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Radical small submodules and hollow modules

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Abstract

In this work, we give a new generalization for two important classes in the category of modules. There are Rad-small and Rad-hollow R-module. In this work, some properties and results of these submodule are given: A non-zero direct summand of any R-mod D is not Rad-small. Moreover, another characterization of Rad-hollow modules.

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1. Introduction

In this paper, R represents the associative ring with identity, and D is an until left modules (in the short, mod). A submodule (in short, sub-mod) N of an R-module D is called small if N+V=D for some sub-mod V of D implies that V=D and is denoted by $N \ll D$ [1]. Recall that a non-zero mod D is said to be hollow, if for all proper sub-mod of D is small sub-mod [2]. The crossover of all maximal sub-mod from D is called radical of D or the sum of all of its small sub-mod of D, denoted as Rad(D). Rad(D)=D when D has no maximal submodules. In this work, we use Rad(D) to study a new generalization for two

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important classes in the category of modules are called Rad-small submodules and Rad-hollow R-mod. In section 2 we introduce Rad-small submodules with some properties of this kind submodules are given. In section 3 we extend the notions of hollow R-modules is given, it is called Rad-hollow R-modules and given some properties of Rad-hollow modules.

2. Radical Small Submodules

Definition 2.1: Let D be an R-module and N be a submodule of D, if $D = N + K$, then $K + Rad(D) = D$ where $K \leq D$ then D is called radical small, shortly (Rad-small)

An ideal I of R is called Rad-small, if I is Rad-small sub-mod of R as R-mod.

Remarks and Examples 2.2

1- One can easily show that small sub-mod is a Rad-small sub-mod. However, the opposite in general may not be correct. Actually Q as Z-module is the sum of two proper sub-mod, then it's exists A and B two proper sub-mod of Q in which $A + B = Q$ [3, P.187], so that both of A and B are not small submodule of Q but A and B are Rad-small in Q since $B + Rad(Q) = Q$ also $A + Rad(Q) = Q$, where $Rad(Q) = Q$.

2- Clearly, (0) is Rad-small sub-mod of any R-mod.

3- If $Rad(D) = (0)$ and N is a Rad-small sub-mod, then N is a small sub-mod.

4- If D is Z-regular (F-regular), then all Rad-small submodule is small. [4] Recall that a module D is Z-regular if any cyclic submodule of D is projective and direct summand.

Also, in [5] Fieldhouse introduce the notion of F-regular, an R-mod is said to be F-regular, if every sub-mod of D is pure. In [6], the concept of pure submodule was studied. A sub-mod N of an R-module D is called pure, if $N \cap ID = ID$ for each ideal I in R.

5- A non-zero direct summand of any R-mod D is not Rad-small. As an illustration, the submodule $(\overline{2})$ is a direct summand of Z_6 as Z-mod but not Rad-small since $(\overline{2}) + (\overline{3}) = Z_6$ but $(\overline{3}) + Rad(Z_6) \neq Z_6$.

6- If $N \ll_R D$ and A is proper sub-mod in N, then $A \ll_R D$.

Proof: Let $D = A + B,$ where B is sub-mod of D, since $A < N$ then $D = N + B$, since $N \ll_R D$, implies $D = B + Rad(D)$ .Then $A \ll_R D$.

7- Let $N \leq L \leq D$ and $\ll_R L$, hence it's not necessary that $L \ll_R D$. As an illustration : In Z as Z-mod $0 \leq 2Z \leq Z$, hence $0 \ll_R 2Z$, while 2Z is not Rad-small in Z, since $2Z + 3Z = Z$ and $2Z + Rad(Z) \neq Z$

Proposition 2.3: If N and K are sub-mod of D, then $N \ll_R D$ and $K \ll_R D$ *if and only if* $N + K \ll_R D$.

Proof: Suppose $N \ll_R D$ and $K \ll_R D$, also let $N + K + H = D$ for any sub-mod H of D. $N \ll_R D$, hence $D = K + H + Rad(D)$. Also, $K \ll_R D$, hence $H + Rad(D) = D$.

Conversely, if $N + K \ll_R D$. To prove that $N \ll_R D$, let $N + H = D$ then $N + K + H = D$. But $N + K \ll_R D$, hence $D = H + Rad(D)$, therefor $N \ll_R D$. Simillary, we can prove $K \ll_R D$.

Corollary 2.4: *Let A_i be submodules of an R-mod D, $1 \le i \le n$. Then $A_i \ll_R D$, $\forall i$ if and only if $\sum_{i=1}^{n} A_i \ll_R D$.*

Proof: It is clear by Proposition (2.3).

Lemma 2.5: [3] *Suppose that $f: D \longrightarrow D'$ is an R- homomorphism, where D and D' be R-mod. Then $f(Rad(D)) \le Rad(D')$. If f is an epimorphism and $kerf \ll D$, then $f(Rad(D)) = Rad(D')$.*

Proposition 2.6: *Suppose that $f: D \longrightarrow D_1$ be an R-epimorphism such that $kerf \ll D$ and $A \ll_R D$, then $f(A) \ll_R D_1$.*

Proof: Let $B \le D_1$ such that $f(A) + B = D_1$. To show that $A + f^{-1}(B) = D$. Since $A \ll_R D$, then $f^{-1}(B) + Rad(D) = D$.

Now, $f(f^{-1}(B)) + f(Rad(D)) = f(D) = (f(D) \cap B) + Rad(D_1) = f(D)$. But f is epimorphism, so $(D_1 \cap B) + Rad(D_1) = D_1$, hence $B + Rad(D_1) = D_1$. Thus $f(A) \ll_R D_1$

Corollary 2.7: *Let A, B are submodules of D, also $B \le A$ with $B \ll D$. If $A \ll_R D$, then $A/B \ll_R D/B$.*

Proof: Assume that $\pi: D \longrightarrow D/B$ be the natural epimorphism. $A \ll_R D$ and $kerf \ll D$, then by proposition (2.6), $\pi(A) = A/B \ll_R D/B$.

Proposition 2.8: Let A and B be submodules of D, with $A \le B \le D$. Then $B \ll_R D$ *if and only if* $A \ll_R D$ also $B/A \ll_R D/A$.

Proof: $\Rightarrow$ Suppose that $C \le D$ and $A + C = D$. $A \le B$, hence $B + C = D$. But $B \ll_R D$, hence $C + Rad(D) = D$. Now, suppose that $/A + C/A = D/A$, where C/A is a sub-mod of D/A, to prove $D/A = C/A + Rad(D/A)$. Since $(B + C)/A = D/A$, then $B + C = D$. But $B \ll_R D$, hence $C + Rad(D) = D$ and $(C + Rad(D))/A = \frac{D}{A}$. So $C/A + (Rad(D))/A = D/A$. Since $Rad(D)/A \subseteq Rad(D/A)$by lemma (2.5), then $C/A + Rad(D/A) = D/A$.

$\Leftarrow$ Suppose $B/A \ll_R D/A$ and $A \ll_R D$, to prove $B \ll_R D$. Let $L \le D$ and $B + L = D$, hence $D/A = (B + L)/A$, since $A \le B$, then $A + B = B$ so $(B + L)/A = (B + L + A)/A = B/A + (L + A)/A = D/A$. Since $/A \ll_R D/A$, then $(L + A)/A + Rad(D/A) = D/A$. But $Rad(D/A) = (Rad(D))/A$

by lemma (2.5). Hence, $(L + A)/A + (Rad(D))/A = D$, then $L + A + Rad(D) = D$, but $A \ll_R D$, implies $L + Rad(D) = D$. Therefor, $B \ll_R D$.

Recall that " an R-mod is called distributive if for all $H, K, N, \leq D$, $H \cap (K + N) = (H \cap K) + (H \cap N)$ " [7].

Proposition 2.9: Let D be a distributive module also A be a direct summand in D such that $Rad(D) \cap A = Rad(A)$, also $B \leq A \leq D$. If $B \ll_R D$, then $B \ll_R A$.

Proof: Let $L \leq A$ such that $B + L = A$. To prove $= L + Rad(A)$. Since A a direct summand of D, then $D = A \oplus V$, where $V \leq D$, so then $(B + L) \oplus V = D$, thus $B + (L \oplus V) = D$, but $B \ll_R D$, hence $(L \oplus V) + Rad(D) = D$, so $[(L + V) + Rad(D)] \cap A = D \cap A$, hence $[L + (V + Rad(D)] \cap A = D \cap A$, we get $L + [(V + Rad(D) \cap A] = A$. But D is a distributive module, hence $L + [(V \cap A) + Rad(D) \cap A = A$. But $V \cap A = (0)$, then $L + (A \cap Rad(D)) = A$. Also $Rad(D) \cap A = Rad(A)$ .Therefor $A = L + Rad(A)$ where $B \ll_R D$.

Corollary 2.10: *Let D be a distributive and projective module, such that $A \leq B \leq D$ and N is a direct summand in D with $A \ll_R D$, then $A \ll_R B$*

Proof: Since D is a projective R-mod, hence $Rad(D) \cap B = Rad(B)$, hence the result follows by Proposition 2.9.

3. Radical Hollow Modules

This section is devoted to define the radical hollow modules. It is a generalization of hollow-mod. We discuss the main characteristics of this class of modules furthermore proceed over a few of its characteristics of this modules.

Definition 3.1: A non-zero R-mod D is said to be radical hollow (for short Rad-hollow), if each proper sub-mod in D is Rad-small of D .

A ring R is called Rad-hollow, if each proper ideal in R is Rad-small.

Remarks and Examples 3.2

1- It is clear that, each hollow module is radical hollow

2- Generally, the converse of (1) need not be accurate, for example:

A set of rational numbers $\mathbb{Q}$ is Rad-hollow as a Z-mod, since $Rad(\mathbb{Q}) = \mathbb{Q}$. But $\mathbb{Q}$ is not hollow, [2].

3- Z_{P^∞} is Rad-hollow Z-module, since Z_{P^∞} is hollow-mod, [2].

4- The Z as Z-mod is not Rad-hollow, also Z does not hollow as Z-mod, since there exists proper sub-mod which is not small in Z.

5- The semisimple module cannot be radical hollow since the radical small submodule of a semisimple module equal zero and every submodule of a semisimple module is direct summand

6- If $d(D) = 0$, then every Rad-hollow module D is hollow.
7- If D is Z-regular hollow R-mod, then it is Rad-hollow.

Proposition 3.3: Let $f: D \to N$ be an epimorphism with $ker \ll D$ and D is Rad-hollow R-mod, then N is Rad-hollow R-mod.

Proof: Suppose $f: D \longrightarrow N$ is an epimorphism with D be an Rad-hollow module and N be any module. If H is a proper sub-mod of N, such that $H + K = N$ where $K \leq N$. To prove $K + Rad(N) = N$. Now, $f^{-1}(H) < D$ since if $f^{-1}(H) = N$ implies $f(f^{-1}(H))) = f(D) = N$ and hence $D = N$ and this is a contradiction. So is $f^{-1}(H) \leq D$. Now $f^{-1}(H) + f^{-1}(K) = f^{-1}(N)) = D$. But D is Rad-hollow R-mod, thus $f^{-1}(H) \ll_{\mathrm{R}} \mathrm{D}$ hence $D = f^{-1}(K) + Rad(D)$ and $f(f^{-1}(K)) + f(Rad(D)) = f(D) = N$, since f is an epimorphism $f(f^{-1}(K))) = K$ hence $K + f(Rad(D)) = N$. Since f is epimorphism with $ker \ll D$, then by lemma (2.5) $f(Rad(D)) = Rad(N)$ hence $K + Rad(N) = N$. Therefore N is Rad-hollow R-mod.

Corollary 3.4: *Let D be an R-module. If D is Rad-hollow, then D/H is Rad-hollow module for every submodule $H \ll D$.*

Proof: Since $\pi : D \longrightarrow D/H$ is a natural epimorphism, and D is Rad-hollow with $r\pi = H \ll D$, hence by proposition (3.3), D/H is Rad-hollow.

Recall that "a sub-mod N of an R-mod D is called fully invariant if $f(N) \subseteq N$. for each $f \in End(D)$", [8, p.4] and " D is called duo module if every submodule of D is Fully invariant" [9, 10].

Proposition 3.5: Let $D = M_1 \oplus M_2$ and D be a duo module. If M_1 and M_2 are Rad-hollow module, then D is Rad-hollow module. provided $A \cap M_i \neq M_i \ \forall i = 1,2$.

Proof: Let $A_1 \leq M_1$ and $A_2 \leq M_2$, since M_1 and M_2 are Rad-hollow modules, then $A_1 \ll_R M_1$ and $A_2 \ll_R M_2$. To show that $A_1 \oplus A_2 \ll_R M_1 \oplus M_2 = D$, then by remark and examples (2.2(6)), $A_1 \ll_R D$ and $A_2 \ll_R D$ hence by proposition (2.3) $A_1 \oplus A_2 \ll_R D$.

Proposition 3.6: For any non-zero R-mod D. Then D is Rad-hollow if and only if $H \ll D$ and D/H is Rad-hollow.

Proof: $\Rightarrow$ If $H = 0$, then it is directly.

$\Leftarrow$ Suppose $\frac{D}{H}$ is Rad-hollow, if A is a sub-mod of D, and $D = A + B$, where B is a sub-mod of D to prove $D = Rad(D) + B$. Now, $\frac{D}{H} = \frac{(A+B)}{H} + \frac{(B+H)}{H}$, $D \neq B + H$ $(H \ll_R D)$, so $\frac{(B+H)}{H} \neq \frac{D}{H}$. Since $\frac{D}{H}$ is Rad-hollow, then $\frac{D}{H} = Rad(\frac{D}{H}) + \frac{(B+H)}{H}$. But $\pi: D \to D/H$ is an epimorphism and $ker\pi = H \ll$

D, hence by lemma (2.5) $\pi\big(Rad(D)\big) = \frac{Rad(D)}{H} = Rad(\frac{D}{H})$ .Thus $\frac{D}{H} = \frac{Rad(D)}{H} + \frac{(B+H)}{H}$, therefore, $D = B + H + Rad(D)$. But $H \ll D$, so $D = Rad(D) + B$. Thus D is Rad-hollow module.

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