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Predicable R-semimodule with some properties

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Abstract

In this paper divided into two parts, related to the concept of projective semimodule we tried to clarify the general concepts of this common topic and other concepts related to it. The first part includes the definition of preradicale semimodule and the properties related to it, and we gave examples on the subject similar to what is found in the module and expanded on the definitions, examples, proofs and results obtained by solving the problems arising from this expansion in the scope of work from the concept of the projective module to the projective semimodule. The second part we put some illustrative examples on the subject of preradicale semimodule and referred to the concept of Jacobson and radical because of its importance in understanding and solving the issues related to it and we put forward several expanded definitions than they were in the subject of the module and the basic results that we needed to prove some properties. Secondly, we studied the Socle radical of semimodule and proved some properties.

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1. Introduction

In this work we have provided basic definitions in building extended concepts that were obtained from the transition from the concept of the module to its extended concept semimodule to what was present in the modules (see [1] and [2]), we also demonstrated some results that we used as a tool to solve problems associated with the expansion process. In this chapter, we will present the general concept of a projective semimodule, after having presented in the previous chapters the detailed concepts and the multiple and different results of the concept of a projective semimodule. The first part includes the definition of preradicale semimodule and the properties related to it, and we gave examples on the subject similar to what is found in the module and expanded on the definitions, examples, proofs and results obtained by solving the problems arising from this expansion in the scope of work from the concept of the projective module to the projective semimodule [3], [4], [6-8]. The second part we put some illustrative examples on the subject of preradicale semimodule and referred to the concept of Jacobson and radical because of its importance in understanding and solving the issues related to it and we put forward several expanded definitions than they were in the subject of the module and the basic results that we needed to prove some properties. Secondly, we studied the Socle radical of semimodule and proved some properties.

2. Preradical on semimodules.

Proposition 1.2: [3] Let B be an R-semimodule, then:

1- $J(B)\leq B$

2- Let B, C are subtractive R-semimodule and $\alpha{:}B\to C$ is R-homomorphism and ek-regular R-homomorphism, then $\alpha(J(B))\subseteq J(C)$.

Definition 1.3: [2] Let be A an R-semimodule is called simple if it has no-proper non-zero R-semimodule and A is said to be semisimple if it is a direct sum of its simple subsemimodules.

Examples 1.10: [3]

1) Every simple semimodule is second semimodule.
2) Let A be an R-semimodule then $\mathrm{soc}(A) \subseteq \mathrm{sec}(A)$.
3) Sec(Z)=0 where Z be Z-semimodule.
4) Sec(Q)=Q where Q be a Z-semimodule.

Definition 1.11: [5] A functor $P{:}_{R}M\to {}_{R}M$ is called preradical on M if the following two conditions are satisfied:

1- $P(A)\leq A$ for all $A\in {}_{R}M$

2- For all $\alpha \in \mathrm{Hom}(A, B)$, then $\alpha(P(A)) \leq P(B)$

In this definition (and what follows) the category , can be replaced by M (The subcategory of subtractive R-semimodules and R- homomorphism) or by

M_{ek} (the subcategory of subtractive R-semimodules and ek-regular homomorphisms).

Example 1.12: By the definition Jacobson radical is a preradical on the category ${}_RM$, and soc, sec by definition [3]. are preradical on the category.

Remark 1.13: An R-semimodule A is called a second semimodule where $A\neq 0$,and ${}_R\mathrm{ann}(A) = {}_R\mathrm{ann}(A/U)$ for all U sub set of A. U is second semimodule of A if it is subsemimodule and U is second semimodule. The second radical of is denoted by $\mathrm{Sec}(A) = \sum\{ U: U$ is second R-semimodule of $A\}$.

Example 1.14: If A simple R-semimodules, then A is second R-semimodules, and $\mathrm{Soc}(A)\subseteq \mathrm{Sec}(A)$.

Example 1.15: If $A = \mathbb{Z}$, where $\mathbb{Z}$ be a $\mathbb{Z}$-semimodule, then $\mathrm{Sec}(A) = 0$ and if $A=\mathbb{Q}$, where $\mathbb{Q}$ be a $\mathbb{Z}$-semimodule, then $\mathrm{Sec}(A) = A$.

Remark 1.16: Let A and B be subtractive R-semimodules and $\alpha \in {}_R\mathrm{Hom}(A,B)$ be k-regular, if U is second R-subsemimodule of A, then $\alpha(U)$ is second R-subsemimodule of B.

Section 2 ρ-Projective semimodule

In this section we will introduce a new generalization of the concept of project, which is a projective relative preradical, and we will study some of the properties of the projective semimodule related to this generalization. We start by the following definition:-

Definition 2.1: Let ρ be a Preradical on a left S-semimodule M. M is said to be projective relative to the Preradical ρ (ρ -projective), if for each surjective homomorphism $\alpha{:}A\to B$, where A, B are any two semimodules, and for each homomorphism β: $P\to B$, there exists a homomorphism γ: $P\to A$ such that $\pi\alpha\gamma=\pi\beta$ where $\pi{:}B\to B/\rho(B)$ is the natural map.

$$\begin{tikzcd}
 & P \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \\
A \arrow[r, "\alpha"'] & B \arrow[d, "\pi"] \\
 & B/J(B)
\end{tikzcd}$$

Example and Remark 2.**2:**

1) If P any R-semimodule projective, then P is ρ-A-projective for every semimodule A.
2) If P any k-projective semimodule, then P is ρ-A-projective for every semimodule A.
3) The preradicals functors :(a) Jacobson radical (b) soc (c) sec all this (a), (b), (c) are preradical on the category $_RM$.
4) If P any R-semimodule N-projective, then P is a spatial case of ρ-projective simply by taking ρ=J (J is the Jacobson radical functor on category $_RM$.
5) If P is an R-semimodule with $\rho(P)$=0, then P is projective if and only if P is ρ-projective.
6) If P any R-semimodule with $\rho(P) = P$, then P is ρ-projective semimodule:
   (a) If P any R-semimodule which has no maximal subsemimodule $J(P)$=P, then P is ρ-projective.
   (b) If P any R-semimodule with soc(P) = P (i.e P is semisimple semimodule), then P is ρ-projective.

Proposition 2.3: If $P_1 \cong P_2$ where P_1, P_2 be two R-semimodules and P_1 is ρ-A-projective, then P_2 is ρ-A-projective.

Proof: Let $P_1 \cong P_2$ **,** and P_1 is ρ-A-projective.

Consider the diagram and in it:

There are two R-homomorphisms θ: $P_1 \rightarrow P_2$, σ: $P_2 \rightarrow P_1$, α:$A \rightarrow B$ surjective A, B are any two R-semimodule and for each R-homomorphism β: $P_1 \rightarrow B$, there exists h: $P_1 \rightarrow A$ such that $\pi\, \alpha h$= $\pi\, \beta$……(1), where π: $B \rightarrow B/\rho\ (B)$ is the natural map. Let h_1=h σ……….(2).

Where h_1: $P_2 \rightarrow A$. Therefor $\pi\alpha h_1$= $\pi\alpha\ h\,\sigma$ by (2), then by (1) $\pi\alpha h_1$= $\pi\ \beta\ \sigma$, thus P_2 is ρ-A-projective.

Converse Let $P_1 \cong P_2$ **,** and P_2 is ρ-A-projective.

Consider the diagram and in it:

There are two R-homomorphisms θ: $P_1 \rightarrow P_2$, σ: $P_2 \rightarrow P_1$, α:$A \rightarrow B$ surjective A, B are any two R-semimodule and for each R-homomorphism β: $P_1 \rightarrow B$, there exists h_1: $P_2 \rightarrow A$ such that $\pi\ \alpha h_1$= $\pi\ \beta\ \sigma$……(1),

where π: $B \rightarrow B/\rho\ (B)$ is the natural map. Let h=$h_1\theta$……..(2).

Therefor $\pi\ \alpha\ h$= $\pi\ \alpha h_1\theta$ by (2), thus by (1) $\pi\ \alpha\ h$= $\pi\ \beta\ \sigma\ \theta$, since $P_1 \cong P_2$, then $\pi\ \alpha\ h$= $\pi\ \beta$, thus P_1 is ρ-A-projective.

Lemma 2.4: *If θ: $B \rightarrow C$ is an isomorphism for any two R-homomorphism $\pi_2\theta$=$\theta^*\pi_1$, then there exist θ^* : $B/\rho(B) \rightarrow C/\rho\ (C)$ is an isomorphism, where π_1: $B \rightarrow B/\rho\ (B)$, and π_2: $C \rightarrow C/\rho\ (C)$ are two natural maps.*

Proof: In the diagram

θ is an isomorphism, implies

$\theta(\rho\ (B))= \rho\ (C)$ Proposition 1.6.

$\theta^*(B/\rho\ (B)) = \theta^*(\pi_1\ (B))= \pi_2(\theta(B))$.

$$\begin{tikzcd}
 & & P_2 \arrow[lld, dashed, "h_1"'] \arrow[d, shift right, "\sigma"'] \\
 & & P_1 \arrow[u, shift right, "\theta"'] \arrow[ll, "h"'] \arrow[d, "\beta"] \\
A \arrow[rr, dashed, "\alpha"] & & B \arrow[d, "\pi"] \\
 & & B/\rho\ (B)
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "\theta"] \arrow[d, "\pi_1"'] & C \arrow[d, "\pi_2"] \\
B/\rho\ (\mathrm{B}) \arrow[r, "\theta^*"'] & C/\rho\ (\mathrm{C})
\end{tikzcd}$$

where $\pi_2\theta=\theta^*\pi_1$ by assumption, then $\theta^*(B/\rho\ (B))= C/\rho\ (C)$, so θ^* is onto. Now we must prove that θ^* is onto, there for let $\theta^*(b_1/\rho\ (B)) = \theta^*(b_2/\rho\ (B))$, implies $\theta^*(\ \pi_1(b_1)) = \theta^*(\pi_2(b_2)) \Rightarrow \theta(b_1)/J(C) = \theta(b_2)/\rho\ (C) \Rightarrow \theta(b_1) + s_1= \theta(b_2) + s_2$ for some $s_1, s_2\ \in \rho\ (C)$.

Since $\theta(J(B)) = \rho\ (C) \Rightarrow s_1= \theta(t_1)$, $s_2= \theta(t_2)$, for some $t_1, t_2 \in B$.

$\theta(b_1) + \theta(t_1) = \theta(b_2)+ \theta(t_2) \Rightarrow \theta(b_1+t_1) = \theta(b_2+t_2)$, since θ is one to one by assumption, then $b_1+t_1 = b_2+t_2$, implies $b_1/\rho\ (B) = b_2/\rho\ (B) \Rightarrow \theta^*$ is one to one, there for θ^* is isomorphism.

Proposition 2.5: Let P be left R-semimodule. Then the following are equivalent:

(1) P is ρ-projective.

(2) For each left R-semimodule A, $K\leq A$ and every homomorphism $\beta: P \longrightarrow A/K$, there exists θ: $P\longrightarrow A$ such that, $\pi\varphi\theta= \pi\beta$ wherφ:A $\longrightarrow A/K$ and π:$A/K\longrightarrow A/K\ /\ \rho\ (A/K)$ are natural maps.

Proof: In the diagram let $N= A/K$ and β: $P\longrightarrow N$ any homomorphism where φ:A $\longrightarrow N$ and π:$N\longrightarrow N\ /\ \rho\ (N)$ are natural maps.

Since P is ρ-projective, then there exists θ: $P\longrightarrow A$ such that, $\pi\varphi\theta= \pi\beta$.

$$\begin{tikzcd}
 & P \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \\
A \arrow[r, "\alpha"'] & B \arrow[d, "\pi"] \\
 & B/J(B)
\end{tikzcd}$$

(2)⟹(1)

: By assumption, there exist $h{:}P \to A$ such that $\pi_2 \gamma h = \pi_2 \theta f$, where $\gamma = \theta g$ by the diagram:

We take the right side $\pi_2 \gamma h = \pi_2\, \theta g h = \theta^* \pi_1 g\, h$ [by Lemma.2.4]…..(1)

We take the left side $\pi_2 \theta f = \theta^* \pi_1\, f$ [by Lemma.2.4]…..(2)

There for from (1) and (2)

$\theta^* \pi_1 g\, h = \theta^* \pi_1\, f$, since θ^* is isomorphism, then $\pi_1 g\, h = \pi_1\, f$, so P is is ρ-projective.

References

[1] Z. A. Aljebory and A. M. A. Alhossaini, "Fully dual stable semimodules," *JIKhS,* vol. 1, no. 1, pp. 92–100 (2017).

[2] S. H. A. Alsaebari and A. M. A. Alhossaini, "On preradical of semimodules," *Open Access Baghdad Sci. J.*, vol. 15, no. 4, pp. 472–478 (2018).

[3] Mustafa A. Mustafa, Saif N. Al-Khazraji, Raghad M. Hasan, and Hiba H. Abdullah, "A novel fuzzy M-transform technique for sustainable ground water level prediction," *Appl. Geomatics,* vol. 16, no. 1, pp. 9–15 (2024).

[4] L. Bican, P. Jambor, T. Kepka, and P. Nemec, "Preradicals," *Comment. Math. Univ. Carolinae,* vol. 15, no. 1, pp. 75–83 (1974).

[5] J. S. Golan, Semirings and Their Applications. Dordrecht, The Netherlands: Kluwer Academic (1999).

[6] S. M. Kadham, “Acute interstitial pneumonia image enhancement using fuzzy partial transforms,” *Appl. Geomatics*, vol. 16, no. 1, pp. 35–39 (2024).

[7] Y. Katsov, T. G. Nam, and N. X. Tuyen, “On subtractive semisimple semirings,” *Algebra Colloq.*, vol. 16, no. 3, pp. 415–426 (Sep. 2009).

[8] M. M. Abbood, H. H. Ebrahim, A. Al-Fayadh, and A. J. Obaid, “Measure defined on Γ–algebra and some of their generalizations,” *J. Interdiscip. Math.*, vol. 26, no. 5, pp. 821–827 (2023), doi: 10.47974/JIM-1502.

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