mathcal{H} \subseteq \mathcal{N}$ so by proposition (2.6) we get $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ and $\mathcal{H} \subseteq^{s.a.m} \mathcal{N}$. But the opposite is not true as: $\mathcal{N} = \mathbb{Z}_6$ as $\mathbb{Z}$ -module. The sub modules $\mathcal{D} = (\bar{2})$ and $\mathcal{H} = (\bar{3})$ are SAM-sub modules of $\mathcal{N}$ where $\nexists \mathcal{W} \subseteq^{s.a.e} \mathcal{N}$ s.t $\mathcal{D}, \mathcal{H} \subseteq \mathcal{W}$. But $\mathcal{D} \cap \mathcal{H} = (0) \not\subseteq^{s.a.m} \mathcal{N}$.

Recall that: An R-module $\mathcal{N}$ is referred to be semi-simple if $\mathcal{N}$ has no proper essential sub module, [10-13].

In the following definition we introduce a specification of this concept.

Definition 2.9: An R-module $\mathcal{N}$ is referred to be small annihilator semi-simple module (briefly SA-SE-module) if $\mathcal{N}$ has no proper SAE-sub module of $\mathcal{N}$. That is if $\mathcal{D} \subseteq^{s.a.e} \mathcal{N}$ then $\mathcal{D} = \mathcal{N}$.

Remarks and Examples (2.10):

1. Every SA-SE-module is semi-simple.

Proof: Let $\mathcal{N}$ is SA-SE-module. Suppose that $\mathcal{N}$ is not semi-simple so $\mathcal{N}$ has a proper essential sub module. Where every essential sub module is SAE-sub module then $\mathcal{N}$ has an SAE-sub module C!. Hence $\mathcal{N}$ is semi-simple.

2. Every SA-SE-module is A-semi-simple module.

Proof: Let $\mathcal{N}$ be an SA-SE-module. Suppose that $\mathcal{N}$ is not A-semi-simple module so $\mathcal{N}$ has a proper ann-essential sub module. Since every AE-sub

module is SAE-sub module then $\mathcal{N}$ has an SAE-sub module which is contradiction. Hence $\mathcal{N}$ is ann-semi-simple module.

3. Let $\mathcal{N} = \mathbb{Z}_{30}$ as $\mathbb{Z}_{30}$ -module, $\mathcal{N}$ is SA-SE-module. Where $\mathcal{N}$ has no proper SAE-sub module.

Proposition 2.11: Let $\mathcal{N}$ be an R-module then $(\bar{0}) \subseteq^{s.a.m} \mathcal{N}$ iff $\mathcal{N}$ is SA-SE-module.

Proof: Let $\mathcal{N}$ is SA-SE-module. So $\mathcal{N}$ has no proper SAE-sub module so $\nexists \mathcal{D} \subsetneq^{s.a.e} \mathcal{N}$ s.t $(\bar{0}) \subseteq \mathcal{D}$. Hence $(\bar{0}) \subseteq^{s.a.m} \mathcal{N}$.

Converse: Since $(\bar{0}) \subseteq^{s.a.m} \mathcal{N}$ so $\mathcal{N}$ has no proper SAE-sub module. Thus $\mathcal{N}$ is SA-SE-module.

Corollary 2.12: Let $\mathcal{N}$ be an R-module. Then the following statements are equivalent:

1. $\mathcal{N}$ is SA-SE-module.
2. $(\bar{0})$ is SAM-sub module.
3. Every proper sub module of $\mathcal{N}$ is SAM-sub module.

Proof: (1) $\rightarrow$ (2) Clear by [proposition (2.11)].

(2) $\rightarrow$ (3) Let $\mathcal{D} \subsetneq \mathcal{N}$. Since $(\bar{0}) \subseteq \mathcal{D} \subseteq \mathcal{N}$ and $(\bar{0}) \subseteq^{s.a.m} \mathcal{N}$ then by proposition (2.7) hence $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$.

(3) $\rightarrow$ (1) Since every sub module is SAM-sub module so $\mathcal{N}$ has no proper SAE-sub module. Hence $\mathcal{N}$ is SA-SE-module.

Remark 2.13: Every SAM-sub module is almost maximal sub module.

Proof: Let $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ s.t $\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{N}$ where $\mathcal{H} \subseteq^e \mathcal{N}$ then by [3, remarks and example (1.2) (1)] we get $\mathcal{H} \subseteq^{s.a.e} \mathcal{N}$. Since $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ so $\mathcal{H} = \mathcal{N}$ thus $\mathcal{D}$ is almost maximal sub module.

The contrast cannot be considered valid. As, $\mathcal{N} = \mathbb{Z}_{42}$ as $\mathbb{Z}$ -module. $(\bar{6})$ is almost maximal sub module of $\mathcal{N}$. But $(\bar{6}) \not\subseteq^{s.a.m} \mathcal{N}$. see the following example

Example 2.14:

$\mathbb{Z}_{42}$	Essential	Ann-Essential	SAE-Sub module	Maximal	AM-sub module	SAM-Sub module	Almost Maximal
$\mathbb{Z}_{42}$	✓	✓	✓	✗	✗	✗	✗
$(\bar{2})$	✗	✓	✓	✓	✓	✓	✓
$(\bar{3})$	✗	✓	✓	✓	✓	✓	✓
$(\bar{6})$	✗	✓	✓	✗	✗	✗	✓

$(\bar{7})$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$(\bar{14})$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$
$(\bar{21})$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$
$(\bar{0})$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$

Proposition 2.15: Let $\mathcal{N}$ be an R-module s.t R is semi-simple ring and let $\mathcal{D} \subseteq \mathcal{N}$ then $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ iff $\mathcal{D}$ is almost maximal sub module of $\mathcal{N}$.

Proof: $\Leftarrow$ Let $\mathcal{D}$ is almost maximal s.t $\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{N}$ where $\mathcal{H} \subseteq^{s.a.e} \mathcal{N}$. Since R is semi-simple ring so by [3, remarks and examples (1.2) (4)] $\mathcal{H} \subseteq^e \mathcal{N}$. But $\mathcal{D}$ is almost maximal then $\mathcal{H} = \mathcal{N}$, hence $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$.

$\Rightarrow$ Clear by remark (2.13).

Proposition 2.16: Let $\mathcal{N}$ be a uniform R-module and $(0) \neq \mathcal{D} \subseteq \mathcal{N}$. Then $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ iff $\mathcal{D}$ is almost maximal sub module.

Proof: $\Leftarrow$ Let $\mathcal{D}$ is almost maximal of $\mathcal{N}$ and $\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{N}$ since $\mathcal{H} \subseteq^{s.a.e} \mathcal{N}$. Since $\mathcal{N}$ is uniform so $\mathcal{H} \subseteq^e \mathcal{N}$ since $\mathcal{D}$ is almost maximal so $\mathcal{H} = \mathcal{N}$. Hence $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$.

$\Rightarrow$ Clear by remark (2.13).

Definition 2.17: Let $\mathcal{N}$ be an R-module. $\mathcal{N}$ is referred to be small annihilator local module (briefly SAL-module) if $\mathcal{N}$ has only one SAM-sub module.

Remarks and Examples 2.18:

- 1) Every A-local module is SAL-module.
- 2) Let $\mathcal{N} = \mathbb{Z}_4$ as $\mathbb{Z}$ -module. $\mathcal{N}$ is SAL-module. Since $\mathcal{N}$ has only one SAM-sub module.
- 3) Let $\mathcal{N} = \mathbb{Z}_{24}$ as $\mathbb{Z}$ -module. $\mathcal{N}$ is not SAL-module because $(\bar{2}), (\bar{3}) \subseteq^{s.a.m} \mathcal{N}$.

Proposition 2.19: Let $\mathcal{N}$ be a non-zero multiplication and SAL-module, and $\mathcal{D} \subseteq^{s.a.m} \mathcal{N}$ where $\mathcal{D} \neq (0)$ then $\mathcal{D} \subseteq^{s.a.e} \mathcal{N}$.

Proof: Let $\mathcal{H} \subseteq \mathcal{N}$ s.t $\mathcal{D} \cap \mathcal{H} = (0)$. Where $\mathcal{N}$ is multiplication module so be [11] $\mathcal{H}$ is contained in a maximal sub module then by [remarks and examples (2.2)(1)] $\mathcal{H}$ is contained in SAM-sub module. But $\mathcal{N}$ is SAL-module so $\mathcal{N}$ has only one SAM-sub module which is $\mathcal{D}$ so $\mathcal{H} \subseteq \mathcal{D}$ then $\mathcal{H} = (0)$ and $\text{ann}(\mathcal{H}) = \text{ann}(0) = \mathcal{R} \subseteq^e \mathcal{R}$. Hence $\mathcal{D} \subseteq^{s.a.e} \mathcal{N}$.

Conclusion

Based on the findings of this study we conclude the following: $\mathcal{D}$ is SAM-sub module in $\mathcal{N}$ if $\mathcal{D}$ Maximal. $\mathcal{D}$ is SAM-sub module if $\mathcal{D}$ is AM-sub module.

The direct sum of SAM-sub modules is SAM-sub module. If $\mathcal{D} \cap \mathcal{H}$ is SAM-sub module in $\mathcal{N}$ then both of $\mathcal{D}$ and $\mathcal{H}$ are SAM-sub module. And, $\mathcal{N}$ is Semi-simple if $\mathcal{N}$ is SA-SE-module. $\mathcal{N}$ is Ann-Semi-simple if $\mathcal{N}$ is SA-SE-module. The zero sub module of $\mathcal{N}$ is SAM-sub module iff $\mathcal{N}$ is SA-SE-module. And, $\mathcal{N}$ is SAL-module if $\mathcal{N}$ is A-local module.

References

- [1] F. Kasch, Modules and Rings. London: Academic Press (1982).
- [2] Y. A. Qasim and S. M. Yaseen, "Annihilator Essential Submodules and Related Concepts," M.Sc. Thesis, College of Science, University of Baghdad, Iraq (2021).
- [3] H. W. Duaa and S. N. Mahdi, "Small Annihilator Essential Submodules," *Mustansiriyah J. Pure Appl. Sci.*, to appear (2024).
- [4] N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, Extending Modules, Pitman Research Notes in Mathematics Series 313, Longman, New York (1994).
- [5] M. S. Nayef, Z. M. A. Al-Majeed, Z. R. Shaker, and A. S. Dawood, "Supplement t-extending modules," *J. Discrete Math. Sci. Cryptogr.*, vol. 25, no. 2, pp. 1591-1598 (2024).
- [6] A. Barnard, "Multiplication Modules," *J. Algebra*, vol. 71, pp. 174-178 (1981).
- [7] S. H. Ali and M. S. Nayef, "Supplement Continuous and Quasi-Supplement Continuous Modules," *Iraqi J. Sci.*, vol. 65, no. 10, pp. 5610-5616 (2023).
- [8] Z. T. Al-Zubaidy, "On Purely Extending Modules," M.Sc. Thesis, College of Science, University of Baghdad, Iraq (2005).
- [9] Z. A. Abdulrazzaq, M. S. Nayef, and M. Hbaib, "On Semi-Continuous and Quasi-Semi-Continuous Modules," *J. Discrete Math. Sci. Cryptogr.*, to appear (2025).
- [10] K. R. Goodearl, Ring Theory. New York: Marcel Dekker (1976).
- [11] M. A. Inaam, H. Hadi, and R. K. Ali, "On Almost Maximal Submodules," *Ibn Al-Haithim J. Pure Appl. Sci.*, vol. 1, pp. 190-197 (2008).

- [12] M. M. Abbood, H. H. Ebrahim, A. Al-Fayadh, and A. J. Obaid, "Measure Defined on Γ -algebra and Some of Their Generalizations," *J. Interdiscip. Math.*, vol. 26, no. 5, pp. 821–827 (2023), doi: 10.47974/JIM-1502.
- [13] M. S. Fiadh, B. N. Shihab, and A. Issa, "Restricted Visible Submodules and Fully Restricted Visible Module," *Babylonian J. Math.*, vol. 2024, pp. 48-52 (2024). [Online]. Available: <https://doi.org/10.58496/BJM/2024/006>.

Received December, 2024