

## Purely co-closed and purely co-essential sub-modules

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### Abstract

In this article, the idea for this study is to present two concepts co-closed and purely co-essential sub-modules that generalize co-closed and co-essential sub-modules. We study the relationship between purely co-closed and purely co-essential sub-modules by adding some of the properties of co-closed and co-essential sub-modules to them.

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### 1. Introduction

In this work,  $D$  is unitary left  $R$ -module,  $R$  is commutative ring with identity. It is established, for each ideal  $I$  for  $R$ , the sub-modules  $C$  belonging to  $D$  is considered pure if  $ID \cap C = IC$  [1]. if  $D$  module & suppose  $C$  the proper sub-modules of  $D$  is called small. In summary ( $C \ll D$ ) if any sub-module  $B$  of  $D$  such that  $C + B = D$  then  $B = D$ , [4-6], [12-13]. A sub-module  $H$  is called co-closed in  $D$  denoted by  $(H \ll_{c.c} D)$ , if  $K$  is a sub-module of  $H$  with  $\frac{H}{K}$  small in  $\frac{D}{K}$  then  $H = K$ . And,  $K \leq H \leq D$ , then  $K$  is called co essential sub-module of  $H$  ( $H \ll_{c.c} D$ ) if  $\frac{H}{K} \ll \frac{D}{K}$ , [3]. As a generalization of a small sub-module Ahmed, Dhari, and Atiya presented a notion of purely small, which is called Pu-small sub-module. Let  $C$  a sub-module for  $R$ -modules  $D$  named purely small ( $C \ll_{pu} D$ ), if  $C + V = D$ , whenever pure sub-module  $V$  for  $D$ .

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Equivalen;  $C \ll_{\text{Pu}} D$ , if the pure sub-module  $V$  for  $D$  &  $C + V \neq D$ , so  $V = D$ . If  $I + J = R$ , the ideal  $I$  for  $R$  purely small whenever pure ideal  $J$  for  $R$ , [1]. Since every small sub-module is purely small this led us to study, the concepts of purely co-closed and purely co-essential sub-modules as an extension of co-essential sub-modules but purely co-closed is stronger than co-closed sub-module. In the section two, A concept of purely co-closed sub-modules, when a sub-module  $H$  of an  $R$ -module  $D$  is called to be purely co-closed if when  $\frac{H}{K} \ll_{\text{Pu}} \frac{D}{K}$  implies  $K = H$ , where  $K \leq H$ . In section three, we discuss the concept of purely co-essential sub-module. A sub-module  $K$  in  $R$ -module  $D$  a purely co-essential sub-module of  $H$  if  $\frac{H}{K} \ll_{\text{Pu}} \frac{D}{K}$  where  $K \leq H \leq D$ . Also, we give some of their elementary properties, which are within the study of the concepts of purely co-closed and purely co-essential sub-modules.

## 2. Purely Co closed Sub-modules

This section provides a definition and illustrations of the idea of a purely-co closed sub-module and we illustrate it with examples.

A sub-module  $N$  is co-closed where  $K \leq N$  with  $\frac{N}{K} \ll \frac{M}{K}$  implies  $N = K$ , [3]. Definition (2.1): Assume  $N$  is sub-module in  $R$ - module  $M$ , then  $N$  is

Purely-co closed. In summary (Pu-co closed) denote as  $(N \leq_{\text{Pu,cc}} M)$ . for  $K \leq N$ , with  $\frac{N}{K} \ll_{\text{Pu}} \frac{M}{K}$ , then  $N = K$ .

### Remarks and Examples (2.2):

1- Every Pu-co closed is co-closed.

**Proof:** Assume that  $K \leq N \leq M$  s.t  $\frac{N}{K}$  a small sub-module for  $\frac{M}{K}$ , then  $\frac{N}{K}$  a Pu-small sub-modules for  $\frac{M}{K}$  [3]. Hence  $M$  a Pu-co closed implies  $N = K$ .

2- In general, the converse (1) is not true. For example: Consider  $Z$  as  $Z$ -module, let  $N = 2Z$  and  $K = (0)$ , and  $K \leq N$   $\frac{2Z}{(0)} \ll_{\text{Pu}} \frac{Z}{(0)}$ , but  $2Z \neq (0)$ ,  $2Z$  is not Pu- co closed of  $Z$  and  $2Z$  not small of  $Z$  then  $\frac{2Z}{(0)}$  not small in  $\frac{Z_{12}}{(0)}$  implies that  $2Z$  is co-closed in  $Z$ .

3- The sub-module  $(\bar{6})$  Pu-small in  $Z_{12}$  as  $Z$ -module. Now let  $N = (\bar{2})$  and  $K = (\bar{4})$  with  $K \leq N$  then  $\frac{N}{K} \ll_{\text{Pu}} \frac{M}{K}$ . Now  $(\bar{6}) \simeq \frac{(2)}{(4)} \ll_{\text{Pu}} \frac{Z_{12}}{(4)} \simeq (\bar{3})$ , then  $(\bar{6}) \ll_{\text{Pu}} (\bar{3})$ . But  $(\bar{2}) \neq (\bar{4})$  implies  $(\bar{6})$  is not co-closed. But  $(\bar{2})$  is co-closed in  $Z_{12}$ . Since  $(\bar{2})$  is not small in  $Z_{12}$ .

4- Let  $M = Z_6$  as  $Z$ -module,  $N = (\bar{2})$  and  $K = (\bar{0})$ ,  $\frac{(2)}{(0)}$  is not Pu- small in  $\frac{Z_6}{(0)}$ , implies  $(\bar{2}) \neq (\bar{0})$ ,  $(\bar{2})$  is co closed and Pu- co closed, since  $(\bar{2})$  is not Pu -small.

**Proposition 2.3:** Every direct summand of  $M$  is a Pu-co closed in  $M$  if  $M \neq 0$  and is an  $R$ -module.

**Proof:** Let  $M = N \oplus K$  &  $L \leq N$  with  $\frac{N}{L}$  is Pu-small in  $\frac{M}{L}$ , since  $M = N \oplus K$ , then  $\frac{M}{L} = \frac{N}{L} + \frac{K+L}{L}$ , so  $L \leq L + K \leq M$ ,  $K$  a pure sub - modules, therefore  $\frac{K+L}{L}$  a pure in  $\frac{M}{L}$  by [7, Remark (3.1.5), p 56]. To show  $N = L$ . Now by the second isomorphic theorem  $\frac{K+L}{L} \cong \frac{K}{K \cap L}$ , when  $K$  a pure sub - modules, then from [7, Remark (3.1.5), p 56]  $\frac{K}{K \cap L}$  be pure sub - modules in  $\frac{M}{L}$ . But  $\frac{N}{L}$  be Pu-small in  $\frac{M}{L}$ , hence  $\frac{K+L}{L} = \frac{M}{L}$ ,  $M = N + K = K + L$  it is simple to demonstrate that  $N = L$ . Thus  $N$  Pu-co closed in  $M$ .

**Remark 2.4:** The only proper Pu-co closed sub-module in a Pu-hollow module is the zero sub-module.

**Proof:** Assume  $N$  proper Pu-co closed sub - modules in  $M$  &  $M$  a Pu-hollow module. Now,  $N \ll_{Pu} M$ , therefore,  $\frac{N}{(0)} \ll_{Pu} \frac{M}{(0)}$ , hence  $N = 0$ .

**Example 2.5:**

- (1)  $\{\overline{0}, \overline{2}\}$  is not Pu-co closed sub-module in the  $Z$ -module  $Z_4$ .
- (2) The only proper Pu-co closed sub-module in the modules  $Z_{p^\infty}$  and  $Z$  the zero sub-modules.
- (3) Every sub-module of the module  $Z_6$  is Pu-co closed. Consider  $Z_6$  as  $Z$  module is Pu-hollow,  $\{\overline{0}, \overline{2}, \overline{4}\}$  is not Pu-small in  $Z_6$ , then  $\{\overline{0}, \overline{2}, \overline{4}\}$  is Pu-co closed.

**Corollary 2.6:** If  $M$  finite generated faithful multiplications  $R$  - module &  $K \leq N \leq M$ . Then  $\frac{N}{K}$  is a Pu-small in  $\frac{M}{K}$  if and only if  $\left[\frac{N}{K} : \frac{M}{K}\right]$  is a Pu-small in  $\frac{R}{[K:M]}$  where  $\left[\frac{N}{K} : \frac{M}{K}\right]$  considered as an ideal of  $\frac{R}{[K:M]}$ .

**Proof:** It verifies that  $\frac{M}{K}$  is multiplication  $\frac{R}{[K:M]}$  -module. So  $\frac{M}{K}$  satisfied Proposition (2.8) [8]. Consequently, the outcome is as follows.

**Proposition 2.7:** Assume  $N$  Pu-co closed sub - module for  $M$ . Then  $\frac{N}{L}$  Pu-co closed sub-module in  $\frac{M}{L}$ .

**Proof:** Assume that  $K$  of  $N$  containing  $L$ , &  $\frac{N}{K} \ll_{Pu} \frac{M}{K}$  by the second isomorphic theorem, [11]. Therefore  $\frac{N}{K} \ll_{Pu} \frac{M}{K}$ . now  $N$  Pu-co closed implies that  $N = K$  and  $\frac{N}{K} \cong \frac{N}{K}$  and  $\frac{M}{K} \cong \frac{M}{K}$  hence  $\frac{N}{L} = \frac{K}{L}$  implies that  $\frac{N}{L} \leq_{Pu.cc} \frac{M}{L}$  for every  $L \leq N$ .

**Proposition 2.8:** If  $M$  finitely generated faithful multiplications  $R$ -modules &  $N$  sub-module &  $N$  Pu-co closed iff  $[N:M]$  Pu-co closed.

**Proof:**  $\Rightarrow$ ) suppose  $N$  Pu-co closed. To prove  $I \leq [N:M] \leq R$

With  $\frac{[N:M]}{I}$  is Pu-small in  $\frac{R}{I}$ . Since  $[IM:M] = I$ , [7]. Then  $\frac{[N:M]}{[IM:M]}$  is Pu-small in  $\frac{R}{[IM:M]}$ . But one can show that  $\frac{[N:M]}{[IM:M]} = \left[ \frac{[N:M]M}{[IM:M]M} : \frac{M}{[IM:M]M} \right]$ , therefore  $\left[ \frac{[N:M]M}{[IM:M]M} \right] = \left[ \frac{M}{[IM:M]M} \right]$  is Pu-small in  $\frac{R}{[IM:M]}$ . Using Corollary (2.6) we get  $\left[ \frac{[N:M]M}{[IM:M]M} \right]$  is Pu-small in  $\frac{M}{[IM:M]M}$  thus  $N = [N:M] = [IM:M]M = IM$ . Then by assumption  $[N:M] = I$ .

$\Leftarrow$ ) Let  $K \leq N \leq M$  with  $\frac{N}{K}$  is Pu-small in  $\frac{M}{K}$ . from (2.6) implies  $\left[ \frac{N}{K} : \frac{M}{K} \right]$  Pu-small in  $\frac{R}{[K:M]}$ . Thus  $\left[ \frac{N}{K} : \frac{M}{K} \right] = \frac{[N:M]}{[K:M]}$ , thus  $\frac{[N:M]}{[K:M]}$  is Pu-small in  $\frac{R}{[K:M]}$ , and since  $[N:M]$  Pu-co closed in  $R$  implies  $[N:M] = [K:M]$ , therefore  $N = K$ . The following gives the basic properties of Pu-co closed sub-modules.

**Proposition 2.9:** If  $M$  F-regular  $R$ -module. Let  $N_1 \leq N_2 \leq M$ , if  $N_1 \ll_{Pu} N_2$  and  $\frac{N_2}{N_1}$  is Pu-co closed in  $\frac{M}{N_1}$ , then  $N_2 \leq_{Pu,cc} M$ .

**Proof:** Assume  $L \leq N_2$  such that  $\frac{N_2}{L} \ll_{Pu} \frac{M}{L}$ .

Let  $\pi: M \rightarrow \frac{M}{N_1}$  is the natural epimorphism, so by [3].  $K \leq_P M$  and since  $\frac{L+N_1}{N_1} \leq_P \frac{N_2}{N_1}$  implies that  $\frac{\frac{N_2}{N_1}}{\frac{L+N_1}{N_1}} \ll_{Pu} \frac{\frac{M}{N_1}}{\frac{L+N_1}{N_1}}$ . Since  $\frac{N_2}{N_1}$  is Pu-co closed in  $\frac{M}{N_1}$ , so  $\frac{L+N_1}{N_1} = \frac{N_2}{N_1}$  and hence  $N_2 = L + N_1$ . but  $M$  F-regular &  $L \leq_P M$  &  $L \leq N_2$  then  $L \leq_P N_2$ . But  $N_1 \ll_{Pu} N_2$ , thus  $N_2 = L$ , therefore  $N_2 \leq_{Pu,cc} M$ .

**Proposition 2.10:** Let  $N_1 \leq N_2 \leq M$  and if  $N_1$  is Pu-co closed, so  $N_1$  Pu-co closed in  $N_2$ .

**Proof:** Suppose  $L \leq N_1$  s.t  $\frac{N_1}{L} \ll_{Pu} \frac{N_2}{L} \leq \frac{M}{L}$  so by [10].  $\frac{N_1}{L} \ll_{Pu} \frac{M}{L}$ . Since  $N_1$  Pu-co closed in  $M$ , by f t zero sub-module, so  $L = N_1$ , therefore  $N_1$  is Pu-co closed in  $N_2$ .

**Proposition 2.11:** Suppose  $N$  is a non-zero. If  $N \leq_{Pu,cc} M$ , then  $N$  not Pu-small in  $M$ .

**Proof:** Assume  $N \ll_{Pu} M$  &  $N \leq_{Pu,cc} M$ . Because  $\{0\} \leq N$  and  $N \simeq \frac{N}{\{0\}} \ll_{Pu} \frac{M}{\{0\}} \simeq M$ . Then  $N = \{0\}$  this contradiction. so  $N$  not Pu-small in  $M$ .

### 3. Purely Coessential Sub-modules

As the generalization of co-essential sub-modules, we define and go over some of the features of the purely co-essential sub-modules in this section.

Suppose  $A, B$  sub-module in  $M$  with  $A \leq B \leq M$ . Recall that  $A$  called co-essential (or co-small) of  $B$  in  $M$ , if  $\frac{B}{A} \ll \frac{M}{A}$ . See [10].

As a generalization of the coessential sub-module, we present the notion of the purely coessential sub-module.

**Definition 3.1:** Assume  $M$   $R$ -module, &  $A$  belongs to  $M$  as a sub-module & called purely co-essential of  $B$  in summary (Pu-coessential) if  $\frac{B}{A} \ll_{\text{Pu}} \frac{M}{A}$  ( $A \leq_{\text{Pu.ce}} B$ ).

#### Remark and Example 3.2:

- 1- Every co-essential is Pu-coessential. A sub - module of  $M$ , since  $A$  is co-essential  $\frac{B}{A} \ll \frac{M}{A}$ , but every small is Pu-small, [3]. Which implies  $\frac{B}{A} \ll_{\text{Pu}} \frac{M}{A}$ . Therefore  $A$  is Pu-co-essential.
- 2- In general, the opposite of (1) not right. for example:- Let  $M = Z$  as  $Z$ -module  $2Z \simeq \frac{(2)}{(0)} \ll_{\text{Pu}} \frac{Z}{(0)} \simeq Z$ .  $(0)$  is Pu-co-essential of  $2Z$  in  $Z$ , but  $(0)$  is not co-essential of  $2Z$  in  $Z$ . Since  $2Z \simeq \frac{(2)}{(0)}$  not small in  $\frac{Z}{(0)} \simeq Z$ . Since  $2Z + 3Z = Z$  but  $3Z \neq Z$ .
- 3- Let  $M = Z_6$  as  $Z$ -module.  $(\bar{0})$  is not Pu-coessential of  $(\bar{3})$  in  $Z_6$ , since  $\frac{(\bar{3})}{(\bar{0})} \simeq (\bar{3})$  is not Pu-small in  $\frac{Z_6}{(\bar{0})} \simeq Z_6$  [8].
- 4- Assume  $A$  sub-module iff  $\{0\} \leq_{\text{Pu.ce}} A$ , then  $A \ll_{\text{Pu}} M$ .

**Proposition 3.3:** Assume  $A \leq B \leq C \leq M$  &  $A, B$ , and  $C$  sub-module for  $R$ -module  $M$ . Then  $B \leq_{\text{Pu.ce}} C$  in  $M$  iff  $\frac{B}{A} \leq_{\text{Pu.ce}} \frac{C}{M}$  in  $\frac{M}{A}$ .

**Proof:** Showing that  $\frac{B}{A}$  is Pu-coessential sub module of  $\frac{C}{A}$ .  $B \leq_{\text{Pu.ce}} C$ . Thus.

Accordingly,  $\frac{C}{B}$  is Pu-small of  $\frac{M}{B}$ . So  $\frac{C}{B} \simeq \frac{\frac{C}{B}}{\frac{A}{B}} = \frac{\frac{C}{A}}{\frac{B}{A}}$  then  $\frac{\frac{C}{A}}{\frac{B}{A}} \simeq \frac{M}{B}$  according to the second isomorphism theorem, [11].  $\frac{C}{B} \ll_{\text{Pu}} \frac{M}{B}$  and  $\frac{B}{A} \leq_{\text{Pu.ce}} \frac{C}{A}$  in  $\frac{M}{A}$ .

$\Leftarrow$  Let  $\frac{B}{A} \leq_{\text{Pu.ce}} \frac{C}{A}$  in  $\frac{M}{A}$ , Hence  $\frac{\frac{C}{A}}{\frac{B}{A}} \ll_{\text{Pu}} \frac{\frac{M}{A}}{\frac{B}{A}}$ , since  $\frac{\frac{C}{A}}{\frac{B}{A}} \simeq \frac{C}{B}$  and  $\frac{\frac{M}{A}}{\frac{B}{A}} \simeq \frac{M}{B}$ , according to the second isomorphism theorem, [11]. Thus,  $\frac{C}{B} \ll_{\text{Pu}} \frac{M}{B}$  and  $B \leq_{\text{Pu.ce}} C$ .

**Proposition 3.4:** If  $A \leq B \leq M$  &  $A \leq_P M$ . If  $M = B + C$  then  $M = A + C$ , for every pure sub-module  $C$  of  $M$  implies  $A \leq_{\text{Pu.ce}} B$ .

**Proof:** If  $\frac{M}{A} = \frac{B}{A} + \frac{L}{A}$  and suppose that  $\frac{L}{A}$  pure for  $\frac{M}{A}$ , so  $M = B + L$ , since  $A$  pure &  $\frac{L}{A}$  pure so by [7, Remark (3.1.5), p56].  $L$  pure. Now, from the assumption  $M = A + L$ , so  $M = L$  and  $\frac{M}{A} = \frac{L}{A}$ , so  $\frac{B}{A} \ll_{\text{Pu}} \frac{M}{A}$  and  $A \leq_{\text{Pu.ce}} B$ .

Recall that If the sum of any of two pure sub-modules is again pure, it is said an  $R$ -module has the pure sum property (PSP) [2].

**Proposition 3.5:** If  $M$  module with PSP property &  $A \leq B \leq M$  &  $A \leq_p M$ . If  $B = A + S$  where  $S \ll_{\text{Pu}} M$ , then  $A \leq_{\text{Pu.ce}} B$ .

**Proof:** Assume that a pure sub-module  $C$  of  $M$  with  $M = B + C$ . Since  $B = A + S$ , therefore  $M = (A + S) + C = (A + C) + S$ , Now  $A$  and  $C$  are pure sub-modules of  $M$ . But  $M$  has PSP property thus  $(A + C)$  a pure sub-module of  $M$ , [2]. Since  $S \ll_{\text{Pu}} M$ , so that  $M = A + C$ . Therefore  $A \leq_{\text{Pu.ce}} B$  by Preposition (3.4).

**Corollary 3.6:** If  $M$  module with PSP property & suppose that  $A \leq B \leq M$  &  $A$  pure. If  $M = A + S$  and  $S \cap B$  is a Pu-small sub-module in  $M$ , then  $A \leq_{\text{Pu.ce}} B$ .

**Proof:** The sub-module  $B = B \cap M = B \cap (A + S) = A + S \cap B$  (by modular low). Since  $S \cap B \ll_{\text{Pu}} M$  and  $A \leq_p M$  implies  $A \leq_{\text{Pu.ce}} B$  by Preposition (3.5).

**Remark 3.7:** In general, the opposite of corollary (3.6) is not right. For example: In  $Z_{12}$  as  $Z$ -module.  $(\bar{6})$  is not Pu-coessential of  $(\bar{3})$  and  $Z_{12} = (\bar{3}) + (\bar{2})$ , but  $(\bar{6}) + (\bar{2}) = (\bar{2}) \neq Z_{12}$ . Since  $(\bar{6}) \simeq \frac{(\bar{3})}{(\bar{6})}$  and  $\frac{Z_{12}}{(\bar{6})} \simeq (\bar{2})$ ,  $(\bar{6}) \ll_{\text{Pu}} (\bar{2})$ , so  $(\bar{6})$  Pu-coessential of  $(\bar{3})$ .

**Proposition 3.8:** Assume  $M$  is  $R$ -module with PSP property & Let  $A \leq B \leq C \leq M$ , where  $A, B$  are pure, so  $A \leq_{\text{Pu.ce}} C$  iff  $A \leq_{\text{Pu.ce}} B$  &  $B \leq_{\text{Pu.ce}} C$ .

**Proof:**  $\Rightarrow$  Assume that  $A \leq_{\text{Pu.ce}} C$  in  $M$ , then  $\frac{C}{A} \ll_{\text{Pu}} \frac{M}{A}$  hence  $\frac{B}{A} \ll_{\text{Pu}} \frac{M}{A}$  by Preposition (2.3) [8]. So  $A \leq_{\text{Pu.ce}} B$  in  $M$ . Now  $\frac{C}{A} \ll_{\text{Pu}} \frac{M}{A}$  and  $\frac{C}{A} \ll_{\text{Pu}} \frac{M}{A}$ . Also,  $B$  is pure so  $\frac{B}{A}$  is pure. Thus by [3]  $\frac{C}{B} \simeq \frac{\frac{C}{A}}{\frac{B}{A}}$  is Pu-small in  $\frac{M}{A}$ . Therefore  $B \leq_{\text{Pu.ce}} C$  in  $M$ .

Conversely: Suppose that  $A \leq_{\text{Pu.ce}} B$  in  $M$  and  $B \leq_{\text{Pu.ce}} C$  in  $M$ . To show that  $A \leq_{\text{Pu.ce}} C$  in  $M$ . Assume that  $\frac{M}{A} = \frac{C}{A} + \frac{U}{A}$ . Since  $\frac{U}{A}$  is a pure sub-module of  $\frac{M}{A}$ , and  $A$  is a pure in  $M$ , then by [7, Remark (3.1.5), p56],  $U$  pure. Now  $M = C + U$ , then  $\frac{M}{B} = \frac{C+U}{B} = \frac{C}{B} + \frac{U+B}{B}$ , since  $M$  has the PSP property, then  $U + B$  is pure in  $M$ , hence by [7, Remark (3.1.5), p56],  $\frac{U+B}{B}$  is pure in  $\frac{M}{B}$ . However, as  $\frac{C}{B}$  is Pu-small in  $\frac{M}{B}$ , therefore  $\frac{M}{B} = \frac{U+B}{B}$  so  $M = U + B$ , then  $\frac{M}{A} =$

$\frac{U}{A} + \frac{B}{A}$ , thus  $\frac{U}{A}$  is pure &  $\frac{B}{A}$  is Pu-small in  $\frac{M}{A}$  then  $\frac{M}{A} = \frac{U}{A}$ , thus  $M = U$ , hence  $\frac{M}{A} = \frac{U}{A}$  and  $A \leq_{\text{Pu.ce}} C$ .

**Lemma 3.9:** Assume  $\pi: M \rightarrow \frac{M}{A}$  be the natural epimorphism with  $\text{Ker } \pi = A$  is pure. If  $B \ll_{\text{Pu}} M$  then  $\pi(B) = \frac{B}{A} \ll_{\text{Pu}} \frac{M}{A}$ .

**Proof:** It is clear by [3, 9].

**Proposition 3.10:** Assume  $A \leq B \leq M$ . If  $A \leq_{\text{Pu.ce}} B$ , with  $A$  is pure, then  $\pi(A) \leq_{\text{Pu.ce}} \pi(B)$ .

**Proof:** Let  $\pi: M \rightarrow \frac{M}{A}$  be a natural epimorphism. Since  $A \leq_{\text{Pu.ce}} B$ , Thus, according to Lemma (3.9),  $\frac{B}{A} \ll_{\text{Pu}} \frac{M}{A}$  &  $\pi(A) \leq_P \frac{M}{A}$ .  $\pi(\frac{B}{A}) = \frac{\pi(B)}{\pi(A)} \ll_{\text{Pu}} \frac{M}{\pi(A)}$ . In this way,  $\pi(A) \leq_{\text{Pu.ce}} \pi(B)$ .

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