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On compactly packed acts over monoid

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Abstract

We claim that a proper subact $\dot{N}$ have been compactly packed (c.P) in generalization idea of c.P modules to S-Acts. whether for all family of prime subact $\{P_\alpha\}_{(\alpha\in\lambda)}$ for some $\beta\in\lambda$ $P_\beta \supseteq \dot{N}$ when $\cup_{(\alpha\in\lambda)}P_\alpha, \supseteq N$. We refer to an S-Act $\dot{M}$ as c.P. if every subact is compactly packed. We study various properties of c.P S-Acts.

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1. Introduction

Put $\dot{M}$ is a unitary module defined on commutative ring with 1. I is said to be $c.P$ ideal if to every family $\{P_\alpha\}_{\alpha\in\lambda}$ of prime ideals with $I \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, there is $\beta \in \lambda$, such that $I \subseteq P_\beta$. In [1] Reis and Viswanathan called rings is $c.P$ in which every ideal is $c.P$. The generalization of S-Acts .have been introduced. It is well known that a subact $\dot{N}$ which is proper is designate prime, whether $rx \in \dot{N}$, $r \in R$ also $x \in \dot{M}$ behold either $r \in [\dot{N}:\dot{M}]$ or $x \in \dot{N}$, [2]. Thus we say that a subact which is proper $\dot{N}$ is $.P$, whether for all family $\{P_\alpha\}_{\alpha\in\lambda}$ of prime subact with $\dot{N} \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, $\dot{N} \subseteq P_\beta$ for some $\beta \in \lambda$ [3].

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2. Compactly Packed S-Acts.

Definition 1.1: $\dot{N}$ is a proper (appropriate) subact which is c.P, whether N have been includ in the union of a family of prime subact , thereafter $\dot{N}$ is contained in one of the members of the family. And $\dot{M}$ is $c.P$ S-Act whether all proper subact is c.P.

Put $\dot{N}$ is a subact, whether found prime subact that include $\dot{N}$, therefore the intersection of all prime subact including $\dot{N}$ have been said the $\dot{M}$-radical alsostand for rad$\dot{N}$. Whether not exist prime subact including $\dot{N}$, thus rad $\dot{N}$ = $\dot{M}$. A subact $\dot{N}$ have been said a radical subact whether rad $\dot{N}$ = $\dot{N}$.

Theorem 1.2: *Put $\dot{M}$ is an S-Act. The next statements are valent:*

i. *$\dot{M}$ is c.P.*

ii. *For all proper(appropriate) subact $\dot{N}$, there is $a\in\dot{N} \ni rad(\dot{N}) = rad(Sa)$.*

iii. *All proper(appropriate) subact $\dot{N}$, whether a family of subact $\{\dot{N}_\alpha\}_{(\alpha\in\lambda)}$ also $\bigcup_{(\alpha\in\lambda)} \dot{N}_\alpha \supseteq \dot{N}$, thus rad $(\dot{N}_\beta) \supseteq \dot{N}$ for some $\beta\in\lambda$,.*

iv. *For all proper(appropriate) subact $\dot{N}$, whether $\{\dot{N}_\alpha\}_{(\alpha\in\lambda)}$ is a family of radical subact and $\bigcup_{(\alpha\in\lambda)}\dot{N}_\alpha \supseteq \dot{N}$ thus for some $\beta\in\lambda$, $\dot{N}_\beta \supseteq \dot{N}$.*

Proof: (i→ii) Put $\dot{N}$ is a proper(appropriate) subact. Presume rad$\dot{N}\not\subset$rad(Sa) for all $a\in\dot{N}$, dwell a prime subact P_a enclud Sa also $\dot{N}\not\subset P_a$. However $\bigcup_{(a\in\dot{N})} P_a \supseteq \bigcup_{(a\in N)} S_a = \dot{N}$, This means $\dot{M}$ which is not compactly packed which contradicts (i).

(ii→i) Put $\dot{N}$ is a proper(appropriate) subact also put a family of subact $\{\dot{N}_\alpha\}_{(\alpha\in\lambda)} \ni \bigcup_{(\alpha\in\lambda)} \dot{N}_\alpha \supseteq \dot{N}$. By (ii) $\exists\, a\in\dot{N} \ni$rad$\dot{N}$ = rad(Sa). Therefore $a \in \bigcup_{(\alpha\in\lambda)} \dot{N}_\alpha$ also thus for some $\beta\in\lambda$. $a \in \dot{N}_\beta$, so $\dot{N}_\beta \supseteq Sa$, also rad$(\dot{N}_\beta) \supseteq$ rad (Sa) = rad $(\dot{N}) \supseteq \dot{N}$

(iii→iv) & (iv→i) are evident.

An S-Act M have been said a multiplication, whether all subact N has the form IM = N for an ideal I of S. In fact N = [N:M]M. [4].Recall that if $\dot{M}$ is a multiplication S-Act and $\dot{N}$ is a subact with $\bigcup_{(\alpha\in\lambda)} \dot{N}_\alpha, \supseteq \dot{N}$ wherever is a prime subact and λ is a finite set, thus $\dot{N}_\beta \supseteq \dot{N}$ for some $\beta\in\lambda$ [5]. Whether $\dot{M}$ is a multiplication containing finite number of prime subact then $\dot{M}$ is compactly packed.

The example that follows provides an S-Act that isn't $c.P$.

Example 1.3: Put Ĕ is an infinite set. Let S be commutative Boolean monoid (P(Ĕ),Δ,∩), wherever P(Ĕ) is power set , also Δ is the usual operation. Let Ù = {Á: Á is finite set of Ĕ}. Since S is commutative Boolean monoid, then for each Á∈ Ù, ⟨Á⟩ is radical ideal [6], then ⟨Á⟩=∩{ƥ: ƥ is prime ideal containing Á}. Because Ù is not principal ideal then Ù⊄⟨Á⟩, that is, there exists a prime $ƥ_{Á}$ containing Á and Ù⊄$ƥ_{Á}$, but Ù $=\bigcup_{(Á\in Ù)}$ ⟨Á⟩$\subseteq\bigcup_{(Á\in Ù)}$ $ƥ_{Á}$, then Ù is not compactly packed and hence S is not compactly packed S-Act.

Put $\dot{M}$ is an S-Act. Have been said a subact $\dot{N}$ pure whether for all ideal Á , ÁN,= AM$\cap$N [7].

Proposition 1.4: Put $\dot{M}$ is an S-Act and every subact is pure , then $\dot{M}$ is c.P iff each proper subact $\dot{N}$ of $\dot{M}$ is cyclic.

Proof: The sufficiency is clear. Pet N is a proper (appropriate) subact. Because M is compactly packed thus by theorem 1.2, $\exists a \in$ N, $\ni$ rad N = rad Ra . But every subact is pure, then by [8], N=Ra.

Theorem 1.5: *M holds the ACC on radical subact. If M is c.P S-Act at least it has one maximal subact.*

Proof: Put $\cdots \supseteq \dot{N}_2 \supseteq \dot{N}_1$ is an ascending chain of radical subact and put $\bigcup_i \dot{N}_i$ = L. Whether $\dot{M}$ = L also a maximal subact Á , thus $\bigcup_i \dot{N}_i \supseteq$ Á. Because $\dot{M}$ is c.P therefore $\dot{N}_j \supseteq$ Á for some j. Therefore Á $\subseteq \dot{N}_j$ and thus $\bigcup_i \dot{N}_i \subseteq \dot{N}_j$, that is $\dot{N}_j \supseteq \dot{M}$this is out of the question. Therefore L is a proper(appropriate) subact. Thus $\dot{N}_j \supseteq$Lfor some j and therefore $\cdots = \dot{N}j+_2 = \dot{N}j+_1.= N_j \supseteq \cdots \dot{N}_2 \supseteq \dot{N}_1$, thus the ACC is hold for radical subact.

Because each finitely generated(f.g) S-Act and all multiplication S-Act owns a proper (appropriate) maximal subact, [8, 9] .so van:-

Corollary 1.6: *M holds the ACC on radical subact, if $\dot{M}$ is f.g or multiplication c.P S-Act.*

A prime subact ϸ have been said a minimal prime subact of a subact $\dot{N}$ if ϸ $\supseteq \dot{N}$ also there exist no smaller prime subact with this property. Remember that if $\dot{M}$ is an S-Act that holds ACC on radical subact therefore the radical of any proper (appropriate) subact $\dot{N}$ is the (cross) intersection of a finite number of minimal prime subact of $\dot{N}$ [9].

We require the following lemma in order to derive another corollary:

Lemma 1.7: *If $\dot{M}$ is a proper (appropriate) multiplication S-Act that holds ACC on radical subact, then for all proper (appropriate) subact $\dot{N}$ there exists a finite number of minimal prime subact of $\dot{N}$.*

Proof: Put $\dot{N}$ is a proper (appropriate) subact, then rad$\dot{N}$ is the intersection of a finite number of minimal prime subact say ϸ$_1$,ϸ$_2$,…,ϸ$_n$. We shall prove that these ϸ_i's are the only minimal prime subact of $\dot{N}$. Suppose Ù is a minimal prime subact. It is clear that rad $\dot{N}\subseteq$Ù that is $\bigcap^n_{(i=1)}$ ϸ$_i \subseteq$ Ù and hence $\bigcap^n_{(i=1)}$ [ϸ$_i$:$\dot{M}$] = [$\bigcap^n_{(i=1)}$ ϸ$_i$: $\dot{M}$] $\subseteq$ [Ù$_i$:$\dot{M}$], and [Ù : $\dot{M}$] is prime ideal [8] then there exists j$\in$\{1,2,…,n\} such that [ϸj:$\dot{M}$] $\subseteq$ [Ù:$\dot{M}$], but $\dot{M}$ is a multiplication S-Act thus ϸ_j$\subseteq$Ù because Ù is minimal prime subact.

Corollary 1.8: *If $\dot{M}$ is a multiplication c.P , thus for every proper(appropriate) subact $\dot{N}$ there exist a finite number of minimal prime subact of $\dot{N}$.*

Put prime ꞵ in Ṁ. The height of ꞵ equals n (denoted by ht(ꞵ)=n) if there exists a chain of distinct prime subact of ꞵi of Ṁ of the form ꞵ=ꞵ_0⊃ꞵ_1⊃⋯⊃ꞵ_n and it is the longest chain such that ꞵ=ꞵ_0. The Krull dimension of Ṁ, denoted by dimṀ, is defined as: dimṀ={ht(Ù):Ù is prime subact ofṀ} [10]. Following [10, 11,12,13,14], the Prime Avoidance Theorem for modules states as follows: Put U is module, K_1, K_2 ,….., K_n $K_i \leq U, i = 1, \dots n$ also K ≤ (submodule) U, ∃K ⊆ K1 ∪ K2 ∪ ….∪ Kn. Preusme that at most two of the Ki;s are not prime, and (Kj : M) ⊈ $\sqrt{K_l : M)}$)(whenever j ≠ l; Therefore K ⊆ Ki for some i∈ { 1,2,…. n}. We examine how this theory can be extended to the Primal Avoidance theory for acts over monoid .

Theorem 1.9: *Let Ṁ be a multiplication S-Act. If dimṀ=0 then Ṁ is c.P iff Ṁ has finite number of prime subact.*

Proof: Suppose that Ṁ is c.P. If {0} is prime subact then the necessity is trivial. If {0} is not prime, then every prime subact is minimal in Ṁ and by (corollary 1.8) the number of prime subact is finite. The sufficiency hold from the Prime Avoidance Theorem.

A partial converse of theorem 1.5 can be found in the subsequent theorem.

Theorem 1.10: *Put Ṁ is an S-act and every f.g subact is cyclic. Whether Ṁ holds ACCon radical subact, thenṀ is c.P.*

Proof: Put Ṅ is proper (appropriate) subact. By [8], there exists a f.g subact Ù of Ṁ ∋ radṄ = radÙ and hence Ù is cyclic submodule. And by theorem 1.2 Ṁ is c.P.

Definition 1.11: An S-act Ṁ is called semilocal, if Ṁ/Rad(Ṁ) is semisimple.

Proposition 1.12: Put Ṁ is a multiplication semilocal S-Act with dim Ṁ≤1. Whether M holds ACC on radical subact, then Ṁ is c.P.

Proof: We have two cases, first if {0} is prime subact then every non-zero prime subact of Ṁ is maximal, and hence the number of prime subact of Ṁ is finite. On the other hand if {0} is not prime, let Á be the set of all prime subact of Ṁ and let Ĕ = {Ù∈Á: Ù is maximal subact of Ṁ}, Ķ={ꞵ∈Á:ꞵ∉Ĕ}. By lemma 1.6 we have Ķ is finite set and hence Á is finite. In any case, we get Ṁ is c.P from the Prime Avoidance Theorem

Definition 1.13: An S-Act Ṁ has held the Cyclic Subact Condition (CSC) whether for all x∈Ṁ and all prime subact Ķ minimal over Sx, therefore ht(K) ≤1.

Proposition 1.14: Suppose that Ṁ is a c.P S-Act. If the CST is held for Ṁ, then dim Ṁ≤1.

Proof: Put Ķ be a maximal subact of Ṁ, then by (1.2) there exists a∈Ṁ ∋ Ķ = rad Sa. Thus Ķ is a minimal prime subact over Sa. By CSC, ht(K) ≤ 1, therefore dim M≤1.

Proposition 1.15: Put f: M→M' is an epi.. Whether M is c.P then so is M'. The converse is true when M is f.g or (multiplication) S-Act and ker f ⊆ rad{0}.

Proof: It is clear.

References

[1] O. Khudhayer and K. Haibat, "Restrict Nearly Primary Submodules," *Wasit J. Pure Sci.,* vol. 1, no. 3, pp. 70–80 (2022).

[2] M. M. AL-Ashker, A. E. Ashour, and A. A. Abu Mallouh, "On Primal Compactly Packed Modules," *Palestine J. Math.,* vol. 3(Spec 1), pp. 481–488 (2014).

[3] M. A. Naghipoor, "On prime and radical subacts of acts over semigroups," *Commun. Algebra,* vol. 51, no. 5, pp. 1779–1799 (2023).

[4] M. S. Abbas and S. Adnan, "The product of Gamma Subacts of Multiplication Gamma Act," *J. Phys.: Conf. Ser.,* vol. 1804, no. 1, pp. 012105, 1–7 (2021).

[5] M. S. Abbas and S. Adnan, Multiplication Gamma Act, M.Sc. thesis, Dept. Math., Coll. Sci., Al-Mustansaryi Univ., Baghdad, Iraq (2020).

[6] A. E. A., Prime and Semiprime Submodules, M.Sc. thesis, Coll. Sci., Univ. of Baghdad, Baghdad, Iraq (1996).

[7] S. A. K. Muheyaddin, New Types of Regular Fuzzy Modules and Pure Fuzzy Submodules, Ph.D. thesis (2017).

[8] M. Kilp, U. Knauer, and A. Mikhalev, Monoids, Acts and Categories, Berlin, Germany: Walter de Gruyter (2000).

[9] A. A. Estaji and M. Shabani, "A Note on Multiplication S-Act," *Far East J. Math. Sci.* (FJMS), pp. 1–12 (Jan. 2010).

[10] Z. Al-Ani, Compactly Packed Modules and Coprimely Packed Modules, M.Sc. thesis, Coll. Sci., Univ. of Baghdad, Baghdad, Iraq (1998).

[11] M. M. AL-Ashker, A. E. Ashour, and A. A. Abu Mallouh, "On Primal Compactly Packed Modules," *Palestine J. Math.*, vol. 3(Spec 1), pp. 481–488 (2014).

[12] S. Sharma and A. J. Obaid, "Mathematical modelling, analysis and design of fuzzy logic controller for the control of ventilation systems using MATLAB fuzzy logic toolbox," *J. Interdiscip. Math.*, vol. 23, no. 4, pp. 843–849 (2020), doi: 10.1080/09720502.2020.1727611.

[13] R. Das, J. Mishra, S. Mishra, and P. K. Pattnaik, "Design of mathematical model for the prediction of rainfall," *J. Interdiscip. Math.,* vol. 25, no. 3, pp. 587–613 (2022), doi: 10.1080/09720502.2021.2016853.

[14] N. Akhtar, M. M. S. Beg, and M. M. Hussain, "Sparse two level topic model for extraction of general summary words," *J. Interdiscip. Math.,* vol. 23, no. 1, pp. 303–310 (2020), doi: 10.1080/09720502.2020.1721925.

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