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On semi-continuous and quasi-semi-continuous modules

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Abstract

In our research paper, we will introduce some classes of modules named (semi-continuous modules and quasi-semi-continuous modules) (briefly, SECO-modules and Q-SECO-modules), which are proper generalizations of both continuous and quasi-continuous modules, respectively. Also, we studied numerous properties of those concepts and discussed the connections between semi-extending and semi-continuous modules and some of the ideas of well-known modules. Some known results on (quasi-)continuous

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modules are generalized to our concepts. In addition, we have discussed the hereditary property of (SECS-module, SECO-module, and Q-SECO-module).

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1. Introduction

In this study, we will assume that $\mathcal{M}$ is a unitary module over a ring $\mathcal{R}$ with unit**.** A submodule $\mathcal{V} \sqsubseteq \mathcal{M}$ is named prime, if $r\, m \in \mathcal{V}$, then either $m \in \mathcal{V}$ or $r \in [\,\mathcal{V}_{\mathcal{R}}\colon \mathcal{M}]$ for $r \in \mathcal{R}$ and $m \in \mathcal{M}$ [12]. The module $\mathcal{M}$ is named fully-prime (FP-module), if all proper sub-modules in $\mathcal{M}$ is prime [5]. An $\mathcal{R}$-module$\mathcal{M} \neq (0)$ is named semi-uniform, if $\forall\ (0) \neq \mathcal{U} \sqsubseteq \mathcal{M}$ implies $\mathcal{U}$ is semi-essential in $\mathcal{M}$ [11], where $(0) \neq \mathcal{U} \sqsubseteq \mathcal{M}$ is named semi-essential (for shortly, SE-submodule(briefly,$\mathcal{U} \sqsubseteq_{sem} \mathcal{M}$), if $\mathcal{U} \cap \mathcal{V} \neq (0)$, $\forall\ (0) \neq \mathcal{V} \sqsubseteq \mathcal{M}$ [11, 18].

A non-zero submodule $\mathcal{H}$ in $\mathcal{M}$is named essential in$\mathcal{M}$ (briefly, $\mathcal{H} \sqsubseteq_e \mathcal{M}$), if $\mathcal{H} \cap \mathcal{W} \neq 0$, $\forall\ (0) \neq \mathcal{W} \sqsubseteq \mathcal{M}$. An $\mathcal{R}$-module $\mathcal{N} \neq (0)$is named uniform, if $\forall\ (0) \neq \mathcal{H} \sqsubseteq \mathcal{N}$ then $\mathcal{H} \sqsubseteq_e \mathcal{N}$[8],[10]. The module $\mathcal{M}$ is named entirely essential (shortly FES-module), if every SE-sub module is necessary [3]. It is apparent that every essential sub-module is a SE-sub module, but the opposite does not hold permanently. It follows that each uniform is semi-uniform see [11, 19].

Now, Follows,[13] for an $\mathcal{R}$ -module $\mathcal{M}$ we have the following conditions:

(C_1): If, $\forall\, \mathcal{U} \sqsubseteq \mathcal{M}$, then $\mathcal{U} \sqsubseteq_e \mathcal{V} \sqsubseteq_{\oplus} \mathcal{M}$.

(C_2): If, $\mathcal{U} \sqsubseteq \mathcal{M}\ and\ \mathcal{U} \cong \mathcal{V} \sqsubseteq_{\oplus} \mathcal{M}$ then $\sqsubseteq_{\oplus} \mathcal{M}$.

(C_3): If, $\mathcal{U} \sqsubseteq_{\oplus} \mathcal{M}\ and\ \mathcal{X} \sqsubseteq_{\oplus} \mathcal{M}\ such\ that\ \mathcal{U} \cap \mathcal{X} = 0\,, then\ \mathcal{U} \oplus \mathcal{X} \sqsubseteq_{\oplus} \mathcal{M}$ then $\mathcal{U} \sqsubseteq_{\oplus} \mathcal{M}$. According to [13, p.18], if the module $\mathcal{M}$ has *C_2-condition*, then it has *C_3-condition.*

Now, also follows [13] the module $\mathcal{M}$ is named continuous (briefly, CO-module), if the conditions and C_2 are held in $\mathcal{M}$. while M is named quasi-continuous (briefly, QCO-module) if the conditions C1 and C_3 are held in M. But $\mathcal{M}$ if it has a C1-condition [7], it is said to be an extending module (briefly, CS-module). Thus, by all the above, we can obtain the following diagram.

Injective ⇒ Q-injective⇒ CO-module ⇒ QCO-module ⇒CS-module*.*

Well known, the opposite of the above diagram is not held see [13]. For more details of these concepts see [1], [7],[13] ,[14], [16-19].

Abbas in [2] was presented that $\mathcal{M}$ is named semi-extending if, $\forall\, \mathcal{U} \sqsubseteq \mathcal{M}\,, \exists\, \mathcal{V} \sqsubseteq_{\oplus} \mathcal{M}\ such\ that\ \mathcal{U} \sqsubseteq_{sem} \mathcal{V}$. Along with our work, we will claim that

$\mathcal{M}$ is SECS-module (or, $\mathcal{M}$ is satisfy condition SEC_1) if, $\mathcal{M}$ is a semi-extending module. CS modules are contained in SECS modules.

Now we are ready to present our main concepts.

2. Semi-Continuous and Quasi-Semi-Continuous Modules

In this part, we will take advantage of the condition provided by M. R. Abbas in [2] together with C_2 and C_3 conditions to introduce a class of new modules

Definition2.1: Let $\mathcal{M}$ be a module.

(i) $\mathcal{M}$ is named semi-continuous (shortly SECO-module), if the conditions SEC_1 and C_2 are held in $\mathcal{M}$

(ii) $\mathcal{M}$ is named quasi-semi-continuous (shortly Q-SECO-module),, if the conditions SEC_1 and C_3 are holds in $\mathcal{M}$.

Remarks and some examples 2.2:

(1) All (Q-) SECO-modules are SECS-modules.

(2) The opposite of (1) may not be generally held. For example, the Z–module Z is SECS-module (because it is semi-uniform). But it isn't an SECO module as it has no C_2-condition, clear, $2Z \cong Z$, But $2Z \not\leq_{\oplus} Z$.

(3) According to [13], if a module has (C_2), then it is satisfied (C_3)]. So, obviously, every SECO module is a Q SECO module.

(4) The opposite of (3) is not true in general. It is well known that Z–module Z is Q-SECO-module. Meanwhile, by (2), it isn't SECO-module.

(5) All semi-simple R-modules are (Q-) SECO-modules.

(6) It is clear that, every CS-module-modules and Q-CO-modules are SECS-module, SECO-modules and Q-SECO-modules respectively. While, when $\mathcal{M}$ is fully prime, then all these concepts are equivalents.

(7) Let $R = Z$, and $\mathcal{M} = Z_{P^\infty}$.Then $\mathcal{M}$ is SECS-module. Moreover, $\mathcal{M}$ is (Q)-SECO-module (since $\mathcal{M}$ is Q-injective) . We conclude from these remarks the following implications:

$$\begin{array}{ccccc} \textbf{\textit{CO-modules}} & \Rightarrow & \textbf{\textit{Q-CO-modules}} & \Rightarrow & \textbf{\textit{CS-modules}} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \textbf{\textit{SECO-modules}} & \Rightarrow & \textbf{\textit{Q- SECO-modules}} & \Rightarrow & \textbf{\textit{SECS-modules}} \end{array}$$

Proposition 2.3: If each non-zero sub-module of a module $\mathcal{M}$ is an essential submodule, then $\mathcal{M}$, is Q-SECO-module.

Proof: Assume $\forall\ (0) \neq \mathcal{U} \sqsubseteq_e \mathcal{M}\ then$ by [8] $\mathcal{M}$ is uniform then by [2 $\mathcal{M}$ hs (SEC_1). Now, if $\mathcal{M} = \mathcal{A} \oplus \mathcal{V}$ with $\mathcal{A} \cap \mathcal{V} = (0)$. Since $\mathcal{M}$ is uniform. Thus, either $\mathcal{A} = 0$ or $\mathcal{V} = 0$. Hence $\mathcal{M} = \mathcal{A} \oplus (0)$ or $\mathcal{M} = (0) \oplus \mathcal{V}$. So, in both ways $\oplus \mathcal{V} \leq_{\oplus} M$. Thus, $\mathcal{M}$ is Q-SECO-module.

Now, we need to recall the following lemmas.

Lemma 2.4: [8] *Every simple module is uniform.*

Lemma 2.5: [2, Corollary (2.2.10) *Every semi-uniform FP-module is uniform.*

Corollary 2.6: *Every simple module is Q-SECO-module.*

Proof: directly from lemma (2.4) and proposition (2.3).

Corollary 2.7: *Each semi-uniform FP-module is Q-SECO-module.*

Proof: Directly from lemma (2.5) and proposition (2.3).

For example, $\mathcal{M} = 15Z/Z$ as $Zthe$ -module is Q-SECO-module, which isn't uniform. This means that the opposite of proposition (2.3) is not generally held.

For indecomposable SECS-module $\mathcal{S}$. If $\mathcal{S}$ is an FES module, then $\mathcal{S}$ is a uniform module [2; Cor. (4.2.4)].

In the following result, we present a condition that makes the opposite of (2.3) true.

Proposition 2.8: For FES-module $\mathcal{M}$. Then $\mathcal{M}$ is indecomposable Q-SECO-module iff $\mathcal{M}$ is uniform.

Proof: **(⇒)** Let $\mathcal{M}$ is indecomposable Q-SECO-module. Then, by remarks and examples (2.2) (1), we obtain $\mathcal{M}$ is indecomposable SECS-module, but $\mathcal{M}$ is fully essential, so M is uniform by [2, Cor. (4.2.4)].

(⇐) Let $\mathcal{M}$ be a uniform R-module. Clearly, from [proposition (2.3)] $\mathcal{M}$ is quasi-semi-continuous. Also, by [8], $\mathcal{M}$ is indecomposable.

Following [Remarks (2.2) (3)] and [proposition (2.8)], we can directly obtain the proof of the next result.

Corollary 2.9: *Let $\mathcal{M}$ is FES-module. Then $\mathcal{M}$ is indecomposable Q-SECO-module iff $\mathcal{M}$ is uniform.*

But the opposite of (2.9) may not be holed in permanently, as in: Let $\mathcal{M} = Z$ as Z–module. obvious that $\mathcal{M}$ is uniform and $\mathcal{M}$ is indecomposable. But by [2, Remark (2.2) (2)]. $\mathcal{M}$ is not SECO-module.

Remark 2.10: A semi-uniform module may not be SECO-module. Where, we have that Z as Z-module is semi-uniform, but it is't SECO-module, since it has no C_2-condition.

3. SECO-modules and Q-SECO-modules and their hereditary properties.

Here we ask when the property of SECS-module, SECO-module and Q-SECO-module is hereditary. Part of the answer will be present through the following results. But first we need recall the next Lemmas:

Lemma 3.1: [15, Lemma (2.4.3)] *If* $\mathcal{B} \sqsubseteq \mathcal{N} \sqsubseteq \mathcal{M}$, *with* $\mathcal{B} \sqsubseteq_{\oplus} \mathcal{M}$ *then* $\mathcal{B} \sqsubseteq_{\oplus} \mathcal{N}$.

Lemma 3.2: [2, Prop. (3.1.9) *For a FP-module* $\mathcal{M}$, *if* $<0> \neq \mathcal{S} \sqsubseteq_{stc} \mathcal{W}$ *and* $\mathcal{W} \sqsubseteq_{stc} \mathcal{M}$, *then* $\mathcal{S} \sqsubseteq_{stc} \mathcal{M}$.

Lemma 3.3: [2, Theorem (4.2.5)] *A module* $\mathcal{M}$ *is SECS-module, iff* $\forall \mathcal{W} \sqsubseteq_{stc} \mathcal{M}$ *implies ,* $\mathcal{W} \sqsubseteq_{\oplus} \mathcal{M}$.

Now we fully prepared to present the following results.

Proposition 3.4: Every Stc-submodule of fully prime SECS-module is SECS-module.

Proof: Assume that $\mathcal{M}$ be SECS-module with $\mathcal{U} \sqsubseteq_{stc} \mathcal{M}$. Now let $\sqsubseteq_{stc} \mathcal{U}$, and since $\mathcal{M}$ is fully prime hence, by [Lemma 3.2] $\mathcal{T} \sqsubseteq_{stc} \mathcal{M}$. But we have $\mathcal{M}$ is SECS-module, so, by [Lemma 3.3], $\mathcal{T} \sqsubseteq_{\oplus} \mathcal{M}$. But, $\sqsubseteq . \mathcal{U}$, then by [15, Lemma (2.4.3)] $\mathcal{T} \sqsubseteq_{\oplus} \mathcal{U}$. Therefore, $\mathcal{U}$ is SECS-module.

According to proposition (3.4) and the fact, [if $\mathcal{M}$ is FP-module, and $\mathcal{B} \sqsubseteq_{c} \mathcal{M}$, then $\mathcal{B} \sqsubseteq_{stc} \mathcal{M}$] [2] ;directly we can prove the next corollary.

Corollary 3.5: *Every closed of fully prime SECS-module is SECS-module.*

Proposition 3.6: Every Stc-submodule of fully-prime (Q-)SECO-module is (Q-) SECO-module.

Proof: Let $\mathcal{M}$ be (Q-)SECO-module and $\mathcal{V} \sqsubseteq_{stc} \mathcal{M}$, since $\mathcal{M}$ is SECS-module, then by [proposition (3.4)], $\mathcal{V}$ is SECS-module and hence $\mathcal{V}$ has (SEC_1)-condition). But, by [Lemma 3.3],$\mathcal{V} \sqsubseteq_{\oplus} \mathcal{M}$. But, $\mathcal{M}$ is fully prime (Q)-SECO-module then by [Remark 2.2(6)] $\mathcal{M}$ is (Q)-CO-module and hence $\mathcal{V}$ is (Q)-CO-module this means by [13], $\mathcal{V}$ has (C_3) and (C_2) conditions. Therefore, $\mathcal{V}$ is (Q)-SECO-module.

Now we can explain that the class of (Q)-SECO-modules is also closed under direct summands.

Corollary 3.7: *Every direct summand of fully prime (Q)-SECO-module is (Q)-SECO-module.*

Proof: Immediately from proposition (3.6).

In the following, another answer of the above inherited question.

Proposition 3.8: Take $\mathcal{M}$ is fully essential SECS-module and $\mathcal{T} \sqsubseteq \mathcal{M}$. If, $\cap \mathcal{V} \sqsubseteq_{\oplus} \mathcal{T}$, for each $\mathcal{V} \sqsubseteq_{\oplus} \mathcal{M}$ then $\mathcal{T}$ is SECS-module.

Proof: Suppose that, $\mathcal{M}$ is SECS-module and $\mathcal{A} \sqsubseteq \mathcal{T} \sqsubseteq \mathcal{M}$, then $\exists \mathcal{F} \sqsubseteq_{\oplus} \mathcal{M}$ such that $\mathcal{A} \sqsubseteq_{sem} \mathcal{F}$. Since, $\mathcal{M}$ is fully essential then, $\mathcal{A} \sqsubseteq_{e} \mathcal{F}$. Now, since

$\sqsubseteq \mathcal{F} \cap \mathcal{T} \sqsubseteq \mathcal{F}$, so by [4, P.57], we have $\mathcal{A} \sqsubseteq_e \mathcal{F} \cap \mathcal{T}$ and by hypothesis, $\mathcal{F} \cap \mathcal{T} \sqsubseteq_{\oplus} \mathcal{T}$. Therefore, $\mathcal{T}$ is SECS-module.

Now we recall that, let $\mathcal{F}$ and $\mathcal{X}$ are $\mathcal{R}$-modules. Then $\mathcal{X}$ is named (pseudo-)$\mathcal{F}$-injective modules, (shortly, if $\forall\, \mathcal{H} \sqsubseteq \mathcal{F}$, with $\mu \in [Mon(\mathcal{H}, \mathcal{X})]$ $Hom(\mathcal{H}, \mathcal{X})$can be extended to $g \in Hom(\mathcal{F}, \mathcal{X})$ [13],[6].. Also, $\mathcal{F}$ is quasi-injective, if $\mathcal{F}$ is $\mathcal{F}$-injective. for more see [9] and [1].

Remark 3.9: [2, Remark (4.2.2)(9) All quasi-injective is SECS-module.

Lemma 3.10: [6, Theorem 2.6], *Every pseudo-injective module is satisfying (C_2).*

In the next theorem, we present the conditions which are make the opposite of our implications is holed.

Theorem 3.11: *For a pseudo-injective FP-module $\mathcal{N}$. Then the following concepts are equivalents:*

i - $\mathcal{N}$ is SECO-module.

ii- $\mathcal{N}$ is Q-SECO-module.

iii- $\mathcal{N}$ is SECS-module.

iv- $\mathcal{N}$ is CO-module.

v- $\mathcal{N}$ is Q-CO-module.

vi- $\mathcal{N}$ is CS-module.

Proof: Obvious.

According [9] was proved that every quasi-injective is pseudo-injective. Then next corollary is directly from theorem (3.11).

Corollary 3.12: For each quasi-injective FP-module $\mathcal{N}$. Then the next concepts are equivalents:

1. $\mathcal{N}$ is SECO-module.
2. $\mathcal{N}$ is Q-SECO-module.
3. $\mathcal{N}$ is SECS-module.
4. $\mathcal{N}$ is CO-module.
5. $\mathcal{N}$ is Q-CO-module.
6. $\mathcal{N}$ is CS-module.

5. Discussion and Conclusion

In this study we conclude that the class of CS-module stronger than the class of SECS-module. Also, the class of (Q-)CO-modules are actually contained in (Q-)SECO-modules. Also, we conclude that, these concepts are equivalent under certain conditions.

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