

On f -lifting modules

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Abstract

This article introduces the concept for f -lifting modules definition (the R -modules W named f -lifting for T any submodule in W , there is direct summand A for W s.t. $A \leq_{foc} T$ in W). Some of the characteristics and examples were given, as well as the relationship between it and the following concepts: lifting module, δ -lifting module, f -hollow module and f -supplemented module.

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1. Introduction

Suppose W a unit right R module where R the identity ring. We will review some concepts and mention their source. Zhou and Zhangin [1] introduces small essential submodule: A is a non-zero small essential submodule of W (notation $A \leq_{se} W$), if $A \cap N \neq 0$ for every $(0 \neq) N \ll W$. let $N \ll W$ be a small proper submodule of W , if $N+H \neq W \ \forall \ H \subset W$. In 2016 the small-singular submodules concept was introduced by [2] submitted small-singular modules a generalization of singular module definition by $Z^s(W) = \{a \in W \mid \text{Ann}(a) \leq_{se} R\}$. If $Z^s(W) = W$, then W s-singular module. For $N \leq M \leq W$, then N is called a coessential submodule of M in W (symbolize $N \leq_{co} M$ in W) if $M/N \ll W/N$ [4]. Moreover in [3] it has submitted f -hollow modules if $\forall \ P \subset W$ is f -small. And let $P \ll_f W$ be a f -small proper submodule of W if, $W = P + N$ and W/N s-singular[4]. In [8, 9] it has submitted f -coessential

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submodule, if W is R -module with $P \leq N \leq W$, then P named a f -coessential submodule of N in W (symbolize $P \leq_{fcoe} N$ in W) if $N/P \ll_f W/P$. Following Zhou [5], let $X \leq W$ is called a δ -small submodules (notation $X \ll_\delta W$) if whenever $W = X + S$ with W/S singular, we have $W = S$. And if W a R -module & $P \leq N \leq W$, so P called δ -coessential submodule of N in W (symbolize $P \leq_{\delta ce} N$ in W) if $N/P \ll_\delta W/P$. And module W named δ -lifting where $\forall K \leq W$, $\exists$ a decomposition $W = A \oplus B$ s.t $A \leq K$ and $A \setminus K$ is δ -small in $A \setminus W$ [6]. After the series of research that has been presented from the following concepts: s -singular submodule, f -small submodule, f -Hollow module, f -supplemented module, f -amply supplemented module and f -coessential submodule. We found it necessary to introduce a new concept called f -lifting modules.

2. Main results

Now, we will introduce the concept called f -lifting Modules, which is a generalization to the concept of lifting Modules, with the sum of basic properties.

Definition 2.1: An R -module W is called f -lifting if $\forall T \leq W$, $\exists A \leq_\oplus W$ s.t. $A \leq_{fcoe} T$ in W .

Examples 2.2:

1. Consider $\mathbb{Z}_4, \mathbb{Z}_6$ and $\mathbb{Z}_8$ as $\mathbb{Z}$ -module, clearly these modules are f -lifting modules.
2. Put $\mathbb{Z}_6$ as $\mathbb{Z}_6$ -module. It clearly to see that $\mathbb{Z}_6$ as $\mathbb{Z}_6$ -module not f -lifting Modules. Since every submodules of this module are not f -coessential submodules.
3. Clearly, every lifting module is f -lifting module.
4. Kosan introduce δ -lifting Modules[6], clearly every f -lifting module is δ -lifting Modules. Note that the next example shows the opposite: $\mathbb{Z}_6$ as $\mathbb{Z}_6$ -module.

Theorem 2.3: Suppose G a R -module:

1. If G s -uniform R -module, so every f -lifting module is lifting modules.
2. If G is s -singular R -modules.so every f -lifting module a lifting modules.
3. If G is torsion modules on integral domain R , so every f -lifting module is lifting module.
4. If G prime R -module. So every f -lifting module is lifting module.

Proof: Clear by [4]

Theorem 2.4: An expressions below equivalent for a R -module W :

1. W a f -lifting
2. $\forall S \leq W$, $\exists W = A \oplus B$ such that $A \leq S$ & $S \cap B \ll_f B$.
3. $\forall S \leq W$ can be written as $S = N \oplus M$, where $N \leq_\oplus W$ and $M \ll_f W$.

Proof: (1 $\rightarrow$ 2) Let $S \leq W$, then $\exists A$ decomposition $W = A \oplus B$ s.t. $A \subseteq S$ & $S/A \ll_f W/A$. So $S = S \cap W = S \cap (A \oplus B) = (S \cap B) \oplus A$ by modular law. And now let the map $\alpha: \frac{W}{A} \rightarrow B \ni \alpha(w + A) = b$ s.t. $w = a + b$ where $a \in A$ and $b \in B$. It is clear that α is an isomorphism. And since $S/A \ll_f W/A$ then by [6] we get $\alpha(S/A) \ll_f B$.

$\alpha(S/A) = \{\alpha(s + A) | s \in S\} = \{\alpha((a' + b') + A) | a' \in A \text{ and } b' \in (S \cap B)\} = \{b' | b' \in (S \cap B)\} = S \cap B$, therefore $S \cap B \ll_f B$. Let $M = S \cap B$, then $S = A \oplus M$ Where $A \leq_{\oplus} W$ and $M \ll_f W$. (2 $\rightarrow$ 3) Let $S \leq W$. then there is a decomposition $W = A \oplus B$ such that $A \subseteq S$ and $S \cap B \ll_f B$ so by Pro. (2.2) $S \cap B \ll_f W$. And $S = S \cap W = S \cap (A \oplus B) = A \oplus (S \cap B)$ by modular law. Let $M = S \cap B$, then $S = A \oplus M$ when A a direct summand for W & $M \ll_f W$. (3 $\rightarrow$ 1) Let $S \leq W$. By (3) $S = K \oplus M$ where $A \leq_{\oplus} W$ and $M \ll_f W$. To prove $K \leq_{fco} S$ in W , let $W/K = S/K + X/K$ where $K \subseteq X$ with $\frac{W/K}{X/K} = \frac{W}{X}$ s-singular module, then $W = S + X$ and hence $W = K + M + X = M + X$ and since $\frac{W}{X}$ s-singular (by third isomorphism theorem). But $M \ll_f W$, therefore $W = X$. Thus W is f -lifting module.

With the help of the above theorem, we will prove the property of direct summand

Proposition 2.5: If a module $W = A \oplus B$ is f -lifting, then A is f -lifting.

Proof: Let $H \leq A \subseteq W$ so $H \leq W$ and W is f -lifting modules then by above theorem, can be written as $H = N \oplus M$, where N is a direct summand of W and $M \ll_f W$. And since $N \leq_{\oplus} W$ contained A , then $N \leq_{\oplus} A$ and $M \ll_f W$, $M \subseteq A$ and $A \leq_{\oplus} W$, then $A \leq_{\oplus} A$. Thus A is f -lifting modules.

Proposition 2.6: Every f -hollow module is f -lifting. (Clear Proof)

Proposition 2.7: Every f -lifting module and indecomposable is f -hollow. (Clear Proof)

Proposition 2.8: if W a R -module. so expressions become equivalent:

1. W a f -lifting & indecomposable modules.
2. W a f -hollow & indecomposable modules.
3. W a hollow module.

Proof: (1) $\Rightarrow$ (2) assume $N < W$ now W a f -lifting so $W = K \oplus K'$ where $K \leq N$ & $N \cap K' \leq W$. As W indecomposable, so $K' = 0$ or $K = 0$. where $K' = 0$, we have $K = W$, which implies $W \leq N$ contradiction. So $K = 0$, hence $K' = W$ and $N \cap K' = N \cap W = N \ll_f W$. Thus W is f -hollow. (2) $\Rightarrow$ (3) It is clear by proposition (2.2(1)). (3) $\Rightarrow$ (1) when W hollow, so M indecomposable & from [3] W is f -hollow and by Pro.(3.4) M is f -lifting.

Note 2.9: Generally, direct sums is not hold of f -lifting modules: Consider the $\mathbb{Z}$ -modules $\mathbb{Z}_8$ and $\mathbb{Z}_2$ are f -lifting (by [3] and Pro.2.8 we obtain, Hollow modules $\Rightarrow f$ -Hollow modules $\Rightarrow f$ -lifting). But $G=\mathbb{Z}_8\oplus\mathbb{Z}_2$ is not f -lifting, since the submodule $H=\langle 2,1 \rangle$ does not contain a nonzero direct summand of G and H is not f -small since $H+\langle 1,0 \rangle=G$ and $G/\langle 1,0 \rangle\cong\langle 4,0 \rangle$ is s-singular by [2, Rem.3.3(2)] and [4, Pro.3.6] but $\langle 1,0 \rangle\neq G$.

Proposition 2.10: An R -module G is f -lifting if and only if $\forall H \leq W$ has a f -supplement U in G s.t. $H \cap U \leq_{\oplus} H$.

Proof: Let a module G is f -lifting. Suppose that $H \leq G$, then $\exists$ a decomposition $G=M\oplus A$ s.t. $M\leq_{fcoe} A$ in G . By modular law $H=H\cap W=H\cap(M\oplus A)=M\oplus(H\cap A)$. It is clear that $G=H+A$. Now to prove that $H\cap A\ll_f A$, then let $A=(H\cap A)+U$, where $\frac{N}{U}$ is s-singular. Therefore $G=M\oplus A=M+(H\cap A)+U$. thus $G=H+U$. Then $\frac{G}{M}=\frac{H}{M}+\frac{(U+M)}{M}$. So by isomorphism theorems we get $\frac{G/M}{(U+M)/M}\cong\frac{G}{(U+M)}=\frac{M\oplus A}{(U+M)}\cong\frac{A}{U'}$, since $\frac{A}{U}$ is s-singular, then $\frac{G/M}{(U+M)/M}$ is s-singular. But $M\leq_{fcoe} A$ in G , so $G=M\oplus A$ and hence $A=U$. So H has a f -supplement U in G s.t. $H \cap U \leq_{\oplus} H$. ($\Leftarrow$) Let $H \leq G$. Then by assumption H has a f -supplement U in G s.t. $H \cap U \leq_{\oplus} H$. So, $G=H+U$, $H \cap U \ll_f U$ and $H=(H \cap U)+M$. Since $\frac{G/P}{M/P}$ is s-singular, and then $\frac{G}{M}$ is s-singular, by 3th isomorphism theorem. But $H \cap U \ll_f G$, then $G=M$. So $P \leq_{fcoe} H$ in G . Thus G is f -lifting module.

Proposition 2.11: if W a f -lifting module & H fully invariant submodule in W , then $\frac{W}{H}$ is a f -lifting module.

Proof: Assume $\frac{P}{H} \leq \frac{W}{H}$. Since W is f -lifting module, then $\exists K \leq_{\oplus} W$ s.t. $K \leq_{fcoe} P$ in W . Thus $W=K\oplus X$, for some $X \subseteq W$. And since H is a fully invariant submodule of W , then $\frac{W}{H}=\frac{K\oplus X}{H}=\frac{K+H}{H}\oplus\frac{X+H}{H}$, by [7, Lemma(5.4)]. Now let $\alpha:\frac{W}{K}\rightarrow\frac{W}{K+H}$ be a map defined by $\alpha(w+K)=w+(K+H) \forall w \in W$. It is clear that α is an epimorphism. $\frac{P}{K} \leq \frac{W}{K}$, then $\alpha(\frac{P}{K})=\frac{P}{K+H} \ll_f \frac{W}{K+H}$, by [4]. and hence $K+H \leq_{fcoe} P$ in W . Therefore $\frac{K+H}{H} \leq_{fcoe} \frac{P}{H}$ in $\frac{W}{H}$ by [8] Thus $\frac{W}{H}$ is a f -lifting module.

Lemma 2.12: Let $W=H+K$ be a f -lifting module. If $\frac{W}{H}$ is s-singular, then there exists a direct summand P of W such that $W=H+P$ and $P \leq_{fcoe} K$ in W .

Proof: Let H and K be submodules of W such that $W=H+K$ and $\frac{W}{H}$ is s-singular. now W f -lifting module, so $\exists$ a direct summand P in M s.t. $P \leq_{fcoe} K$ in W .

Then $\frac{W}{P} = \frac{H+P}{P} + \frac{K}{P}$. Since $\frac{W}{H}$ is small-singular, then $\frac{W}{H+P}$ is small-singular by [2.Pro.(1.2)] and hence $\frac{W/P}{H+P/P}$ is small-singular by 3th isomorphism theorem. But $\frac{K}{P} \ll_f \frac{W}{P}$, therefore $\frac{W}{P} = \frac{H+P}{P}$. Then $W=H+P$.

Note 2.13: Every f -lifting module is f -supplemented module. Let us give the following example to show the opposite: consider $W=\mathbb{Z}_8 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$ -module is not f -lifting module (see note 2.9) but W is supplemented and hence f – supplemented modules

Proposition 2.14: If $W=H+U$ a f -lifting module & $\frac{W}{H}$ s -singular, then W is f –amply supplemented.

Proof: Assume that $W=H+U$ be f -lifting module with $\frac{W}{H}$ s -singular. then by above lemma, we get $U \leq_{\oplus} W$. And by Theorem (3.2) $H \cap U = X \oplus Y$ where $X \leq_{\oplus} W$ and $Y \ll_f W$. Since $U \leq_{\oplus} W$ then $X \leq_{\oplus} U$ and $Y \ll_f U$, by [6] Now let $U=X \oplus N$. So, $W=H+U=H+X+N=H+N$.

Claim that $H \cap N \ll_f N$. Now let a projection map $\alpha: X \oplus N \rightarrow N$. So by modular law we get $H \cap U = (H \cap U) \cap U = (H \cap U) \cap (X \oplus N) = X \oplus [(H \cap U) \cap N] = X \oplus (H \cap N)$. Since $Y \ll_f K$, then by [4] we get $\alpha(Y) \ll_f N$ and $\alpha(Y) = \alpha(X \oplus Y) = \alpha(H \cap U)$ hence $H \cap N = \alpha(X \oplus (H \cap U)) = \alpha(H \cap U) \ll_f N$. Thus N is a f – supplement of H in W . Thus W is f –amply supplemented module.

Proposition 2.15: If $W=H_1 \oplus H_2$ a R -module s.t. $\text{Ann}(H_1) + \text{Ann}(H_2) = R$. $K \leq W$ therefore $K=A_1 \oplus A_2$, where $A_1 \subseteq H_1$ and $A_2 \subseteq H_2$. Since H_1 and H_2 are f -lifting modules, $A_i=B_i \oplus C_i$ ($i=1,2$) where $B_i \leq_{\oplus} H_i$ and $C_i \ll_f H_i$.

Therefore $K=A_1 \oplus A_2=(B_1 \oplus C_1) \oplus (B_2 \oplus C_2) = (B_1 \oplus B_2) \oplus (C_1 \oplus C_2)$.

It is easy show $B_1 \oplus B_2 \leq_{\oplus} W$ and $C_1 \oplus C_2 \ll_f W$ by [6].

Proposition 2.16: if $W=H_1 \oplus H_2$ R -module such that every submodules for W is fully invariant. If H_1 and H_2 are f -lifting then W is f -lifting.

Proof: Assume $K \leq W$ then by hypothesis, K is fully invariant and hence $K=(K \cap H_1) \oplus (K \cap H_2)$ then $K \cap H_1 \leq H_1$ and $K \cap H_2 \leq H_2$ and since H_1 & H_2 are f -lifting then there exists a decompositions $K \cap H_1 = A_1 \oplus A'_1$ and $K \cap H_2 = A_2 \oplus A'_2$ where $A_1 \leq_{\oplus} H_1$ & $A_2 \leq_{\oplus} H_2$ and $A'_1 \ll_f H_1$ & $A'_2 \ll_f H_2$ so $A_1 \oplus A_2 \leq_{\oplus} H_1 \oplus H_2 = W$ and $A'_1 \oplus A'_2 \ll_f H_1 \oplus H_2 = W$ thus W is f -lifting.

Proposition 2.17: Let $W=H+K$ be a duo-module s.t. H and K are f -lifting modules. so W a f -lifting module.

Proof: Assume that $H+K=W$ duo-module & let $S \leq W$, so S a fully invariant, hence $S=(S \cap H) \cap (S \cap K)$. And since H and K are f -lifting modules, $\exists$ a decomposition $S \cap H = H_1 \oplus K_1$ and $S \cap K = H_2 \oplus K_2$ where $H_1 \leq_{\oplus} H$ and $H_2 \leq_{\oplus} K$ and $K_1 \ll_f H$ and $K_2 \ll_f K$. Then $H_1 \oplus H_2 \leq_{\oplus} W$ and by [3] $K_1 \oplus K_2 \ll_f W$. Therefore, W is f -lifting module.

3. Discussion and Conclusion

In this research, a new concept was introduced and a set of examples was given showing the realization of this concept and the study of the relationship between it and a set of other concepts as well as the study of the equivalent relationships with a set of characteristics

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