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On the local (a, d)-edge antimagic coloring of gear graph and its subdivision

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Abstract

For any graph $G = (V, E)$, with $|V(G)| = p$ and $|E(G)| = q$. A bijection $f : E(G) \to \{1, 2, 3,\ldots,|E(G)|\}$ is called an local (a,d)-edge antimagic coloring of G if the element of the edge-weight set $w(uv) = f(u) + f(v)$, where $w(e_1) \neq w(e_2)$, $uv \in E(G)$, are distinct and the set of all edge-weights are formed an arithmetic sequence with initial smallest edge-weight value a and different d. The local (a,d)-edge antimagic chromatic number is is the minimum number of colors needed to color G such that a graph G admits the local (a,d)-edge antimagic coloring.The local (a,d)-edge antimagic chromatic number denoted by $\chi'_{(a,d)\text{-ela}}$. In the present, we will obtain the lower and upper bound of $\chi'_{(a,d)\text{-ela}}$ and determine the exact of value of the local (a,d)-edge antimagic coloring chromatic number of gear graph and it's subdivisions.

Subject Classification (2010): *05C15, 05C78.*

Keywords: *Local (a, d)-edge antimagic coloring, (a, d)-edge antimagic coloring chromatic number, Gear graph and its subdivision.*

Introduction

The topic to be studied in this research is the development of local edge antimagic coloring in combination with antimagic labeling, namely local (a,d)-edge antimagic coloring. Before this research, Arumugam et al [5] stated one to the pic about local antimagic coloring of graphs. Antimagic labeling is a bijective mapping from the set of edges or the set of vertex in a graph G to the set of positive numbers such that the sum of any two edges produces a different value. The bijective mapping $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ is called local edge antimagic coloring if two neighboring edges e_1 and e_2, then $w(e_1) \neq w(e_2)$, where w_e is obtained from the sum of the labels of the points incident to the edge [2]. The chromatic number of local edge antimagic coloring is denoted by $\chi_{lea}(G)$, while local edge antimagic coloring is denoted by $\chi'_{lea}(G)$. Local edge antimagic coloring is said to be local (a,d)-edge antimagic coloring if the set of side colors forms an arithmetic sequence with initial value a and difference d. The chromatic number of local (a,d)-edge antimagic coloring is denoted by $\chi'_{(a,d)-ela}(G)$[10].

In graphs there are several types of operations, one of which is the subdivision operation on a graph G which is denoted by $S(G)$. The subdivision graph of graph G is the graph obtained from graph G by adding a point to each

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edge in graph G. The subdivision graph of graph G is denoted by $S(G)$ [6,7]. In the previous discussion, we have discussed local (*a,d*)-edge antimagic coloring. On the topic of local (*a,d*)-edge antimagic coloring, there are several graphs that have been studied, namely path graphs [8], fan graphs, friendship graphs, star graphs [3], join product [9], corona product [4], and comb product [1]. In addition, wheel graphs have also been studied on the same topic, namely local (*a,d*)-edge antimagic coloring [10]. Lie, et al [9] determined modulo local antimagic chromatic number of graphs.

The results

In this section, the results of the research will be presented, namely 4 theorems regarding the chromatic number of *(a,d)*-edge local antimagic coloring on gear graphs and their subdivisions.

Theorem 2.1: *The chromatic number of local (2n,1)-edge antimagic coloring on gear graph* G_n, *with* $n \geq 2$ *is* $n \leq \chi'_{(2n,1)-ela}(G_n) \leq n+2$.

Proof: The graph G_n has the vertex set $V(G_n) = \{z\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and edge set $E(G_n) = \{zx_i; 1 \leq i \leq n\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{x_i y_{i+1}; 1 \leq i \leq n-1\} \cup \{x_n y_1\}$. The cardinality of vertices is $|V(Gn)| = 2n+1$ and the cardinality of edges is $|E(Gn)| = 3n$. To prove the theorem, one must first show the lower bound of $\chi'_{(2n,1)-ela}(Gn)$. According to Sundaramoorthy dan Chettiar (2022), $\chi'_{(a,d)-ela}(G) \geq \chi'_{ela}(G) \geq \chi'(G) \geq \Delta(G)$. So the lower bound $\chi'_{(a,d)-ela}(G_n) \geq \Delta(G_n) = n$.

Second, showing the upper bound of $\chi'_{(a,d)-ela}(G_n)$ is determined by the bijective mapping $f: V(G_n) \to \{1,2,3,\ldots,|V(G_n)|\}$ as follows:

$f(x_i) = i$, untuk $1 \leq i \leq n$
$f(y_i) = 2n - i + 1$, untuk $1 \leq i \leq n$
$f(z) = 2n+1$

Based on the point labeling on the gear graph above, the following side weights are obtained:

$$f(w) = \begin{cases} 2n, & x_i y_{i+1}; 1 \leq i \leq n-1 \\ 2n+1, & y_i x_i; 1 \leq i \leq n \\ 3n, & x_n y_1 \\ 2n+i+1, & x_i z; 1 \leq i \leq n \end{cases}$$

From the edge weight function above, the set of edge weights $W = \{2n, 2n+1, 2n+2, \ldots, 3n+1\}$ is obtained. Based on the set of edge weights, an arithmetic set with $a = 2n$ and $d = 1$ can be shown. Showing the cardinality of the edge weight $|W|$ is $U_{|W|} = a + (|W|-1)d \Leftrightarrow 3n+1 = 2n + (|W|-1)1 \Leftrightarrow |W| = n+2$. From the obtained edge weight cardinality, we have $\chi'_{(2n,1)-ela}(G_n) \leq n+2$. Based on the proof, the lower bound

$\chi'_{(2n,1)-ela}(G_n) \geq n$ and the upper bound $\chi'_{(2n,1)-ela}(G_n) \leq n+2$ are obtained. So it can be concluded that the chromatic number of local (*a,d*)-edge antimagic coloring on gear graph is $n \leq \chi'_{(2n,1)-ela}(G_n) \leq n+2$.

Theorem 2.2: *The chromatic number of local (3,2)-edge antimagic coloring on gear graph G_n, with $n \geq 2$ is $n \leq \chi'_{(3,2)-ela}(G_n) \leq 2n$.*

Proof: The graph G_n has the vertex set $V(G_n) = \{z\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and edge set $E(G_n) = \{zx_i; 1 \leq i \leq n\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{y_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 y_n\}$. The cardinality of vertices is $|V(Gn)| = 2n + 1$ and the cardinality of edges is $|E(Gn)| = 3n$. To prove the theorem, one must first show the lower bound of $\chi'_{(3,2)-ela}(Gn)$. According to Sundaramoorthy dan Chettiar (2022), $\chi'_{(a,d)-ela}(G) \geq \chi'_{ela}(G) \geq \chi'(G) \geq \Delta(G)$. So the lower bound $\chi'_{(a,d)-ela}(G_n) \geq \Delta(G_n) = n$.

Second, showing the upper bound of $\chi'_{(a,d)-ela}(G_n)$ is determined by the bijective mapping $f: V(G_n) \rightarrow \{1,2,3, \ldots, |V(G_n)|\}$ as follows:

$$
\begin{aligned}
&f(x_i) = 2i, && \text{for } 1 \leq i \leq n \\
&f(y_i) = 2i+1, && \text{for } 1 \leq i \leq n \\
&f(z) = 1
\end{aligned}
$$

Based on the point labeling on the gear graph above, the following side weights are obtained:

$$
f(w) = \begin{cases} 4i+1, & x_i y_i; 1 \leq i \leq n \\ 2n+i+1, & y_i x_{i+1}; 1 \leq i \leq n-1 \\ 3n-(n-3), & y_n x_1 \\ 2i+1, & x_i z; 1 \leq i \leq n \end{cases}
$$

From the edge weight function above, the set of edge weights $W = \{3,5,7,9, \ldots, 4n+1\}$ is obtained. Based on the set of edge weights, an arithmetic set with $a = 3$ and $d = 2$ can be shown. Showing the cardinality of the edge weight $|W|$ is $U_{|W|} = a + (|W|-1)d \Leftrightarrow 4n+1 = 3 + (|W|-1)2 \Leftrightarrow 2|W| = 4n \Leftrightarrow |W| = 2n$. From the obtained edge weight cardinality, we have $\chi'_{(3,2)-ela}(G_n) \leq 2n$. Based on the proof, the lower bound $\chi'_{(3,2)-ela}(G_n) \geq n$ and the upper bound $\chi'_{(3,2)-ela}(G_n) \leq 2n$ are obtained. So it can be concluded that the chromatic number of local (*a,d*)-edge antimagic coloring on gear graph is $n \leq \chi'_{(3,2)-ela}(G_n) \leq 2n$. Figure 1 (b) shows an illustration of $\chi'_{(2n,1)-ela}(G_n)$ on a gear graph with $n = 5$.

In Figure 1 (a), a gear graph with $n = 6$ and $V = \{1,2,3, \ldots, 13\}$ produces edge weights $W = \{12,13,14, \ldots, 19\}$. The obtained edge weights form an arithmetic sequence with $a = 12$ and $d = 1$. In Figure 1(b), a gear graph with $n = 5$ and $V = \{1,2,3, \ldots, 11\}$ produces edge weights $W =$

$\{3,5,7,9,11,13,15,27,19,21\}$. The obtained edge weights form an arithmetic sequence with $a = 3$ and $d = 2$.

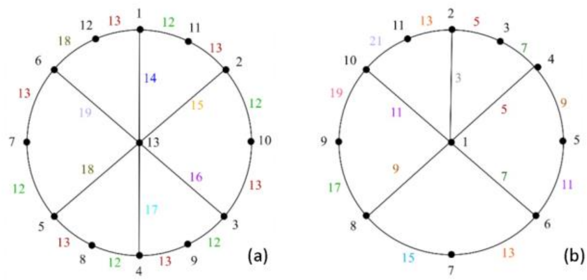

Figure 1
(a) the local $(2\mathbf{n}, 1)$-edge antimagic coloring of G_6; (b) the local $(3, 2)$-edge antimagic coloring of G_5

Theorem 2.3: *The chromatic number of local (3n+3,1)-edge antimagic coloring on gear graph subdivision* $S(G_n)$, *with* $n \geq 2$ *is* $n \leq \chi'_{(3n+3,1)-ela} S(G_n) \leq 2n + 1$.

Proof: The graph $S(G_n)$ has the vertex set $V(S(G_n)) = \{z\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{k_i; 1 \leq i \leq 2n\}$ and edge set $E(S(G_n)) = \{zx_i; 1 \leq i \leq n\} \cup \{x_n k_1\} \cup \{y_i k_{2i}; 1 \leq i \leq n\} \cup \{k_{2i} x_i; 1 \leq i \leq n\} \cup \{x_i k_{2i+1}; 1 \leq i \leq n-1\} \cup \{k_{2i-1} y_i; 1 \leq i \leq n-1\}$. The cardinality of vertices is $|V(S(G_n))| = 4n + 1$ and the cardinality of edges is $|E(S(G_n))| = 5n$. To prove the theorem, one must first show the lower bound of $\chi'_{(3n+3,1)-ela}(S(G_n))$. According to Sundaramoorthy dan Chettiar (2022), $\chi'_{(a,d)-ela}(G) \geq \chi'_{ela}(G) \geq \chi'(G) \geq \Delta(G)$. So the lower bound $\chi'_{(a,d)-ela}(S(G_n)) \geq \Delta(S(G_n)) = n$.

Second, showing the upper bound of $\chi'_{(a,d)-ela}(S(G_n))$ is determined by the bijective mapping $f: V(S(G_n)) \to \{1,2,3,\dots,|V(S(G_n))|\}$ as follows:

$f(x_i) = 4n + (2 - i)$, untuk $1 \leq i \leq n$
$f(y_i) = 3n + (2 - i)$, untuk $1 \leq i \leq n$
$f(k_i) = i + 1$, untuk $1 \leq i \leq 2n$
$f(z) = 1$

Based on the point labeling on the gear graph above, the following side weights are obtained:

$$f(w) = \begin{cases} 3n + 3 + i, & x_n k_1 \\ 3n + 3 + i, & y_i k_{2i}; 1 \leq i \leq n \\ 4n + 3 + i, & k_{2i} x_i; 1 \leq i \leq n \\ 4n + 4 + i, & x_i k_{2i+1}; 1 \leq i \leq n \\ 3n + 3 + (i - 1), & k_{2i-1} y_i; 1 \leq i \leq n - 1 \\ 4n + 3 - i, & x_i z; 1 \leq i \leq n \end{cases}$$

From the edge weight function above, the set of edge weights $W = \{3n+3, 3n+4, 3n+5, \dots, 5n+3\}$ is obtained. Based on the set of edge weights, an arithmetic set with $a = 3n+3$ and $d = 1$ can be shown. Showing the cardinality of the edge weight $U_{|W|} = a + (|W|-1)d \Leftrightarrow 5n+3 = 3n+3+(|W|-1)1 \Leftrightarrow |W| = 2n+1$. From the obtained edge weight cardinality, we have $\chi'_{(3n+3,1)-ela}(S(G_n)) \leq 2n+1$. Based on the proof, the lower bound $\chi'_{(3n+3,1)-ela}(S(G_n)) \geq n$ and the upper bound $\chi'_{(3n+3,1)-ela}(S(G_n)) \leq 2n+$ 1are obtained. So it can be concluded that the chromatic number of local (*a,d*)-edge antimagic coloring on gear graph is $n \leq \chi'_{(3n+3,1)-ela}(S(G_n)) \leq 2n+1$. Figure 2 (a) shows an illustration of $\chi'_{(3n+3,1)-ela}(S(G_n))$ on a gear graph with $n = 4$.

Theorem 2.4: *The chromatic number of local (3,2)-edge antimagic coloring on gear graph subdivision $S(G_n)$, with $n \geq 2$ is $n \leq \chi'_{(3,2)-ela}(S(G_n)) \leq 4n$.*

Proof: The graph $S(G_n)$ has the vertex set $V(S(G_n)) = \{z\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{k_i; 1 \leq i \leq 2n\}$ and edge set $E(S(G_n)) = \{zx_i; 1 \leq i \leq n\} \cup \{x_i k_{2i-1}; i \leq i \leq n\} \cup \{y_i k_{2i-1}; 1 \leq i \leq n\} \cup \{x_i k_{2i-2}; 2 \leq i \leq n\} \cup \{y_i k_{2i}; 1 \leq i \leq n\} \cup \{x_1 k_n\}$. The cardinality of vertices is $|V(S(G_n))| = 4n+1$ and the cardinality of edges is $|E(S(G_n))| = 5n$. To prove the theorem, one must first show the lower bound of $\chi'_{(3n+3,1)-ela}(S(G_n))$. According to Sundaramoorthy dan Chettiar (2022), $\chi'_{(a,d)-ela}(G) \geq \chi'_{ela}(G) \geq \chi'(G) \geq \Delta(G)$. So the lower bound $\chi'_{(a,d)-ela}(S(G_n)) \geq \Delta(S(G_n)) = n$.

Second, showing the upper bound of $\chi'_{(a,d)-ela}(S(G_n))$ is determined by the bijective mapping $f: V(S(G_n)) \to \{1,2,3,\dots,|V(S(G_n))|\}$ as follows:

$$
\begin{aligned}
f(x_i) &= 4n + (2-i), && \text{untuk } 1 \leq i \leq n \\
f(y_i) &= 3n + (2-i), && \text{untuk } 1 \leq i \leq n \\
f(k_i) &= i + 1, && \text{untuk } 1 \leq i \leq 2n \\
f(z) &= 1
\end{aligned}
$$

Based on the point labeling on the gear graph above, the following side weights are obtained:

$$
f(w) = \begin{cases} 6i + (2i-3), & x_i k_{2i-1}; 1 \leq i \leq n \\ 7i + (i-1), & y_i k_{2i-1}; 1 \leq i \leq n \\ 8i + 1, & y_i k_{2i}; 1 \leq i \leq n \\ 7i + (i-5), & x_i k_{2i-2}; 2 \leq i \leq n \\ 4n + 3, & x_1 k_n \\ 4i - 1, & x_i z; 1 \leq i \leq n \end{cases}
$$

From the edge weight function above, the set of edge weights $WW = \{3n+3, 3n+4, 3n+5, \dots, 5n+3\}$ is obtained. Based on the set of edge weights, an arithmetic set with $a = 3n+3$ and $d = 1$ can be shown. Showing the cardinality of the edge weight $U_{|W|} = a + (|W| - 1)d \Leftrightarrow 5n + 3 = 3n + 3 + (|W| - 1)1 \Leftrightarrow |W| = 2n + 1$. From the obtained edge weight cardinality, we have $\chi'_{(3n+3,1)-ela}(S(G_n)) \leq 2n + 1$. Based on the proof, the lower bound $\chi'_{(3n+3,1)-ela}(S(G_n)) \geq n$ and the upper bound $\chi'_{(3n+3,1)-ela}(S(G_n)) \leq 2n + 1$are obtained. So it can be concluded that the chromatic number of local (a,d)-edge antimagic coloring on gear graph is $n \leq \chi'_{(3n+3,1)-ela}(S(G_n)) \leq 2n + 1$. Figure 2 (b) shows an illustration of $\chi'_{(3,2)-ela}(S(G_n))$ on a gear graph with $n = 3$.

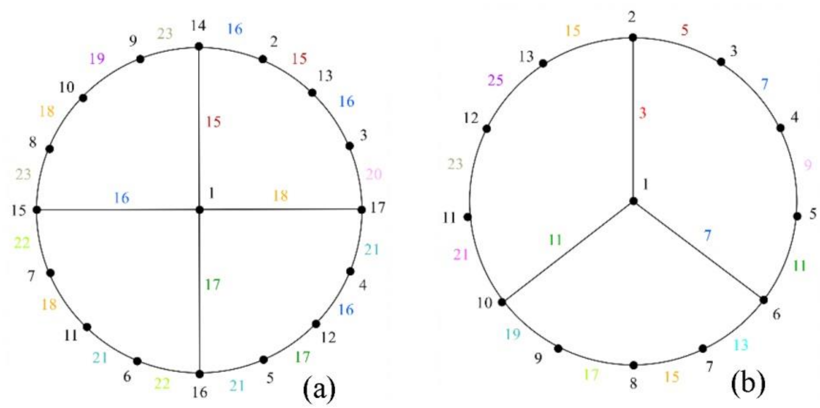

Figure 2
(a) The local $(3n + 3, 1)$-edge antimagic coloring of $S(G_4)$; (b) The local $(3, 2)$-edge antimagic coloring of $S(G_3)$

In Figure 2 (a), a gear graph with $n = 4$ and $V = \{1,2,3,\ldots,17\}$ produces edge weights $W = \{15,16,17,\ldots,23\}$. The obtained edge weights form an arithmetic sequence with $a = 15$ and $d = 1$. In Figure 2 (b), a gear graph with $n = 3$ and $V = \{1,2,3,\ldots,13\}$ produces edge weights $W = \{3,5,7,9,\ldots,23,25\}$. The obtained edge weights form an arithmetic sequence with $a = 3$ and $d = 2$.

Conclusion

Based on the research results and discussions that have been presented in this study, the local chromatic number of local (a,d)-edge antimagic coloring on gear graphs with $d = 1$ and $d = 2$, and subdivisions of gear graphs with $d = 1$ and $d = 2$ are obtained.

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