

Best approximation on graphs

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Abstract

In this paper we define best approximation on graphs when $p \geq 1$. The great Russian mathematician P.L. Chebyshev developed approximation theory. Here we study the best on graphs and the problem of uniqueness approximation.

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1. Introduction

The history of approximation theory is very long, many big names mathematics worked on approximation theory for its big important in classical and contemporary research. So, it is very interesting subject to work on it.

The approximation theory has a great interest to many scientist and researchers in mathematics for over two centuries. Panuty Lvovich Chebyshev was one of them and the first who works in approximation functions. Chebyshev already did something with approximation theory that far different than ancient mathematicians did. For example Archimedes using polygons approximated the circumference of a circle and therefore π . Pierre Simon Laplace approximated planets by ellipsoide and Leonhard Euler approximated the proportion of longitudes and latitudes of maps to the real proportion of the earth. [1]

Everything about it is obvious. In quite clear way, Chebyshev tell us about the aim of his study in approximation theory, and mention the person that results were starting point. There are many examples of spaces with best approximation

like the usual norm $\|(x_k)_{k=1}^n\|_2 = (\sum_{k=1}^n |x_k|^2)^{\frac{1}{2}}$, where $X = \mathbb{R}^n$, the problem has a complete solution for any subspace (or, indeed, any closed convex set) Y . If we consider $X = \mathbb{R}^2$ the norm $\|(x, y)\| = \max\{|x|, |y|\}$, and let the subspace $y = \{(0, y), y \in \mathbb{R}\}$ (i.e, the y -axis). There are many norms consider on $\mathbb{R}^n$.

There is particular important of L_p -norm is $\|(x_i)_{i=1}^n\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}, 1 \leq p \leq \infty$

And

$$\|(x_i)_{i=1}^n\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Then we define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ as a norm. Many research studies the discrete Sobolev seminorms and the corresponding normed spaces in several contexts in bounds for eigenvalues in the Spectral Graph Theory, [2] (and in [3 chapter 11]) Inequalities between discrete Sobolev seminorms, see [4], [5] and [6]. Let G a graph and $\mathbb{G}$ a subgraph in G , f is continuous function on G then we study the best approximation on graphs by the norm

$$\|f\| = \left\{ \left(\sum_{(u,v) \in E} |f(u) - f(v)|^p \right)^{\frac{1}{p}}, 1 \leq p < \infty. [7] \right.$$

Definition 1: Let $\mathbb{G}$ be a subgraph of simple graph G and a vertex $u \in G$, there exist an vertex $v \in \mathbb{G}$ which is “nearest” to u , such that $\|u - v\|_p = \inf_{w \in \mathbb{G}} \|u - w\|_p, p \geq 1$. Then we called v is the best approximation to u .

Theorem 1: Suppose G be a finite graph. So all norms on G are equivalent. That is if $\|\cdot\|$ and $|||\cdot|||$ are norms on G , then there exist $0 < M, N < \infty$ such that

$$M\|w\| \leq |||w||| \leq N\|w\|$$

For all vertices $w \in G$.

Proof: Let G be a finite graph have a set of vertices $\{v_1, v_2, \dots, v_n\}$ and fix a basis $e_1, e_2, \dots, e_n$ for G then we write a norm on G as

$$\| \cdot \| = (\sum |f(u) - f(v)|^p)^{\frac{1}{p}}, \text{ where } p \geq 1$$

We take the norm

$$\|\sum_{i=1}^n v_i e_i\|_1 = \sum_{i=1}^n |v_i| = \| (v_i)_{i=1}^n \|_1 \text{ where } p = 1$$

For $w = \sum_{i=1}^n v_i e_i \in G$.

From the fact that $e_1, e_2, \dots, e_n$ is a basis for G , it's easy to see that $\| \cdot \|_1$ is really a norm on G .

We notice that one inequality is easy to show

$$\|\sum_{i=1}^n v_i e_i\| \leq \sum_{i=1}^n |v_i| \|e_i\| \leq (\max_{1 \leq i \leq n} \|e_i\|) \sum_{i=1}^n |v_i| = \mathcal{M} \|\sum_{i=1}^n v_i e_i\|_1$$

The first side was proved now we must prove the second side.

In the beginning, we notice that set up a correspondence $\mathbb{R}^n$ and G .

Precisely, it is obvious that the map $(v_i)_{i=1}^n \rightarrow \sum_{i=1}^n v_i e_i$ both one-to-one and one to.

Furthermore, the correspondence between $(\mathbb{R}^n, \| \cdot \|_1)$ and $(G, \| \cdot \|_1)$ is an isometry.

Immediately the inequality we 've just formed shows that the function $v \mapsto \|v\|$ is continuous on the space $(G, \| \cdot \|_1)$ since

$$|\|u\| - \|v\|| \leq \|u - v\| \leq \mathcal{M} \|u - v\|_1$$

For any $u, v \in G$. So, the norm $\| \cdot \|$ consider as a minimum value on the compact set

$$A = \{v \in G: \|v\|_1 = 1\}$$

Specially, there exist $N > 0$ such that $\|v\| > N$ wherever $\|v\|_1 = 1$.

Then we obtained the inequality

$$\left\| \frac{v}{\|v\|_1} \right\| \geq N \Rightarrow \|v\| \geq N \|v\|_1.$$

Theorem 2: Let G be a finite connected graph and let $\mu > 0$. So, any closed ball $\{w \in G: \|w\| \leq \mu\}$ is compact.

Proof: Let G a finite graph of vertices $V = \{v_1, v_2, \dots, v_n\}$ and $e_1, e_2, \dots, e_n$ is basis for G . From last theorem we know that there is some constant $N > 0$ such that

$$N \sum_{i=1}^n |v_i| \leq \|\sum_{i=1}^n v_i e_i\|$$

For all $v = \sum_{i=1}^n v_i e_i \in G$. Particularly,

$$N |v_i| \leq \|\sum_{i=1}^n v_i e_i\| \leq \mu \Rightarrow |v_i| \leq \frac{\mu}{N}$$

For $i = 1, \dots, n$.

Thus, $\{w \in G: \|w\|_p \leq \mu\}$ is a closed subset of compact set.

$$\left\{v = \sum_{i=1}^n v_i e_i, |v_i| \leq \frac{\mu}{N}, i = 1, 2, \dots, n\right\}.$$

Theorem 3: Suppose $\mathbb{G}$ be subgraph of a graph G , and let $u \in G$. So, there exists (not necessarily unique) $v^* \in \mathbb{G}$ such that

$$\|u - v^*\|_p = \min_{v \in \mathbb{G}} \|u - v\|_p, \forall v \in \mathbb{G}, p \geq 1$$

Then, there is a best approximation to u by a vertex of $\mathbb{G}$.

Proof: In the beginning notice that $w \in \mathbb{G}$ isolated vertex. The nearest vertex v^* will satisfy $\|u - v^*\|_p \leq \|u\|_p = \|u - w\|_p, p \geq 1$. So, it suffices to look for v^* among the vertices $v \in \mathbb{G}$ to satisfy $\|u - v\|_p \leq \|u\|_p, p \geq 1$. We use the triangle inequality:

$$\|u - v\|_p \leq \|u\|_p \Rightarrow \|v\|_p \leq \|u - v\|_p + \|u\|_p \leq 2\|u\|_p, p \geq 1.$$

Hence, v 's is in the compact set restrict our attention

$$F = \{v \in \mathbb{G}: \|v\|_p \leq 2\|u\|_p\}$$

To ending the proof, we must notice the function $f(v) = \|u - v\|_p, p \geq 1$ is continuous (since Sobolev spaces on continuous function)

$$\begin{aligned} (\sum_{v,k \in \mathbb{G}} |f(v) - f(k)|^p)^{\frac{1}{p}} &= |\|u - v\|_p - \|u - k\|_p| \\ &\leq \|v - k\|_p, p \geq 1 \end{aligned}$$

Then notice a minimum value at some vertex $v^* \in F$.

Proposition 1: Let $f \in G$ where $G, \forall i \in \mathbb{Z}^+$ (where $\mathbb{Z}^+$ positive integer) exist there (not necessarily unique) polynomial $p_i^* \in P_i$ such that $\|f - P_i^*\| \leq \|f - p\|$, where $p \in P_i$.

Proposition 2: If $f \in G$, for positive integer i , there exists a constant $K < \infty$ such that $\|f - \sum_{L=0}^i v_L x^L\| \leq \|f\|$, then

$$\max_{0 \leq L \leq i} |v_L| \leq K.$$

Theorem 4: Suppose $\mathbb{G}$ a connected subgraph of a graph G , and let for all $v \in G$ has a unique nearest vertex $u_v \in \mathbb{G}$. So, the nearest vertex map $v \rightarrow u_v$ is continuous.

Proof: We write $p(v) = u_v$ for the nearest vertex map, and take $v_i \rightarrow v$ in G . We want to prove $p(v_i) \rightarrow p(v)$, and to satisfy that we must have a subsequence of $(p(v_i))$ converges to $p(v)$. Where the sequence (v_i) is bounded in G , let $\|v_i\|_q \leq \mathcal{M}$, for all i and $q \geq 1$ Then

$$\begin{aligned}
\|p(v_i)\|_q &= (\sum_{v_i \in G} |p(v_i) - v_i + v_i|^q)^{\frac{1}{q}}, q \geq 1 \\
&\leq (\sum_{v_i} |p(v_i) - v_i|^q)^{\frac{1}{q}} + (\sum_{v_i \in G} |v_i|^q)^{\frac{1}{q}} \\
&\leq \|p(v_i) - v_i\|_q + \|v_i\|_q \\
&\leq 2\|v_i\|_q \leq 2\mathcal{M}
\end{aligned}$$

So, $(p(v_i))$ is a bounded in $\mathbb{G}$, let we take $(p(v_i))$ converges to some vertex $v^* \in \mathbb{G}$.

Now we must prove $v^* = p(v)$. $\|p(v_i) - v_i\| \leq \|p(v) - v_i\|$.

For any i , and then for $i \rightarrow \infty$ $\|p_* - v\| \leq \|p(v) - v\|$, from the uniquely nearest vertex in $\mathbb{G}$, we must have $v^* = p(v)$.

Conclusion

Best approximation on graphs, best approximation on polynomial, the problem of uniqueness of best approximation can be introduced on graph when $1 \leq p < \infty$.

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