

Pseudo permutation-BCK-algebras with their ideals

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Abstract

This paper provides new ideas like pseudo permutation BCK -algebra, pseudo permutation subalgebra, pseudo permutation ideal, and portion permutation. Moreover, their qualities are discussed. In this work, we show that if $\lambda_{p_0}^\beta$ is a portion permutation of $\mathcal{M} = (X, \odot, \oplus, T)$, then $\lambda_{p_0}^\beta \odot \lambda_i^\beta = \lambda_{p_0}^\beta \oplus \lambda_i^\beta$ for any $\lambda_i^\beta \in X$. Next, some interesting results in pseudo permutation BCK -algebra are given and shown. Furthermore, some instances are provided to support our findings in this paper.

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1. Introduction

Imai and Is'eki introduced BCK -algebras [1]. BCK –algebras are algebraic representations of the BCK –system in combinatory logic, with many interesting applications in Computer program. Georgescu and Iorgulescu [2] presented pseudo- BCK –algebras as an extension of BCK –algebras in their paper. In Walendziak [3] provides an alternative axiomatization of pseudo- BCK algebras. As a result, several academics have looked into generalizations of $BCK/BCI/RHO$ algebras (see [4, 5]). In this work, new ideas like pseudo permutation BCK – algebra, pseudo permutation subalgebra, pseudo permutation ideal, and portion permutation are given. Also, their qualities are discussed. In this work, we show that if $\lambda_{p_0}^\beta$ is a portion permutation of $\mathcal{M} =$

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$(X, \odot, \odot, T)$, then $\lambda_{po}^\beta \odot \lambda_i^\beta = \lambda_{po}^\beta \odot \lambda_i^\beta$ for any $\lambda_i^\beta \in X$. After that, some interesting results in pseudo permutation BCK –algebra are given and shown. In addition, illustrations are presented to back up our results in this work.

2. Preliminaries

Definition 2.1: [4] An algebra $(W, \vartheta, 0)$ is called BCK –algebra if $\vartheta: W \times W \rightarrow W$ and $0 \in W$ such that (1) $((a\vartheta b)\vartheta(a\vartheta c))\vartheta(c\vartheta b) = 0$, (2) $(a\vartheta(a\vartheta b))\vartheta b = 0$, (3) $a\vartheta a = 0$, (4) $0\vartheta a = 0$, (5) If $a\vartheta b = 0$ & $b\vartheta a = 0$, then $a = b$, $\forall a, b, c \in W$.

Definition 2.2: [6] If $f = \prod_{e=1}^{c(f)} h_e \in S_m$ is a permutation in S_m , where S_m is a symmetric–group with $\{h_e\}_{e=1}^{c(f)}$ are pairwise disjoint cycles and $h_e = (t_1^e, t_2^e, \dots, t_{\alpha_e}^e)$, $1 \leq e \leq c(f)$, for some $1 \leq \alpha_e, c(f) \leq m$. Then the permutation sets of f in S_m are given as $h_e^f = \{t_1^e, t_2^e, \dots, t_{\alpha_e}^e\} | 1 \leq e \leq c(f)$ and symbolized as f –sets of $\{h_e\}_{e=1}^{c(f)}$.

Definition 2.3: [7] Assume that $W = \{h_i^f\}_{i=1}^{c(f)} = \{t_1^i, t_2^i, \dots, t_{\alpha_i}^i\} | 1 \leq i \leq c(f)\}$ is a family of f –sets, where $f \in S_m$. For some $h_k^f \in W$ put $T = h_k^f$, where h_k^f with $\sum_{s=1}^{\alpha_k} t_s^k \leq \sum_{s=1}^{\alpha_i} t_s^i$ & $|h_k| \leq |h_i|, \forall (1 \leq i \leq c(f)) \dots (*)$. If $\exists h_v^f, h_g^f \in W$, with $h_v^f \cap h_g^f = \emptyset$ and they are satisfied $(*)$, then $T = h_v^f$ if $\exists t_r^v \in h_v^f$ with $t_r^v < t_s^g, \forall (1 \leq s \leq \alpha_g)$ or $T = h_g^f$ if $\exists t_r^g \in h_g^f$ with $t_r^g < t_s^v, \forall (1 \leq s \leq \alpha_v)$. Also, let $\#: W \times W \rightarrow W$ be a closed on W . We say $(W, \#, T)$ is a permutation BCK –algebra ($PBCK - A$), if (1) $((h_i^f \# h_j^f) \# (h_i^f \# h_b^f)) \# (h_b^f \# h_j^f) = T$, (2) $(h_i^f \# (h_i^f \# h_j^f)) \# h_j^f = T$, (3) $h_i^f \# h_i^f = T$, (4) $T \# h_i^f = T$, (5) If $h_i^f \# h_j^f = T$ and $h_j^f \# h_i^f = T$, then $h_i^f = h_j^f, \forall h_i^f, h_j^f \in W$.

Definition 2.4: [8] Assume that $*$, $\circ$, and $\leq$ are three binary defined on W and $0 \in W$. We say $\Psi = (W, \leq, *, \circ, 0)$ is pseudo– BCK –algebra if it such that $(a_1) (m * n) \circ (m * k) \leq k * n, (m \circ n) * (m \circ k) \leq k \circ n, (a_2) m * (m \circ n) \leq n, m \circ (m * n) \leq n, (a_3) m \leq m, (a_4) 0 \leq m, (a_5)$ if $m \leq n$ and $n \leq m$, then $m = n, (a_6) m \leq n \leftrightarrow m * n = 0 \leftrightarrow m \circ n = 0, m, n, k \in W$.

3. Pseudo Permutation-BCK-Algebras

Definition 3.1: We say $\mathcal{M} = (W, \leq, \odot, \odot, T)$ is pseudo permutation BCK –Algebra [briefly, $(pPBCK - A)$] where $T \in W$, $\odot$ & $\odot$ are binary on W and $\leq$ is defined on W as $(\lambda_i^\beta \leq \lambda_j^\beta \leftrightarrow \lambda_i^\beta \odot \lambda_j^\beta = T \leftrightarrow \lambda_i^\beta \odot \lambda_j^\beta = T)$, if $\mathcal{M}$ such that: (P1) $(\lambda_i^\beta \odot \lambda_j^\beta) \odot (\lambda_i^\beta \odot \lambda_m^\beta) \leq \lambda_m^\beta \odot \lambda_j^\beta, (\lambda_i^\beta \odot \lambda_j^\beta) \odot (\lambda_i^\beta \odot \lambda_m^\beta) \leq \lambda_m^\beta \odot \lambda_j^\beta, (P2) \lambda_i^\beta \odot (\lambda_i^\beta \odot \lambda_j^\beta) \leq \lambda_j^\beta, \lambda_i^\beta \odot (\lambda_i^\beta \odot \lambda_j^\beta) \leq \lambda_j^\beta, (P3) \lambda_i^\beta \leq \lambda_j^\beta,$

(P4) $T \leq \lambda_i^\beta$, (P5) If $\lambda_i^\beta \leq \lambda_j^\beta$ and $\lambda_j^\beta \leq \lambda_i^\beta$, then $\lambda_i^\beta = \lambda_j^\beta$, $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in W$.

Remark 3.2: (i) From [(P3)], we consider the opeston of (P5), if $\lambda_i^\beta = \lambda_j^\beta$, then $\lambda_i^\beta \circledast \lambda_j^\beta = \lambda_i^\beta \odot \lambda_j^\beta = \lambda_j^\beta \circledast \lambda_i^\beta = \lambda_j^\beta \odot \lambda_i^\beta = T$, (ii) Every $(pPBCK - A)$ $\mathcal{M} = (X, \circledast, \odot, T)$ satisfying $\lambda_i^\beta \circledast \lambda_j^\beta = \lambda_i^\beta \odot \lambda_j^\beta$, for all $\lambda_i^\beta, \lambda_j^\beta \in X$ is a $(PBCK - A)$, (iii) If $\mathcal{M} = (X, \circledast, \odot, T)$ is a $(pPBCK - A)$, then $\mathcal{M}^P = (X, \odot, \circledast, T)$ is also a $(pPBCK - A)$, (iv) Any $(pPBCK - A)$ $\mathcal{M} = (X, \leq, \circledast, \odot, T)$, can be thought of as an algebraic universal $\mathcal{M} = (X, \circledast, \odot, T)$.

Example 3.3: Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 7 & 10 & 1 & 9 & 4 & 2 & 3 & 5 & 8 & 6 \end{pmatrix}$ be a permutation in S_{11} . Thus, $f = (11\ 6\ 4\ 1)(10\ 8\ 3)(5\ 9)(2\ 7)$ & $W = \{\lambda_i^f\}_{i=1}^4 = \{\{1, 4, 6, 11\}, \{3, 8, 10\}, \{5, 9\}, \{2, 7\}\}$ with $T = \{2, 7\}$. Assume $\circledast, \odot: W \times W \rightarrow W$ are two binary operations, where they are defined by tables (1)-(A) and (1)-(B). Take $\lambda_1^f = \{2, 7\}, \lambda_2^f = \{1, 4, 6, 11\}, \lambda_3^f = \{3, 8, 10\}$, and $\lambda_4^f = \{5, 9\}$.

Table (1)-(A)
The Term $(W, T, \circledast)$.

$\circledast$	λ_1^f	λ_2^f	λ_3^f	λ_4^f
λ_1^f	λ_1^f	λ_1^f	λ_1^f	λ_1^f
λ_2^f	λ_2^f	λ_1^f	λ_1^f	λ_1^f
λ_3^f	λ_3^f	λ_3^f	λ_1^f	λ_1^f
λ_4^f	λ_4^f	λ_3^f	λ_3^f	λ_1^f

Table (1)-(B)
The Term $(W, T, \odot)$.

$\odot$	λ_1^f	λ_2^f	λ_3^f	λ_4^f
λ_1^f	λ_1^f	λ_1^f	λ_1^f	λ_1^f
λ_2^f	λ_2^f	λ_1^f	λ_1^f	λ_1^f
λ_3^f	λ_3^f	λ_3^f	λ_1^f	λ_1^f
λ_4^f	λ_4^f	λ_4^f	λ_2^f	λ_1^f

Then $\mathcal{M} = (W, \circledast, \odot, T)$ is a $(pPBCK - A)$.

Proposition 3.4: Let $\mathcal{M} = (X, \circledast, \odot, T)$ be a $(pPBCK - A)$. Then, for all $\lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in X$, the following hold: (i) If $\lambda_i^\beta \leq T \Rightarrow \lambda_i^\beta = T$, (ii) If $\lambda_i^\beta \leq \lambda_j^\beta$, then $\lambda_m^\beta \circledast \lambda_j^\beta \leq \lambda_m^\beta \circledast \lambda_i^\beta$ and $\lambda_m^\beta \odot \lambda_j^\beta \leq \lambda_m^\beta \odot \lambda_i^\beta$, (iii) If $\lambda_i^\beta \leq \lambda_j^\beta$ & $\lambda_j^\beta \leq \lambda_m^\beta$, then $\lambda_i^\beta \leq \lambda_m^\beta$, (iv) $(\lambda_i^\beta \circledast \lambda_j^\beta) \odot \lambda_m^\beta = (\lambda_i^\beta \odot \lambda_m^\beta) \circledast \lambda_j^\beta$, (v) $\lambda_i^\beta \circledast \lambda_j^\beta \leq \lambda_i^\beta$, $\lambda_i^\beta \odot \lambda_j^\beta \leq \lambda_i^\beta$, (vi) $\lambda_i^\beta \circledast \lambda_j^\beta \leq \lambda_m^\beta \Leftrightarrow \lambda_i^\beta \odot \lambda_m^\beta \leq \lambda_j^\beta$, (vii) $(\lambda_i^\beta \circledast \lambda_j^\beta) \circledast (\lambda_m^\beta \circledast \lambda_j^\beta) \leq \lambda_i^\beta \circledast \lambda_m^\beta$, $(\lambda_i^\beta \odot \lambda_j^\beta) \odot (\lambda_m^\beta \odot \lambda_j^\beta) \leq \lambda_i^\beta \odot \lambda_m^\beta$, (viii) If $\lambda_i^\beta \leq \lambda_j^\beta$, then $\lambda_i^\beta \circledast \lambda_m^\beta \leq \lambda_j^\beta \circledast \lambda_m^\beta$ & $\lambda_i^\beta \odot \lambda_m^\beta \leq \lambda_j^\beta \odot \lambda_m^\beta$, (ix) $\lambda_i^\beta \circledast T = \lambda_i^\beta \odot T$.

Proof: (i) If $\lambda_i^\beta \leq T$, then $T \odot \lambda_i^\beta = (\lambda_i^\beta \circledast T) \odot \lambda_i^\beta = (\lambda_i^\beta \circledast (\lambda_i^\beta \odot \lambda_i^\beta)) \odot \lambda_i^\beta = T$ by (P2), that is, $T \leq \lambda_i^\beta$. Hence $\lambda_i^\beta = T$ by (P5).

(ii) Assume $\lambda_i^\beta, \lambda_j^\beta \in X$ with $\lambda_i^\beta \leq \lambda_j^\beta$. Then $(\lambda_m^\beta * \lambda_j^\beta) \odot (\lambda_m^\beta * \lambda_i^\beta) \leq \lambda_i^\beta * \lambda_j^\beta = T$, thus $(\lambda_m^\beta * \lambda_j^\beta) \odot (\lambda_m^\beta * \lambda_i^\beta) = T$ by (P1 and (i)). Therefore $(\lambda_m^\beta * \lambda_j^\beta) \leq (\lambda_m^\beta * \lambda_i^\beta)$ by (P1). Now $(\lambda_m^\beta \odot \lambda_j^\beta) * (\lambda_m^\beta \odot \lambda_i^\beta) \leq \lambda_i^\beta \odot \lambda_j^\beta = T$, and thus $(\lambda_m^\beta \odot \lambda_j^\beta) * (\lambda_m^\beta \odot \lambda_i^\beta) = T$ by (P1 and (i)). This implies $(\lambda_m^\beta \odot \lambda_j^\beta) \leq (\lambda_m^\beta \odot \lambda_i^\beta)$ by (P1). (iii) It follows from (ii), (i) and (P6). (vi) Since $\lambda_i^\beta * (\lambda_i^\beta \odot \lambda_m^\beta) \leq \lambda_m^\beta$ by (P2), it follows from (ii), (P1) and (P2), we get $(\lambda_i^\beta \odot \lambda_m^\beta) * \lambda_j^\beta = (\lambda_i^\beta * \lambda_j^\beta) \odot \lambda_m^\beta$. (v) Since $(\lambda_i^\beta * \lambda_j^\beta) \odot \lambda_i^\beta = (\lambda_i^\beta \odot \lambda_i^\beta) * \lambda_j^\beta = T * \lambda_j^\beta = T$ by (vi), (P3) and [Definition (3.1)]. Thus $(\lambda_i^\beta * \lambda_j^\beta) \leq \lambda_i^\beta$. In the same way we find $\lambda_i^\beta \odot \lambda_j^\beta \leq \lambda_i^\beta$. (vi) If $\lambda_i^\beta * \lambda_j^\beta \leq \lambda_m^\beta$ by [Definition (3.1)] and (iv), then $T = (\lambda_i^\beta * \lambda_j^\beta) \odot \lambda_m^\beta = (\lambda_i^\beta \odot \lambda_m^\beta) * \lambda_j^\beta$, and so $\lambda_i^\beta \odot \lambda_m^\beta \leq \lambda_j^\beta$. Conversely, if $\lambda_i^\beta \odot \lambda_m^\beta \leq \lambda_j^\beta$, then $T = (\lambda_i^\beta \odot \lambda_m^\beta) * \lambda_j^\beta = (\lambda_i^\beta * \lambda_j^\beta) \odot \lambda_m^\beta$. Hence $\lambda_i^\beta * \lambda_j^\beta \leq \lambda_m^\beta$. (vii) By (P1) and (vi), then $(\lambda_i^\beta * \lambda_j^\beta) \odot (\lambda_i^\beta * \lambda_m^\beta) \leq \lambda_m^\beta * \lambda_j^\beta$. This imply $(\lambda_i^\beta * \lambda_j^\beta) * (\lambda_m^\beta * \lambda_j^\beta) \leq \lambda_i^\beta * \lambda_m^\beta$. In similarly way we have $(\lambda_i^\beta \odot \lambda_j^\beta) \odot (\lambda_m^\beta \odot \lambda_j^\beta) \leq \lambda_i^\beta \odot \lambda_m^\beta$. (viii) it follows from (vii), (P2), (P1) & (P5).

Proposition 3.5: In a $(pPBCK - A)$ $\mathcal{M}$, for all $\lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in X$ the following properties hold: (i) If $\lambda_i^\beta \leq \lambda_j^\beta$ then $T * \lambda_i^\beta = T \odot \lambda_j^\beta$, (ii) $T \odot (T * (T \odot \lambda_i^\beta)) = T \odot \lambda_i^\beta$, $T * (T \odot (T * \lambda_i^\beta)) = T * \lambda_i^\beta$, (iii) $T * (\lambda_i^\beta * \lambda_j^\beta) = (T \odot \lambda_i^\beta) \odot (T * \lambda_j^\beta)$, (iv) $T \odot (\lambda_i^\beta \odot \lambda_j^\beta) = (T * \lambda_i^\beta) * (T \odot \lambda_j^\beta)$,

Proof: (i). Let $\lambda_i^\beta \leq \lambda_j^\beta$. Then $\lambda_i^\beta \odot \lambda_j^\beta = T$ and therefore $T * \lambda_i^\beta = (\lambda_i^\beta \odot \lambda_j^\beta) * \lambda_i^\beta = (\lambda_i^\beta \odot \lambda_i^\beta) \odot \lambda_j^\beta = T \odot \lambda_j^\beta$. (ii). It is held from (P2). (iii) It is obvoise by applying (iv) and (P3) (iv) The proof is similar to the proof of (iii).

Definition 3.6: Let $\mathcal{M} = (X, *, \odot, T)$ be a $(pPBCK - A)$ and $\emptyset \neq S \subseteq X$. S is a pseudo permutation subalgebra of $\mathcal{M}$ ($pPBCK - SA$) if it satisfies $\lambda_i^\beta * \lambda_j^\beta \in S$ and $\lambda_i^\beta \odot \lambda_j^\beta \in S$, $\forall \lambda_i^\beta, \lambda_j^\beta \in S$. Also, S is said to be proper, if $S \neq X$.

Definition 3.7: Let $\mathcal{M} = (X, *, \odot, T)$ be a $(pPBCK - A)$ and $I \subseteq X$. I is called a pseudo permutation ideal of $\mathcal{M}$ ($pPBCK - I$) if it such that: (I1) $T \in I$, (I2) If $\lambda_i^\beta * \lambda_j^\beta \in I$, $\lambda_i^\beta \odot \lambda_j^\beta \in I$, and $\lambda_j^\beta \in I$, then $\lambda_i^\beta \in I$, $\forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Remarque 3.8: For any $(pPBCK - A)$ $\mathcal{M}$. The set $K(X) = \{\lambda_i^\beta \in X / T \leq \lambda_i^\beta\}$ is a $(pPBCK - SA)$ and $(pPBCK - I)$ of $\mathcal{M}$.

Proposition 3.9: Let I be a $(pPBCK - I)$ of a $(pPBCK - A)$ $\mathcal{M}$. Then for any $\lambda_i^\beta, \lambda_j^\beta \in X$, if $\lambda_j^\beta \in I$ and $\lambda_i^\beta \leq \lambda_j^\beta$, then $\lambda_i^\beta \in I$.

Proof: Assume $\lambda_j^\beta \in I$ and $\lambda_i^\beta \leq \lambda_j^\beta$, then $\lambda_i^\beta \odot \lambda_j^\beta = T$ and $\lambda_i^\beta \odot \lambda_j^\beta = T$ by [Definition 3.1], but $T \in I$ [From (I1)] and I is a $(pPBCK - I)$. Thus $\lambda_i^\beta \in I$.

Proposition 3.10: Let $\mathcal{M} = (X, \odot, \odot, T)$ be a $(pPBCK - A)$. If I is a $(pPBCK - I)$ of $\mathcal{M}$, then it is a $(pPBCK - SA)$ of $\mathcal{M}$.

Proof: The proof is held from [Proposition 3.4-(v)] and [Proposition 3.9].

Example 3.11: Let $\mathcal{M}$ be a $(pPBCK - A)$ from [Example 3.3]. Then $H_1 = \{2, 7\}$, $H_2 = \{\{2, 7\}, \{1, 4, 6, 11\}\}$ and W are $(pPBCK - SA)$ and $(pPBCK - I)$ of $\mathcal{M}$. But $G_1 = \{\{2, 7\}, \{3, 8, 11\}\}$, $G_2 = \{\{2, 7\}, \{5, 9\}\}$, $G_3 = \{\{2, 7\}, \{1, 4, 6, 11\}, \{3, 8, 10\}\}$ are all $(pPBCK - SA)$ of $\mathcal{M}$ but not $(pPBCK - I)$ because $\{1, 4, 6, 11\} \# \{3, 8, 10\} = \{2, 7\} \in G_1$, but $\{1, 4, 6, 11\} \notin G_1$. In the same way you can check G_2, G_3 . Note that the converse of [Proposition 3.110] does not hold.

4. Portion Permutation of $(pPBCK - A)$.

Definition 4.1: Let $\mathcal{M}$ be a $(pPBCK - A)$. Let $T \neq \lambda_{po}^\beta \in X$, we say λ_{po}^β is a portion permutation (PP) of $(pPBCK - A)$ $\mathcal{M}$, if $\lambda_i^\beta \leq \lambda_{po}^\beta$ then $\lambda_i^\beta = \lambda_{po}^\beta$, for every $T \neq \lambda_i^\beta \in X$ and the collection of all (PP) of $\mathcal{M}$ is the center of $\mathcal{M}$ and denoted by $C_{pp}(\mathcal{M})$. Also $C_{pp}(\mathcal{M})$ is a $(pPBCK - SA)$ of $\mathcal{M}$ and $(pPBCK - I)$.

Example 4.2: Let $\mathcal{M}$ be a $(pPBCK - A)$ from [Example 3.3]. Then $\{1, 4, 6, 11\}$ is the only (PP) of $\mathcal{M}$.

Example 4.3: Let $\mathcal{F} = \{1, 2, 3, \dots, n\}$. Let $\beta = \begin{pmatrix} 1 & 2 & 3 & \dots & \dots & \dots & n \\ 1 & 2 & 3 & \dots & \dots & \dots & n \end{pmatrix}$ be a permutation in S_n . Therefore, we have $X = \{\lambda_i^\beta\}_{i=1}^n$, where $\lambda_i^\beta = \{i\}$, $1 \leq i \leq$

n . Define $\odot$ and $\odot$ on X by: $\lambda_i^\beta \odot \lambda_j^\beta = \lambda_i^\beta \odot \lambda_j^\beta = \begin{cases} T = \{1\}, & \text{if } i \leq j \\ \lambda_i^\beta, & \text{if } j = 1 \\ \lambda_n^\beta, & \text{if } j < i \end{cases}$,

$\forall \lambda_i^\beta, \lambda_j^\beta \in X$. Then $\mathcal{M} = (X, \odot, \odot, T)$ is a $(pPBCK - A)$. Here, only λ_2^β is (PP), but for any $\lambda_k^\beta \in X - \{T, \lambda_2^\beta\}$ is not (PP). Owing for any $\lambda_g^\beta \in X$, such that $3 \leq g \leq n$, there is $T \neq \lambda_d^\beta \in X$ and $d < g$. From definition $\odot$ and $\odot$ of X , we have $\lambda_d^\beta \odot \lambda_g^\beta = T$ and $\lambda_d^\beta \odot \lambda_g^\beta = T$. But $\lambda_d^\beta \neq \lambda_g^\beta$.

Proposition 4.4: Let $\mathcal{M} = (X, \odot, \odot, T)$ be a $(pPBCK - A)$. If λ_{po}^β is a (PP) of $\mathcal{M}$, then $\lambda_{po}^\beta \odot \lambda_i^\beta = \lambda_{po}^\beta \odot \lambda_i^\beta, \forall \lambda_i^\beta \in X$.

Proof: The proof is held from [Proposition 3.4-(v)] and [Definition 4.1]. This finishes the proof. From Proposition 4.4 we conclude the following.

Corollary 4.5: Let $\mathcal{M}$ be a $(pPBCK - A)$. If every $T \neq \lambda_{po}^\beta \in X$ is a (PP) of $\mathcal{M}$, then $\mathcal{M}$ is a $(PBCK - A)$.

Proposition 4.6: Let $\mathcal{M}$ be a $(pPBCK - A)$ and let $\lambda_{po}^\beta \in X$. Then the following are equivalent: (i) λ_{po}^β is (PP) , (ii) $\lambda_i^\beta \odot (\lambda_i^\beta * \lambda_{po}^\beta) = \lambda_{po}^\beta$, (iii) $T \odot (T * \lambda_{po}^\beta) = \lambda_{po}^\beta$, (iv) $\lambda_{po}^\beta * \lambda_i^\beta = (T * \lambda_i^\beta) \odot (T * \lambda_{po}^\beta)$, (v) $\lambda_{po}^\beta * \lambda_i^\beta = T \odot (\lambda_i^\beta * \lambda_{po}^\beta)$, $\forall \lambda_i^\beta \in X$.

Proof: (i) $\Rightarrow$ (ii). It follows By (P2). (ii) $\Rightarrow$ (iii). Obvious from (ii). Thus (iii) holds. (iii) $\Rightarrow$ (iv). By (iii) and [Proposition 3.4 -(iv)] we have (iv) holds. (iv) $\Rightarrow$ (v). Applying [Proposition 3.5-(iii)] we get (v) holds. (v) $\Rightarrow$ (i). Let $\lambda_i^\beta \leq \lambda_{po}^\beta$. Then $\lambda_i^\beta * \lambda_{po}^\beta = T$ and hence $\lambda_{po}^\beta * \lambda_i^\beta = T \odot (\lambda_i^\beta * \lambda_{po}^\beta) = T$. Thus $\lambda_{po}^\beta \leq \lambda_i^\beta$. Consequently, $\lambda_i^\beta = \lambda_{po}^\beta$. Imply λ_{po}^β is (PP) .

4. Conclusion

In this work, we present and investigate new classes of pseudo BCK-algebras with their Ideals using permutations. In the future, we will examine their extensions in neutrosophic sets and provide some interesting results.

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