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On some features isomorphisms of (*L, W*)-modules

Ali Aldulaimi *
Ministry of Youth & Sports
General Directorate of Coordination & Follow up
Baghdad
Iraq

Ines Ellouze †
Department of Mathematics
Faculty of Science
University of Sfax
Sfax
Tunisia

Abstract

Assume two random rings, L and W. It is well-established to be the L-module architecture within the form of an algebraic structure is a vector-based generalization. Similar to the ring organization, several previous academics have dismissed the existence of L-module homomorphisms, as well as their kinds, characteristics, and basic assumption of L-module isomorphisms. Conversely, the (*L, W*)-module framework is a generalization of the L-module framework. Studies and discourse on (*L, W*)-modules remains in the early stages, nevertheless. Thus, the definition of (*L, W*)-module homomorphisms, their kinds, their features, and the basic assumption for (*L, W*)-module isomorphisms are all presented in the current article.

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1. Introduction

When stated differently, every ring in this article is arbitrarily chosen so M is an additional Abelian subgroup. A ring homomorphism is an operator that exists in ring concept that maintains the double operations of addition as well as multiplication. Assume that the rings $(W,+,\cdot)$ and $(\acute{W}, \acute{+}, \acute{\cdot})$ have singularity.

* *E-mail:* **alimilano1992@gmail.com (Corresponding Author)**
† *E-mail:* **inesellouze@yahoo.fr**

A ring homomorphism $h : W \to W_0$is defined as follows: $h(\xi + \zeta) = h(\xi) \acute{+} h(\zeta)$ and $h(\xi \cdot \zeta) = h(\xi) \acute{\cdot} h(\zeta)$, based on [1]. If each $\xi, \zeta \in W$ fulfills Additionally, [1] covered the various kinds of ring homomorphism as well as its characteristics.

Moreover, the basic assumption of ring isomorphisms invariably concludes arguments pertaining to ring homomorphisms. Additionally, a more thorough examination of ring homomorphisms has been covered in [1–9]. Later on, all ring theory theories—including those concerning ring homomorphisms—were incorporated into module concept. Suppose W be a single ring. Assuming that $\mathscr{H}$ and $\acute{\mathscr{H}}$ are left modules above W. In light of [2], Whenever $h(\lambda + \kappa) = h(\lambda) \acute{+} h(\kappa)$ and $h(\kappa m) = \kappa h(m)$ are satisfied for any $\lambda, \kappa \in \mathscr{H}$ and $\kappa \in W$, then the expression $h: \mathscr{H} \to \acute{\mathscr{H}}$ is referred to as a module homomorphism. Similar to ring homomorphisms, there are three different forms of W-module homomorphisms. The extension of ring homomorphism qualities is known as W -module homomorphism features. Furthermore, the basic concept of W-module isomorphisms closes up the investigation of W-module homomorphism. Furthermore, a number of writers have thoroughly examined the idea of W-module homomorphism, including [2-9].

Assume two random rings, R and S. The component has been modified and expanded to become the $(L, W) -$ module throughout times. In [10-12], a (L, W) −modules were proposed to represent an extension of (L, W) −bimodules. Inasmuch as there are central autonomous items within the two rings L and W, a (L, W) −module consists of a (L, W) −bimodule construction. (L, W) −submodules of M are defined in [11-12] to represent the addition subgroups N of M that means for every $\mathscr{r}$ in L, $\mathscr{w}$ in N, and $\mathscr{s}$ in W, $\mathscr{r}\mathscr{s}\mathscr{w} \in \mathcal{K}$. Moreover, additional scholars have started advancing findings on the (L, W) −module. Included these are studies found in [13–18]. Therefore, in Part 2, we establish a (L, W) −module homomorphism and provide certain kinds of (L, W) −module homomorphisms as well as several of their properties. In the third section, the types of W −module homomorphism and the fundamental idea of W −module isotropic habits are further developed to (L, W) −module. In addition, we provide the homomorphism in (L, W) −modules and several scenarios related to for them. In the last section we will present conclusions.

2. A Few Characteristics of Homomorphisms of (L, W) −Modules

We introduce the (L, W) −submodule and the (L, W) −module framework with some properties related it.

Definition 2.1: Take $\mathcal{K}$ have an abelian ring within addition and multiplication, and allow L and W be rings. If $\cdot_{-}\ \Omega_{-}$exists, subsequent $\mathcal{K}$ is a (L, W)-module. The characteristics of $M \times L \times W \to \mathcal{K}$ are as follows: for every $e, e' \in L$, $d, c \in \mathcal{K}$, and $\mathscr{v}, \mathscr{v}' \in W$,

(i) $(m + n) \cdot e \cdot \mathscr{v} = m \cdot e \cdot \mathscr{v} + n \cdot e \cdot \mathscr{v}$

(ii) $m \cdot (e + e') \cdot v = m \cdot e \cdot v + m \cdot e' \cdot v$
(iii) $m \cdot e \cdot (v + v') = m \cdot e \cdot v + m \cdot e \cdot v'$
(iv) $(m \cdot v \cdot e')ev' = m \cdot (vv') \cdot (ee')$

Typically, we shorten $m \cdot v \cdot e to mve$. $\mathcal{K}$ can also be described as a (L, W)-module according to addition and multiplication. Moreover, given an unfilled set $\mathcal{H} \subseteq \mathcal{K}$ and a (L, W)-module $\mathcal{K}$. If $\mathcal{H}$ is both an additive subgroup of $\mathcal{K}$ and a (L, W)-module according to a similar operation established on $\mathcal{K}$, then $\mathcal{H}$ is referred to as a (L, W)-submodule of $\mathcal{K}$.

The definition of (L, W) −module homomorphisms will be provided below.

Definition 2.2: Let $\mathcal{K}$ and $\mathcal{K}'$ be (L, W)-modules. A function $h: \mathcal{K} \to \mathcal{K}'$ is called an (L, W)-module homomorphism if for each $d, c \in M, e \in L$, and $v \in W$ satisfy:
(i) $h(d + c) = h(d) + h(d)$
(ii) $h(d \cdot e \cdot v) = h(d).' e .' v.$

Let's establish a few fundamental characteristics of (L, W) −module homomorphisms prior to looking at below example.

Proposition 2.3: Take $h: \mathcal{K} \to \mathcal{K}'$ be a form of homomorphism of (L, W) −modules. Next,

The null component, $h(0_{\mathcal{K}}) = 0_{\mathcal{K}'}$is preserved by h.

Proof: Given that h is a homomorphism of the (L, W) −module, $h(0_{\mathcal{K}}) + h(0_{\mathcal{K}}) = h(0_{\mathcal{K}} + 0_{\mathcal{K}}) = h(0_{\mathcal{K}}) = h(0_{\mathcal{K}}) + 0_{\mathcal{K}'}$. This suggests that, according to the removal law, $h(0_{\mathcal{K}}) = 0_{\mathcal{K}'}$ (5)

Proposition 2.4: Take $h: \mathcal{K} \to \mathcal{K}'$ be a form of homomorphism of (L, W) −modules. Next, each component in $\mathcal{K}$ has an inverse component that h nurtures, so for every d in $\mathcal{K}$, $h(-d) = -h(d)$.

Proof: Assume $d \in \mathcal{K}$. At that point, $h(d) + h(-d) = h(d + (-d)) = h(0_{\mathcal{K}}) = 0_{\mathcal{K}'}$. The formula for $h(-d) + h(d) = h(-d + d) = h(0_{\mathcal{K}}) = 0_{\mathcal{K}'}$ is similar. $h(-d) = -h(d)$. because $h(a)$includes one of a kind invers.

Proposition 2.5: Assume that $\mathcal{H}$ is a $\mathcal{K}'(L, W)$ −submodule where$h: \mathcal{K} \to \mathcal{K}'$ be a form of homomorphism of (L, W) −modules. Then, Submodule of $\mathcal{K}'$ with (L, W) identity is $h^{-1}(\mathcal{H}) = \{u \in \mathcal{K} \parallel h(u) \in \mathcal{H}\}$.

Proof: Given $\mathcal{K}'$, let H be a (L, W) −submodule. Since $0_{\mathcal{K}} \in h^{-1}(\mathcal{H})$, it is evident that $h^{-1}(\mathcal{H}) \neq \emptyset$. Suppose that $u, v \in h^{-1}(\mathcal{H})$. Next, $h(u)$ and $h(v)$ belong to $\mathcal{H}$ because$h$ is a homomorphism of (L, W) −modules, $h(u - v) =$

$h(u)-h(v)$ so $u-v \in h^{-1}(\mathcal{H})$since $h(u)$ and $h(v)$ belong to $\mathcal{H}$ and $h(u)-h(v) \in \mathcal{H}$. Moreover, allow every $e \in L$ and$n \in W$. $h(uen)= h(u)en$ because h is a homomorphism of (L,W) −modules. Given that $h(u) \in \mathcal{H}$ and that H is a (L,W) −submodule, $uen \in h^{-1}(\mathcal{H})$ since $h(u)en \in \mathcal{H}$. Thus, $h^{-1}(\mathcal{H})$ is demonstrated to be a (L,W) −submodule of $\mathcal{K}$.

Proposition 2.6: Assume that $\mathcal{H}$ is a $\mathcal{K}'(L,W)$ −submodule where$h: \mathcal{K} \to \mathcal{K}'$ be a form of homomorphism of (L,W) −modules. Then,(L,W) −submodules of $\mathcal{K}'$ are represented by $h(\mathcal{P}) = \{h(p) \mid p \in \mathcal{P}\}$ if $\mathcal{P}$ is a (L,W) −submodule of K.

Proof: Suppose $\mathcal{P}$ be a submodule of $\mathcal{K}$ that is (L,W). subsequently by this fact$h(0_{\mathcal{K}}) = 0_{\mathcal{K}'}$and $0_M \in \mathcal{P}$. Therefore, $h(\mathcal{P}) \neq \emptyset$. since $0_{\mathcal{K}'} = h(0_{\mathcal{K}}) \in h(\mathcal{P})$. Given $r = h(p_1)$ and $s = h(p_2)$, let any $r, s \in h(\mathcal{P})$. Given that h is a homomorphism of (L,W) −modules, $r - s = h(p_1) - h(p_2) = h(p_1 - p_2) \in h(\mathcal{P})$. Moreover, allow any e in L and m in W. Because h is a homomorphism of (L,W) −modules, $rem = h(p_1)em = h(p_1 em) \in h(\mathcal{P})$. As a result, $\mathcal{K}'$has a (L,W) −submodule in $h(\mathcal{P})$.

Definition 2.7: Suppose $h: \mathcal{K} \to \mathcal{K}'$ be a homomorphism of (L,W) −modules. The collection has been defined as the image for h, represented as $\mathrm{Im}(h)$ and

$$\mathrm{Im}(h) = \{a' \in \mathcal{K}' \text{and } \exists a \in \mathcal{K} \mid h(a) = a'\}$$

Definition 2.8: Suppose $h: \mathcal{K} \to \mathcal{K}'$ be a homomorphism of (L,W) −modules. Then,$\mathrm{Ker}(h)$ denoted by kernel component of h, is given as the collection

$$\mathrm{Ker}(h) = \{a \in \mathcal{K} \mid h(a) = 0_{\mathcal{K}'}\}.$$

Example 2.9: Suppose R be a module over (2ℝ,2ℝ). An operator $h: \mathbb{R} \to \mathbb{R}$ is a homomorphism of the (2ℝ,2ℝ)-module if and only if $h(r) = 6r$ for every r in Z. Given any u, v in $\mathbb{R}$ and e, d in $\mathbb{R}^2$, we obtain:

$$h(u+v) = 6(u+v) = 6u + 6v = h(u) + h(v)$$

and

$$h(uer) = 6(\ uer\ ) = 6uer = h(u)usr.$$

Thus, we have demonstrated that h is a homomorphism of a an (2ℝ,2ℝ)-module via (3) and (4). It is now evident that $\mathrm{Im}(h) = 6\mathbb{Z}$ and $\mathrm{Ker}(h) = \{0\}$.The features listed below demonstrate that each (L,W) −module homomorphism's kernels and images are (L,W) −submodules.

Proposition 2.10: Suppose $h: \mathcal{K} \to \mathcal{K}'$ be a homomorphism of (L,W) −modules. Hence, a (L,W) −submodule of $\mathcal{K}'$is images of h.

Proof: Obviously, $\mathrm{Im}(h) \subseteq \mathcal{K}'$. So $\mathrm{Im}(h) \neq \emptyset$because $h(0_{\mathcal{K}}) = 0_{\mathcal{K}'} \in \mathrm{Im}(h)$. If d, c are in $\mathcal{K}$ and u, v and $u, v \in \mathrm{Im}(h)$, therefore $u = h(d)$ and $v = h(c)$. Given that h is a homomorphism of (L,W) −modules, $u - v = h(d) -$

$h(c) = h(d - c)$. Thus $u - v \in \mathrm{Im}(h)$because$d, c \in \mathcal{K}$ implies that $d - c \in \mathcal{K}$. Moreover, allow any e in L and a in W and $ues = h(d)\, es = h(\, des)$ because h is a homomorphism of (L, W) −modules. Since $des \in \mathcal{K}$ and $u \in \mathcal{K}$, we have $ues \in \mathrm{Im}(h)$. Hence $\mathrm{Im}(h)$is (L, W) −submodule of $\mathcal{K}'$.

Corollary 2.11: *Suppose $h{:}\,\mathcal{K} \to \mathcal{K}'$ be a homomorphism of (L, W) −modules. Hence, a (L, W) −submodule of $h^{-1}(\mathcal{K})$is images of h.*

Proof: Using Proposition 2.10. we get the result.

3. Some Properties of an Isomorphism of (L, W) −Modules

The kinds of (L, W) −module homomorphisms, a few instances of (L, W) −module homomorphisms, typical (L, W) −module homomorphisms, including the basic concept of (L, W) −module isomorphisms are all covered in this part. R-module homomorphism can take form in module principle, including L-module isomorphism. The various kinds of (L, W) −module homomorphism is now shown below.

Definition 3.1: Suppose h be a homomorphism of (L, W) −modules: $h : \mathcal{K} \to \mathcal{K}'$. Whenever a function h is a bijective, then it is referred to as a (L, W) −module isomorphism. Additionally, if there is an isomorphism between the (L, W) −modules $\mathcal{K}$ and $\mathcal{K}'$ (represented by $\mathcal{K} \simeq \mathcal{K}'$), then the (L, W) −modules $\mathcal{K}$ and $\mathcal{K}'$ are said to be isomorphic.

The following argument provides a necessary and sufficient condition for an injection map with respect to its kernel.

Proposition 3.2: Suppose $h{:}\,\mathcal{K} \to \mathcal{K}'$be a homomorphism (L, W) −modules. If $\mathrm{Ker}(h) = \{0_{\mathcal{K}}\}$, subsequently f is injecting.

Proof: We have the assumption $h(u) = h(v)$ for all u, v that are in $\mathcal{K}$. Thus, $h(u - v) = h(u) - h(v) = h(u) - h(u) = 0_{\mathcal{K}'}$. that case. As a result, $u - v \in \mathrm{Ker}(h)$. We obtain $u - v = 0_M$ because $\mathrm{Ker}(h) = \{0_{\mathcal{K}}\}$. Thus, $u = v$. It has been shown that h is injecting.

Definition 3.3: Allow $\mathcal{H}$ to be a (L, W) −submodule of $\mathcal{K}$, and assume $\mathcal{K}$ be a (L, W) −module. Give $\mathfrak{T}(u) = u + N$ for all u in $\mathfrak{S}$ a function g through $\mathcal{K}$ to a quotient (L, W) −module $\mathcal{K}/\mathcal{H}$. subsequently $\mathfrak{T}$ is a homomorphism for $\mathcal{K}$ onto $\mathcal{K}/\mathcal{H}$ that is a (L, W) −module, and $Ker(\mathfrak{T}) = N$. Furthermore, the presumed (L, W) −module homomorphism about $\mathcal{K}$ onto $\mathcal{K}/\mathcal{P}$ is denoted by the (L, W) −module homomorphism $\mathfrak{T}$.

We are currently presenting a thesis that elucidates the connection connecting homomorphisms of (L, W) −modules and quotient (L, W) −modules.

Proposition 3.4: Consider that$\mathcal{H}$ be a (L, W) −submodule for $\mathcal{K}$ included in $\mathrm{Ker}(h)$, and let $h: \mathcal{K} \to \mathcal{K}'$ be a homomorphism of (L, W) −modules. Let $\mathfrak{T}$ be the homomorphism of $\mathcal{K}$ into $\mathcal{K}/\mathcal{H}$that is natural to (L, W) −modules. Then, $h = \mathfrak{S} \circ \mathfrak{T}$ exists for the special homomorphism $\mathfrak{S}$ from $\mathcal{K}/\mathcal{H}$ to $\mathcal{K}'$. Moreover, if and only if $\mathcal{H} = \mathrm{Ker}(h)$ is $\mathfrak{S}$ injecting.

Proof: To prove Proposition 3.4, We will first prove that the homomorphism $\mathfrak{S}$ exists and is unique, and then we will show that it is equivalent to $\mathfrak{S}$ having injectiveand $\mathcal{H} = \mathrm{Ker}(h)$ Existence and Uniqueness of $\mathfrak{S}$:

Consider the natural (L, W) −module homomorphism $\mathfrak{T}: \mathcal{K} \to \mathcal{K}/\mathcal{H}$defined by $\mathfrak{T}(m) = m + \mathcal{H}$ for all m $\in\mathcal{K}$. Since $\mathcal{H} \subseteq Ker(h)$, we have $h(\mathcal{H}) = 0$, which implies that h factors through the quotient $\mathcal{K}/\mathcal{H}$. That is, there exists a homomorphism $h: \mathcal{K} \to \mathcal{K}'$such that $h = \mathfrak{S} \circ \mathfrak{T}$.To show the uniqueness of $\mathfrak{S}$, suppose there exists another homomorphism $\mathfrak{S}': \mathcal{K}/\mathcal{H} \to \mathcal{K}'$such that $h = \mathfrak{S}' \circ \mathfrak{T}$. Then for any element $m + \mathcal{H} \in \mathcal{K}/\mathcal{H}$, we have $(\mathfrak{S} \circ \mathfrak{T})(m + \mathcal{H}) = \mathfrak{S}(\mathfrak{T}(m + \mathcal{H})) = \mathfrak{S}(m) = (\mathfrak{S}' \circ \mathfrak{T})(m + \mathcal{H})$. Since $h = \mathfrak{S} \circ \mathfrak{T} = \mathfrak{S}' \circ \mathfrak{T}$, it follows that $\mathfrak{S}(m) = \mathfrak{S}'(m)$ for all $m \in \mathcal{K}$. Therefore, $\mathfrak{S}$ and $\mathfrak{S}'$ are identical homomorphisms, proving the uniqueness of $\mathfrak{S}$.Now we will show the equivalence between $\mathfrak{S}$ being injective and $\mathcal{H} = Ker(h)$.If $\mathfrak{S}$ is injective: Assume that $\mathfrak{S}$ is injective. We will show that $\mathcal{H} = Ker(h)$.First, let's prove that N$\mathcal{H} \subseteq Ker(h)$. Take any $n \in \mathcal{H}$. Then by definition of $\mathfrak{T}$, we have $\mathfrak{T}(n) = n + \mathcal{H} = \mathcal{H}$. Therefore, $(h \circ \mathfrak{T})(n) = h(\mathfrak{T}(n)) = h(\mathcal{H}) = 0$, which implies that $n \in Ker(h)$. Since this holds for any $n \in \mathcal{H}$, we conclude that $\mathcal{H} \subseteq Ker(h)$.Next, we need to prove that $Ker(h) \subseteq \mathcal{H}$. Let $m \in Ker(h)$. Then $(\mathfrak{S} \circ \mathfrak{T})(m) = \mathfrak{S}(\mathfrak{T}(m)) = \mathfrak{S}(m + \mathcal{H}) = h(m) = 0$, where the last equality follows from $m \in Ker(h)$. Since $\mathfrak{S}$ is injective, it implies that $m + \mathcal{H} = \mathcal{H}$, which implies $m \in \mathcal{H}$. Hence, $Ker(h) \subseteq \mathcal{H}$.Combining both inclusions, we have $\mathcal{H} = Ker(h)$.If $\mathcal{H} = Ker(h)$, $\mathfrak{S}$ is injective:Assume that $\mathcal{H} = Ker(h)$. We will show that $\mathfrak{S}$ is injective. Let $(m + \mathcal{H}) \in Ker(\mathfrak{S})$, where $m \in \mathcal{K}$. Then $\mathfrak{S}(m + \mathcal{H}) = 0$, which implies $(h \circ \mathfrak{T})(m) = h(\mathfrak{T}(m)) = h(m + \mathcal{H}) = h(m + \mathcal{H}) = 0$. Since $\mathcal{H} = Ker(h)$, it follows that $m + \mathcal{H} = \mathcal{H}$, which implies $m \in \mathcal{H}$. Therefore, $(m + \mathcal{H}) = \mathcal{H}$, which means that $(m + \mathcal{H})$ is the zero element in $\mathcal{K}/\mathcal{H}$. Hence, $Ker(\mathfrak{S}) = \{\mathcal{H}\}$, showing that h is injective.

Theorem 3.5: *Suppose $\mathcal{K}$ and $\mathcal{K}'$ are modules of (L, W). If $h: \mathcal{K} \to \mathcal{K}'$ is a homomorphism of (L, W)-modules, therefore $\mathcal{K}/Ker(h) \simeq Im(h)$.*

Proof: To prove that $\mathcal{K}/Ker(h) \simeq Im(h)$, where $h: \mathcal{K} \to \mathcal{K}'$ is a homomorphism of (L, W) −modules, we need to show that the quotient module $\mathcal{K}/Ker(h)$ is isomorphic to the image of h.Let's proceed with the proof:

Define the map $\phi: \mathcal{K}/Ker(h) \to Im(h)$ as follows:

For any element $k + Ker(h)$ in $\mathcal{K}/Ker(h)$, where $k \in \mathcal{K}$, we define $\phi(k + Ker(h)) = h(k)$.We need to show that ϕ is well-defined, meaning that the choice of representative for the $coset\ k + Ker(h)$ does not affect the value of $\phi(k + Ker(h))$.Suppose we have another representative $k' + Ker(h)$ for the same coset. We want to show that $\phi(k + Ker(h)) = \phi(k' + Ker(h))$.Since k and k' differ by an element in $Ker(h)$, i.e., $k - k' \in Ker(h)$, we have $h(k) - h(k') = h(k - k') = 0_{\mathcal{K}'}$, because h is a module homomorphism and $Ker(h)$ is a submodule. Therefore, $h(k) = h(k')$, which implies that $\phi(k + Ker(h)) = \phi(k' + Ker(h))$. Thus, ϕ is well-defined.Next, we will show that ϕ is a module homomorphism.To prove that ϕ is additive, consider two elements $k_1 + Ker(h)$ and$k_2 + Ker(h)$ in $\mathcal{K}/Ker(h)$, where $k_1, k_2 \in K$. We have:$\phi((k_1 + Ker(h)) + (k_2 + Ker(h))) = \phi((k_1 + k_2) + Ker(h)) = h(k_1 + k_2) = h(k_1) + h(k_2) = \phi(k_1 + Ker(h)) + \phi(k_2 + Ker(h))$.

Thus, ϕ preserves addition. To prove that ϕ is compatible with scalar multiplication, let r be an element in the ring L, and let $k + Ker(h)$ be an element in $\mathcal{K}/Ker(h)$. We have:

$$\phi(r(k + Ker(h))) = \phi((rk) + Ker(h)) = h(rk) = r(h(k)) = r\phi(k + Ker(h)).$$

Therefore, ϕ is compatible with scalar multiplication. Since ϕ preserves addition and scalar multiplication, it is a module homomorphism. We will now show that ϕ is surjective. Take any element $\mathcal{v}$ inIm(h). By definition, there exists an element $\mathcal{u}$ in K such that $h(\mathcal{u}) = \mathcal{v}$. Consider the element $\mathcal{u} + Ker(h)$ in $\mathcal{K}/Ker(h)$. We have: $\phi(\mathcal{u} + Ker(h)) = h(\mathcal{u}) = \mathcal{v}$.

This shows that for any $\mathcal{v}$ in $Im(h)$, there exists an element $\mathcal{u} + Ker(h)$ in $\mathcal{K}/Ker(h)$ such that $\phi(\mathcal{u} + Ker(h)) = \mathcal{v}$. Hence, ϕ is surjective.Finally, we will show that ϕ is injective.Suppose we have two elements $k_1 + Ker(h)\, and\, k_2 + Ker(h)\, in\, \mathcal{K}/Ker(h)$ such that $\phi(k_1 + Ker(h)) = \phi(k_2 + Ker(h))$. This implies that $h(k_1) = h(k_2)$. Since h is a module homomorphism, we have $h(k_1 - k_2) = h(k_1) - h(k_2) = 0$. This means that $k_1 - k_2 \in Ker(h)$, so $k_1 + Ker(h) = k_2 + Ker(h)$. Therefore, ϕ is injective. Therefore, we conclude that ϕ is a bijective module homomorphism, which establishes the isomorphism between $\mathcal{K}/Ker(h)$ and Im(h).Hence, we have proven that $\mathcal{K}/Ker(h) \simeq Im(h)$.

The inverse of the above Theorem3.5 is incorrect. We will give an example to illustrate this case

Example 3.6: Let $\mathcal{K} = Z$ be a (L, W) −module over itself, and let $\mathcal{K}' = Z/2Z$ be a module over itself.Define the homomorphism $h: \mathcal{K} \to \mathcal{K}'$ as follows: For any integer $k \in Z, h(k)$ is the residue class of k modulo 2 in $Z/2Z$.In other words, $h(k) = [k]$. Now, let's examine the kernel and image of h: The $Ker(h)$consists of all elements in $\mathcal{K}$ that are mapped to the zero element in $\mathcal{K}'$.Since $h(k) = [k]\, in\, Z/2Z$, so $Ker(h) = \{k \in Z \mid h(k) = [k] = [0] \text{ in } Z/2Z\} = \{k \in Z \mid k \text{ is an even integer}\}$. Therefore, $Ker(h) = 2Z$. The $Im(h)$, is the set of all elements in $\mathcal{K}'$ that are reached by h.In this case$Im(h) = \{[0], [1]\}$. Now, let's check if the converse of Theorem 3.5 holds for this example:we have $\mathcal{K}/Ker(h) = Z/2Z$ (because $Ker(h) = 2Z$), and

$Im(h) = \{[0],[1]\}$.However, $Z/2Z$ and $\{[0],[1]\}$ are isomorphic as modules, but h is not a homomorphism.To see this, consider $h(3) + h(5) = [3] + [5] = [8] = [0]$ in $Z/2Z$,but $h(3 + 5) = h(8) = [8] = [0]$ in $Z/2Z$.Therefore, h does not preserve addition, and thus, it is not a (L, W) −homomorphism.

Theorem 3.7: Let $\mathcal{K}$ be an $(L, W) -$ module. If $\mathcal{H}$ and P be (L, W)-submodules of $\mathcal{K}$ with $\mathcal{P} \subseteq \mathcal{H}$, then $\mathcal{K}/\mathcal{H} \simeq (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$.

Proof: To prove that $K/H \simeq (K/\mathcal{P})/(H/\mathcal{P})$, where K is an (L, W) module and $\mathcal{H}$ and $\mathcal{P}$ are (L, W) −submodules of $\mathcal{K}$ with $\mathcal{P} \subseteq \mathcal{H}$, we need to show that the quotient module $\mathcal{K}/\mathcal{P}$ is isomorphic to the quotient module $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$.Now let's move on to the evidence:

Define the map $\phi: \mathcal{K} \to (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$ as follows:For any element k in $\mathcal{K}$, we define $\phi(k) = (k + \mathcal{P}) + (\mathcal{H}/\mathcal{P})$.We need to show that ϕ is well-defined, meaning that the choice of representative for the coset $k + \mathcal{P}$ does not affect the value of $\phi(k)$.Suppose we have another representative k' for the same coset. We want to show that $\phi(k) = \phi(k')$.Since k and k' differ by an element in $\mathcal{P}$, i.e., $k - k' \in \mathcal{P}$, we have $(k + \mathcal{P}) - (k' + \mathcal{P}) = (k - k') + \mathcal{P} \in (\mathcal{H}/\mathcal{P})$, because $\mathcal{P} \subseteq \mathcal{H}$. Therefore, $(k + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (k' + \mathcal{P}) + (\mathcal{H}/\mathcal{P})$, which implies that $\phi(k) = \phi(k')$. Thus, ϕ is well-defined. Next, we will show that ϕ is a module homomorphism. To prove that ϕ is additive, consider two elements k_1 and k_2 in K. We have:$\phi(k_1 + k_2) = (k_1 + k_2 + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (k_1 + \mathcal{P}) + (k_2 + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = \phi(k_1) + \phi(k_2)$.Thus, ϕ preserves addition.To prove that ϕ is compatible with scalar multiplication, let r be an element in the ring L, and let k be an element in K. We have:

$$\phi(rk) = (rk + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = r(k + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = r\phi(k).$$

Therefore, ϕ is compatible with scalar multiplication. Since ϕ preserves addition and scalar multiplication, it is a module homomorphism.We will now show that $Ker(\phi) = \mathcal{H}$.Let's first show that $Ker(\phi) \subseteq \mathcal{H}$: Take any element $k \in Ker(\phi)$, which means $\phi(k) = (k + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (\mathcal{P} + \mathcal{H})/\mathcal{P} = 0_{\mathcal{K}'}$ in $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$. This implies that $k + \mathcal{P} \in (\mathcal{H}/\mathcal{P})$, which means $k \in \mathcal{P} + \mathcal{H}$. Since $\mathcal{P} \subseteq \mathcal{H}$, we have $k \in \mathcal{H}$. Let's now show that $\mathcal{H} \subseteq Ker(\phi)$:Take any element $h \in H$. We want to show that $\phi(h) = (h + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (\mathcal{P} + \mathcal{H})/\mathcal{P} = 0_{\mathcal{K}'}$ in $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$.Notice that $(h + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (h + \mathcal{P}) + (\mathcal{P} + \mathcal{H})/\mathcal{P} = (h + \mathcal{P} + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (h + \mathcal{P}) + (\mathcal{H}/\mathcal{P}) = 0_{\mathcal{K}'}$, because $(\mathcal{P} + \mathcal{H})/\mathcal{P} = 0_{\mathcal{K}'}\, in\, (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$.Therefore, we have shown that $Ker(\phi) = \mathcal{H}$. By the Theorem 3.5, we have $\mathcal{K}/Ker(\phi) \simeq Im(\phi)$. Substituting $Ker(\phi)$ with $\mathcal{H}$, we have $\mathcal{K}/\mathcal{H} \simeq Im(\phi)$.We will now show that $Im(\phi) = (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$. Take any element $(k + \mathcal{P}) + (\mathcal{H}/\mathcal{P})\, in\, (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$. We want to show that there exists an element k' in $\mathcal{K}$ such that $\phi(k') = (k + \mathcal{P}) + (\mathcal{H}/\mathcal{P})$. Notice that $k' = k + \wp$, where$\wp \in \mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{H}$, we have $k' \in \mathcal{H}$. Therefore, $\phi(k') = (k' +$

$\mathcal{P}) + (\mathcal{H}/\mathcal{P}) = (k + \mathcal{P}) + (\mathcal{H}/\mathcal{P})$.This shows that for any element $(k + \mathcal{P}) + (\mathcal{H}/\mathcal{P})$ in $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$, there exists an element k' in K such that $\phi(k') = (k + \mathcal{P}) + (\mathcal{H}/\mathcal{P})$. Hence, $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P}) \subseteq Im(\phi)$.On the other hand, since ϕ is a module homomorphism, its image $Im(\phi)$ is a submodule of $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$. Therefore, we have $Im(\phi) \subseteq (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$.Combining the results from the above arguments, we conclude that $Im(\phi) = (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$. Finally, since $\mathcal{K}/\mathcal{H} \simeq Im(\phi)$ and $Im(\phi) = Im(\phi) \subseteq (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$, we have $\mathcal{K}/\mathcal{H} \simeq (\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$.

The inverse of the above Theorem3.7 is incorrect. We will give an example to illustrate this case

Example 3.8: Let $\mathcal{K} = Z$ be a module over itself, and let $\mathcal{H} = 2Z$ and $\mathcal{P} = 4Z$. In this case, $\mathcal{K}/\mathcal{H} = Z/2Z$, and $\mathcal{K}/\mathcal{P} = Z/4Z$. The quotient module $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$ is obtained by taking the quotient of $\mathcal{K}/\mathcal{P}$ by the submodule $\mathcal{H}/\mathcal{P}$. Since $\mathcal{H}/\mathcal{P} = 2Z/4Z = \{0 + 4Z, 2 + 4Z\}$, we have: $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P}) = (Z/4Z) / (\{0 + 4Z, 2 + 4Z\})$. Now, let's examine $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P})$ more closely: $(Z/4Z) / (\{0 + 4Z, 2 + 4Z\}) = \{0 + (\{0 + 4Z, 2 + 4Z\}), 1 + (\{0 + 4Z, 2 + 4Z\}), 3 + (\{0 + 4Z, 2 + 4Z\})\}$.

Simplifying further, we have: $(Z/4Z) / (\{0 + 4Z, 2 + 4Z\}) = \{0 + 4Z, 1 + 4Z, 3 + 4Z\}$. On the other hand, $\mathcal{K}/\mathcal{H} = Z/2Z = \{0 + 2Z, 1 + 2Z\}$. Comparing $(\mathcal{K}/\mathcal{P})/(\mathcal{H}/\mathcal{P}) = \{0 + 4Z, 1 + 4Z, 3 + 4Z\}$ with $\mathcal{K}/\mathcal{H} = \{0 + 2Z, 1 + 2Z\}$, we can see that they are not isomorphic as $(L, W) -$ module.

4. Conclusion

In summary, this investigation of homomorphisms between (L, W) −modules shed light on the composition and properties of (L, W) −modules. The mapping of components among (L, W) −modules whereas preserving their submodule buildings is made possible by these homomorphisms, which also regard the structure of the submodules $(L, W)$and keep the module operation. Standard module homomorphisms and (L, W) −module homomorphisms share numerous features. (L, W) −module homomorphisms come in various forms, with all having unique characteristics, including injection-like, a surjective, and subjective homomorphisms. A bijective homomorphisms known as (L, W) −module isomorphisms create a one-to-one mapping between each component of two (L, W) −modules, thereby emphasizing their fundamental equivalency or similarities. Investigators can learn more about (L, W) −modules and create relationships among various mathematical structures via studying (L, W) −module homomorphisms. There are various varieties of (L, W) −the outcomes and evaluation of homomorphisms between (L, W) −modules add to the body of knowledge in module concepts and offer an outline for studying the characteristics and uses of (L, W) −modules.

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