

Fully Stable Semirings and Related Concepts

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Abstract

In this article, we introduce the notion of fully stable semiring. Several equivalent definitions of this concept exist, and numerous classical constructs satisfy a fully stable semiring, such as the 2-regular semiring. A relation Ψ_K has been observed on a semiring K , which is evidence of stability. Upon further investigation of this concept, a stronger concept emerged depending on annihilator ideal θ , which is called fully θ -stable; the properties of this concept provided a most important property: 'any ideal in a commutative fully θ -stable semiring possesses a unique complement ideal.'

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1. Introduction

In-ring category, Alamelu launched the concept of stability in 1976 [2]. This concept depends on the element in a ring: an element r in R is said to be stable if every homomorphism from (r) into R is a multiplication by an element in R ; in globally, R is said to be stable if every element in R is stable. Many

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important properties have been proved, like that if R is stable, then $\text{End}_R(T)$ is the commutative ring for every ideal in R (proposition 2, [1, 2]).

The concept of stability to the category of modules. It has a more general meaning : An R -submodule U of Y is said to be stable if $\alpha(U) \subseteq U$ for any R -homomorphism $\alpha \rightarrow Y$, globally, R is said to be fully stable if every R -submodule in Y is stable after giving this concept. It was noted that it has a deep meaning and is a stronger of important concepts, like the invariant module, And then. Exploring the characteristics of this concept gives it equivalent conditions of stability, the most important of these equivalents. R - module Y is fully stable if $\theta_K(\theta_K((u)) = (u)$ for all u in Y where $\theta_K(u) = \{w \in R \mid wu = 0\}$ is the annihilator ideal of element u , and many characteristics were studied [1, 4, 7].

Khabazian, Safaeyan, and Vedadi [6] in 2010 introduced a concept of strongly duo modules and rings, equivalent to a fully stable module. Still, they studied differently, which depended on trace homomorphism: an R -module Y is said to be strongly duo if $\text{Tr}(U, Y) = U$ for every R -submodule U of Y . In (Theorem 2.1, [6]), they presented the equivalence conditions; by the same time, part (iii) provided the equivalences to the concept fully stable module.

Vandiver 1935 introduced the notion of semiring, and many authors have studied this structure (for example [5]), which is a generalization of the ring category. The essential difference is the absence of the inverse element with the addition operation. The category of semiring has an essential application in mathematical and computer science; in particular, the semiring category plays a vast part in the encryption of information and many applications. And the importance of the topic. In this article, we will study the concept of stability on a semiring category to see the differences in the behavior of a semiring category

Preliminaries

A semiring $(K, +, \cdot)$ is a nonempty set K such that $(K, +)$ is monoid (in this paper $+$ not necessary commutative) and $(K, \cdot)$ semigroup which $w(e+r) = we + wr$ for all $w, e, r \in K$ and $t0 = 0t = 0$ for all $t \in K$, and K is said to be semi subtractive if for any elements $w_i \in K$, $i = 1, 2$, $w_1 + q = w_2$ for some $q \in K$, K is said to be cancellable K if for all elements $w_1, w_2, w \in K$ such that $w_1 + q = w_2 + q$ implies $w_1 = w_2$. An equivalence relation Ω on K is said to be right K - congruence if and only if for all $(w_1, w_2), (q_1, q_2) \in \Omega$ implies $(w_1 + q_1, w_2 + q_2) \in \Omega$ and if $e \in K$ then $(ew_1, ew_2) \in \Omega$. (P162, [4, 5]). A nonempty subset L of K is said to be right ideal if $(L, +)$ is subsemigroup of K and $kw \in L$ for all $w \in L$ and $k \in K$. A subtractive ideal (or k -ideal) L is an ideal which if $m, m + w \in L$ implies $w \in L$, for any nonempty subset U of K the annihilator of U denoted by $\theta_K(U) = \{w \in K \mid wu = 0 \text{ for all } u \in U\}$, clearly $\theta_K(U)$ is a right ideal. Let L be a right ideal of K , a mapping $\alpha : L \rightarrow K$ is said to be K -homomorphism if $\alpha(w_1 + w_2) = \alpha(w_1) + \alpha(w_2)$ and $\alpha(qw) = q\alpha(w)$. In This paper, every occurrence of the term "ideal" refers to the right ideal.

2. Fully Stable and Relative Concepts

Definition 2.1: Let K be a semiring. An ideal L of K is said to be stable if and only if any K – homomorphism $\alpha : L \rightarrow K$ implies $\alpha(L) \subseteq L$. A semiring K is called fully stable if and only if any ideal in K is stable.

The following theorem gives equivalent definition of stability

Proposition 2.2: Let K be a semiring. Then the following statement are equivalent (i) K is fully stable. (ii) Any K – homomorphism from a principal ideal into K it is a multiplication by element of K . (iii) For any two ideals L, M of K , M is epimorphic image to L implies $M \subseteq L$.

Proof: (i) to (ii) Is a straightforward (ii) to (i). Let L be an ideal in a semiring K , and $\alpha : L \rightarrow K$ be an R – homomorphism, let $w \in L$ implies that $\alpha|_{(w)}(w) \in L$, Hence, we have $\alpha(L) \subseteq L$. (i) to (iii) Let L, M be ideals in a fully stable semiring K and $\alpha : L \rightarrow M$ be K -epimorphism, then $M = \alpha(L) \subseteq L$. (iii) to (i) is straightforward.

Examples and Remarks (2.3).

- (1) For any semiring K , $V(K) = \{e \in K \mid \text{there is an element } r \in K \text{ such that } e + r = 0\}$ is the stable right ideal of K .
- (2) Let K be well ordered nonempty set, and define $w + e = \max(w, e)$, $w * e = \min(w, e)$ for all $a, b \in K$, under this situation K form a fully stable semiring
- (3) Any semi-field is fully stable semiring.
- (4) Analogist with ring category of ring. A fully stable semiring is duo. But the converse is not true in general. See 5.
- (5) $(\mathbb{N}, +, \cdot)$ be a semiring of all natural numbers with usual addition and multiplication. Let $\alpha : 3\mathbb{N} \rightarrow \mathbb{N}$ be $\mathbb{N}$ - homomorphism which define by $\alpha(3e) = 5e$, then $\alpha(3\mathbb{N}) \not\subseteq 3\mathbb{N}$, and hence $\mathbb{N}$ is duo but not fully stable.

For a specific element w in K , the K - congruence $\Psi_K(w) = \{(a, b) \in K \times K \mid aw = bw\}$ imparts significance to a concept of stability, if $\Psi_K(h) = I_K$ for some element h in K , then K is division, and consequently K is fully stable.

In a ring category. R is fully stable if and only if $\theta_R(\theta_R(r)) = (r)$ for any element r in R [6, 7]. However, an additional condition is necessary to establish the equivalence in the semiring category. Thus, we state the following lemma.

Lemma 2.5: Let K be a semiring, if $\theta_K(\theta_K(w)) = (w)$ for all $w \in K$, then K is a fully stable.

Proof: Let $w \in K$, $\alpha : (w) \rightarrow K$ be a K – homomorphism, since $\theta_K(w) \subseteq \theta_K(\alpha(w))$ then $\theta_K(\theta_K(\alpha(w))) \subseteq \theta_K(\theta_K((w)))$, by hypotheses implies $\alpha(w) \in (w)$ then α is multiplication by element of K , by proposition (2.2) K is fully stable.

The following example shows the reverse of lemma 2.5 not true in general.

Example 2.6: Let $(K, \oplus, \odot)$ the commutative semiring such that $K = \{0, q < w < e < r\}$ and $a \oplus b = \max(a, b)$, and $a \odot b = \min(a, b)$. Under this situation, where K forms a commutative semiring, it is evident that any right ideal is represented by $I_w = \{r \in K \mid r \leq w\}$ for some $w \in K$, and $\alpha(I_w) \subseteq I_w$ for any K -homomorphism $\alpha : I_w \rightarrow K$. But $\theta_K(\theta_K(e)) \neq (e)$

Definition 2.6: A semiring K is said to be fully θ -stable if and only if $\theta_K(\theta_K(w)) = (w)$ for all $w \in K$.

Example 2.7: In a semiring $(Z_4, +_4, *_4)$, $\theta_{Z_4}(\theta_{Z_4}(w)) = (w)$ for all $w \in Z_4$, and hence Z_4 is fully θ -stable.

Lemma (2.5) and example (2.6) illustrate the concept of fully θ -stable is stronger than fully stable. In the next theorem, we will consider a condition on a semiring that ensures the equivalence between the concepts of fully θ -stable and fully stable.

Theorem 2.8: Let K be a semiring such that $\Psi_K(r) \subseteq \Psi_K(t)$ for all $t \in \theta_K(\theta_K(r))$. Then the following statement is equivalent

(1). K is fully stable (2). K is fully θ -stable (3). If $\theta_K(w) \subseteq \theta_K(q)$ for some $w, q \in K$, then $q \in (w)$. (4). Any essential right ideal is fully stable, and any two ideals with zero intersection are mutually orthogonal. (5). Every 2-generated right ideal is fully stable.

Proof: (1) to (2) Let $q \in \theta_K(\theta_K(w))$ and define a K -homomorphism $\alpha : (w) \rightarrow K$ by $(rw) = rq$. α is well-defined. Indeed, let $r_1 w = r_2 w$ for $r_1 \neq r_2$ in K , then $(r_1, r_2) \in \Psi_K(w) \subseteq \Psi_K(q)$ and hence $r_1 q = r_2 q$. Then $q = \alpha(w) \in (w)$, which implies $\theta_K(\theta_K(w)) \subseteq (w)$; the other part of inclusion is straightforward. (2) to (3). By hypotheses $(q) = \theta_K(\theta_K(q)) \subseteq \theta_K(\theta_K(w)) = (w)$ (3) to (4). Let L be essential right ideal of K , M be a right ideal of L , and $\beta : M \rightarrow L$ be K -homomorphism. $\theta_K(e) \subseteq \theta_K(\beta(e))$ for $e \in M$ by hypotheses implies $\beta(e) \in L \cap M$. Then L is fully stable ideal and clear every two right ideals with zero intersection are mutually orthogonal (4) to (5). Let L be 2-generated right ideal, M be the right ideal of L , and $\beta : M \rightarrow L$ be K -homomorphism. Let N be a complement k -ideal of M in K . Then β can be extended to $\bar{\beta} : N + M \rightarrow L$. By hypotheses $\beta(M) = \bar{\beta}(M) \subseteq N + M$, and hence $\pi_N \bar{\beta} : M \rightarrow N$ be a K -homomorphism. Since $N \cap M = 0$, then by hypotheses $\pi_N \bar{\beta}$ is a zero mapping. By this, we can conclude that $\beta(M) \subseteq M$. (5) to (1). Let r an element of K , $\beta : (r) \rightarrow K$ be K -homomorphism. Now consider the right ideal $M = (r, \beta(r))$, and $\gamma : (r) \rightarrow M$ be K -homomorphism such that $\gamma(r) = \beta(r)$. By hypothesis implies $\beta((r)) \subseteq (r)$, and by proposition (2.2), K is fully stable.

Lemma 2.9: (1). If K is semi-subtractive and cancellable semiring. Then $\Psi_K(r) \subseteq \Psi_K(t)$ for all $t \in \theta_K(\theta_K(r))$. (2). If K is commutative fully θ -stable. Then $\Psi_K(r) \subseteq \Psi_K(t)$ for all $t \in \theta_K(\theta_K(r))$.

Proof: (1). Let $(r_1, r_2) \in \Psi_K(r)$ for $r_1 \neq r_2$ then there exists s in K with $r_1 = r_2 + s$ and so $r_1 r = (s + r_2)r = sr + r_2 r = r_2 r$ then by cancelability implies $s \in \theta_K(r) \subseteq \theta_K(t)$, and hence $r_1 t = (s + r_2)t = st + r_2 t = r_2 t$ implies $(r_1, r_2) \in \Psi_K(t)$. (2). Let $(w, e) \in \Psi_K(r)$. Now for some $t \in \theta_K(\theta_K(r))$. By hypotheses $t = cr$ for some c in K , and $wcr = ecr$ and hence $(w, e) \in \Psi_K(t)$.

Corollary 2.10: *If K be a semi subtractive with cancellable semiring. Then K is fully stable if and only K is fully θ – stable.*

Proof: Let $q \in \theta_K(\theta_K(w))$, define a K – homomorphism $\alpha: (w) \rightarrow K$ by $\alpha(rw) = rq$. α is well – defined [let $r_1 w = r_2 w$ for $r_1 \neq r_2$ by lemma (2.9) implies $(r_1, r_2) \in \Psi_K(q)$ and then $r_1 q = r_2 q$]. Since $q = \alpha(w) \in (w)$ implies $\theta_K(\theta_K(w)) \subseteq (w)$. Another direction of inclusion is straightforward.

The complement ideal has been introduced in [3]. An ideal L is a complement of ideal M if L is a maximal element with respect to set of zero intersection with M . Zorn's lemma asserts that any ideal has a complement ideal (k - ideal). However, the complement ideal may not be unique.

Theorem 2.11: *If K is commutative fully θ – stable, then any right ideal of K has a unique complement k – ideal.*

Proof: Let L be a nonzero right ideal of K . Consider the subset $C(L) = \{q \in K \mid \text{for all } (0 \neq w) \in L, \text{ for all } e \in K, \text{ there exists } r \in K \text{ such that } wr \neq 0 \text{ and } qer = 0\}$. It clear that $C(L)$ is nonempty subset of K . we will show $C(L)$ is a right ideal of K . Let $a, s \in C(L)$, z be a nonzero element in L , and $x \in K$. Then there exist $t_1 \in K$ such that $zt_1 \neq 0$ and $axt_1 = 0$, since $(0 \neq zt_1) \in L$ and $xt_1 \in K$ then there exist $t_2 \in K$ such that $zt_1 t_2 \neq 0$ and $sxt_1 t_2 = 0$. By this we get $z(t_1 t_2) \neq 0$ and $(a + s)xt_1 t_2 = 0$. That is $a + s \in C(L)$ and clearly show $C(L)R \subseteq C(L)$. Now let if there is nonzero element $c \in L \cap C(L)$ implies $cr \neq 0$ and $c.1.r = 0$ that is a contradiction. Hence $L \cap C(L) = 0$. The remainder of proof just the maximality property of $C(L)$. Now if $L \cap M = 0$ for some nonzero right ideal of K and let $n \in M \setminus C(L)$ then there exist $(0 \neq w) \in L$ and $e \in K$ such that for any $r \in K$ with $ner = 0$ implies $wr = 0$. By this this we can define mapping $\alpha: (ne) \rightarrow (w)$ by $\alpha(nev) = wv$ for all $v \in K$. To prove α is well – defined let $neh_1 = neh_2$ that is $(h_1, h_2) \in \Psi_K(ne)$ and since $w \in \theta_K(\theta_K(ne))$ by hypotheses implies $(h_1, h_2) \in \Psi_K(w)$ and hence $wh_1 = wh_2$ and clearly α is nonzero K – homomorphism. But $L \cap (ne) = 0$ by theorem (2.8) that is contradiction! Zorn's lemma provided existences. Therefore $C(L)$ is a unique complement k – ideal.

Corollary 2.12: *If K is commutative fully stable with $\Psi_K(r) \subseteq \Psi_K(t)$ for all $t \in \theta_K(\theta_K(r))$, then any right ideal of K has a unique complement k – ideal*

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