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On rainbow vertex anti-magic coloring of amalgamation graphs

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Abstract

A rainbow vertex anti-magic coloring represents a relatively new area of exploration in graph theory. This concept extends the idea of rainbow vertex coloring by incorporating elements of anti-magic labeling. Given a function $f : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$, the weight of a vertex $v \in V(G)$ under f is given by $w_f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ denotes the set of edges incident to v. The function f is classified as a vertex anti-magic edge labeling if the weights assigned to all vertices are distinct. A path is referred to as a rainbow path if for any pair of vertices u and v all internal vertices along the $u - v$ path possess distinct weights. The rainbow vertex anti-magic connection number of G, denoted as $rvac(G)$, is the minimum

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number of colors required across all rainbow colorings derived from a rainbow vertex anti-magic labeling of G. This paper presents the computation of the rainbow vertex anti-magic connection number for specific graph families, including the Dutch windmill, diamond graph, octopus graph, and amalgamation graph.

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Keywords: *Rainbow vertex connection, Rainbow vertex anti-magic coloring, Amalgamation graphs.*

1. Introduction

Hartsfield and Ringel [1] proposed the concepts called anti-magic labeling and anti-magic graphs in 1990. An anti-magic labeling of a graph with p vertices and q edges refers to a bijective function f that maps the set of edges to the positive integers $\{1, 2, 3, \dots, q\}$. This labeling ensures that the oriented vertex weights of all p vertices are distinct, where the oriented vertex weight of a vertex is calculated as the total sum of the labels of edges directed into the vertex minus the sum of the labels of edges directed away from it.

The application of Rainbow Connection in a delivery system is an innovative approach that utilizes concepts from graph theory to improve the efficiency and reliability of delivery. Rainbow Connection in the context of graph theory is a condition where all pairs of nodes in a graph are connected by a rainbow path. This path consists of edges that each have a different color. In a delivery system, each node can represent a location such as a warehouse, distribution center, or delivery destination, while the edges represent routes or paths between these locations. By implementing Rainbow Connection, each pair of locations will be connected by a unique path with routes that do not share the same color (route). This helps in avoiding congestion on certain routes and distributing delivery traffic evenly. Using colored graphs to design distribution paths, so that each distribution center has a unique path to another warehouse or distribution point. This ensures that a failure in one path will not disrupt the entire system. By utilizing Rainbow Connection, the delivery system can allocate shipments through different paths, avoiding routes that are already too congested with other shipments. Using a variety of unique rainbow paths for each delivery, making it difficult for outsiders to guess which path will be used. In addition, the Rainbow Connection concept can be used by city planners to ensure that each transportation route is well connected without overburdening certain routes, better distributing traffic.

We study concepts related to labeling and coloring in graph theory. Among the topics in labeling is anti-magic labeling, while rainbow vertex coloring is a key focus in coloring [2]. A graph G with a vertex coloring is considered rainbow vertex-connected if, for every pair of vertices u and v in G, there exists at least one $u - v$ path where all internal vertices have distinct colors. The minimum number of colors required to achieve this property is referred to as the rainbow vertex-connection number of G, denoted by $rvc(G)$.

A rainbow vertex anti-magic coloring is a recently introduced topic in graph theory, combining aspects of rainbow vertex coloring with anti-magic labeling. For a function $f: E(G) \to \{1, 2, \ldots, |E(G)|\}$, the weight of a vertex $v \in V(G)$ with respect to f is defined as $w_f(v) = \Sigma_{e \in E(v)} f(e)$, where $E(v)$ represents the set of edges incident to) f. The function f is termed a vertex anti-magic edge labeling if the weights of all vertices are distinct.

A path is classified as a rainbow path if, for any pair of vertices u and v, all internal vertices along the $u - v$ path have unique weights. If every pair of vertices u and v in G is connected by at least one rainbow $u - v$ path, the labeling f is called a rainbow vertex anti-magic labeling of G. Assigning each edge uv the color corresponding to the vertex weight $w_f(v)$ leads to a rainbow vertex anti-magic coloring of the graph G. The smallest number of colors needed across all rainbow colorings induced by rainbow vertex anti-magic labeling of G is known as the rainbow vertex anti-magic connection number, denoted $rvac(G)$ [3].

In recent year, there were several researchers who had studied rainbow vertex anti-magic coloring. Among them is Marsidi et. al [3]. Some results is divided into observation and Lemma. Their lemma and Observation, we need to proof in this paper.

Observation 1.1: [3] Let G be a connected graph with n vertices and a diameter denoted as $diam(G)$. It has been established that $diam(G) - 1 \leq rvc(G) \leq n - 2$. Furthermore, if G contains c cut vertices, then $rvc(G) \leq c$.

Observation 1.2: [3] The value $rvc(G)$ for a graph G represents the smallest positive integer k such that G admits a rainbow k-coloring. Consequently, $rvc(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.

Observation 1.3: [3] For any connected graph G that is not trivial, it holds that $rvc(G) \geq max\{\chi(G), diam(G)\}$.

Remark 1.4: [3] For a connected graph G, it holds that $rvac(G) \geq rvc(G)$.

There is some results depents rainbow vertex anti-magic coloring for tree graphs [4], and Shell Related Graphs [5]. Agustin et al [6] studied the application to the encryption keystream construction using rainbow vertex anti-magic coloring.

2. Result and Discussion

In this section, we determine the rainbow vertex antimagic coloring of some families graphs namely dutch windmill, diamond graph, octopus graph, and amalgamation graph.

Lemma 2.1: *For* $n \geq 3, rvc(D_n^3) = 3.$

Proof: Let D_n^3 be a dutch windmill graph with $V(D_n^3) = \{a\} \cup \backslash\{x_{i,j};\ 1 \leq i \leq n, 1 \leq j \leq 3\}$ and $E(D_n^3) = \{ax_{i,1}, 1 \leq i \leq n\} \cup \{ax_{i,3}, 1 \leq i \leq n\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq n, 1 \leq j \leq 2\}$. The vertex set and edge set of D_n^3 have cardinalities $3n + 1$ and $3n$, respectively. The diameter of D_n^3 is 4. A vertex coloring $c: V(D_n^3) \rightarrow \{1, 2, 3\}$ is defined as follows:

$$c(x_{i,1}) = c(x_{i,2}) = 1;\ c(a) = 2;\ c(x_{i,3}) = 3$$

Every path is rainbow vertex connected with 3-coloring, In such a way that the maximum value of the rainbow vertex connection number of D_n^3 is $rvc(D_n^3) \leq 3$. Based on **Observation 1.1** that $rvc(D_n^3) \geq diam(D_n^3) - 1 = 4 - 1 = 3$. Thus, $rvc(D_n^3) = 3$. An example of rainbow vertex coloring for the Dutch windmill graph is shown in **Figure 1.**

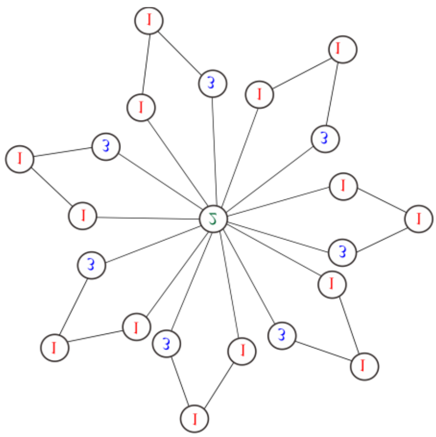

Figure 1
The illustration rainbow vertex coloring of D_7^3

Lemma 2.2: *For* $n \geq 2, rvc(D_n) = 1$

Proof: Let D_n be a diamond graph with $V(D_n) = \{p\} \cup \{q\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_i;\ 1 \leq i \leq n\}$ and $E(D_n) = \{pq\} \cup \{px_i, 1 \leq i \leq n\} \cup \{qx_i, 1 \leq i \leq n\} \cup \{py_i, 1 \leq i \leq n\} \cup \{qy_i, 1 \leq i \leq n\}$. The cardinality of the vertex set and edge set of D_n, respectively are $2n + 2$ and $4n + 1$. The diameter of D_n is 2. Define a vertex coloring $c: V(D_n) \rightarrow \{1\}$ as follows:

$$c(p) = c(q) = c(x_i) = c(y_i) = 1$$

Every path is rainbow vertex connected with this coloring, such that the upper bound of rainbow vertex connection number of D_n is $rvc(D_n) \leq 1$. Based on **Observation 1.1** that $rvc(D_n) \geq diam(D_n) - 1 = 2 - 1 = 1$. Thus $rvc(D_n) = 1$.

Lemma 2.3: *For* $n \geq 3, rvc(O_n) = 1$

Proof: Let O_n be a octopus graph with $V(O_n) = \{a\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_j;\ 1 \leq j \leq n\}$ and $E(O_n) = \{ax_i;\ 1 \leq i \leq n\} \cup \{x_i x_{i+1};\ 1 \leq i \leq n - 1\} \cup \{y_j, 1 \leq j \leq n\}$. The cardinality of the vertex set and vertex set of O_n. respectively are $2n + 1$ and $2n$. The diamter of O_n is 2. Define a vertex coloring $c: V(O_n) \to \{1\}$ as follows:

$$c(a) = c(x_i) = c(y_j) = 1$$

Every path is rainbow vertex connected with this coloring, such that the upper bound of rainbow vertex connection number of O_n is $rvc(O_n) \leq 1$. Based on **Observation 1.1** that $rvc(O_n) \geq diam(O_n) - 1 = 2 - 1 = 1$. Thus $rvc(O_n) = 1$.

Lemma 2.4: *For* $m \geq 3$ *and* $n \geq 3, rvc(Amal(Tb_m, v, n)) = 1$.

Proof: Let $Amal(Tb_m, v, n)$ be a vertex amalgamation product of triangular book with vertex set $V\left(Amal(Tb_m, v, n)\right) = \{z\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_{i,j};\ 1 \leq i \leq n \backslash\& 1 \leq j \leq m\}$ and edge set $E\left(Amal(Tb_m, v, n)\right) = \{zx_i;\ 1 \leq i \leq n\} \cup \left\{zy_{i,j};\ 1 \leq i \leq n, 1 \leq j \leq m\right\} \cup \{x_i y_{i,j};\ 1 \leq i \leq n, 1 \leq j \leq m\}$. So, we get $|V(Amal(Tb_m, v, n))| = nm + n + 1$ and $|E(Amal(Tb_m, v, n))| = 2nm + n$. The diameter of $Amal(Tb_m, v, n)$ is 2. Define a vertex coloring $c: V\left(Amal(Tb_m, v, n)\right) \to \{1\}$ as follows:

$$c(z) = c(x_a) = c(y_{a,b}) = 1$$

Every path is rainbow vertex connected with this coloring, such that the upper bound of rainbow vertex connection number of $Amal(Tb_m, v, n)$ is $rvc(Amal(Tb_m, v, n)) \leq 1$. Based on **Observation 1.1** that $rvc(Amal(Tb_m, v, n)) \geq diam\left(Amal(Tb_m, v, n)\right) - 1 = 2 - 1 = 1$. Thus $rvc(Amal(Tb_m, v, n)) = 1$.The illustration Rainbow Vertex Coloring of $Amal(Tb_3, v, 4)$ can see in **Figure 2**.

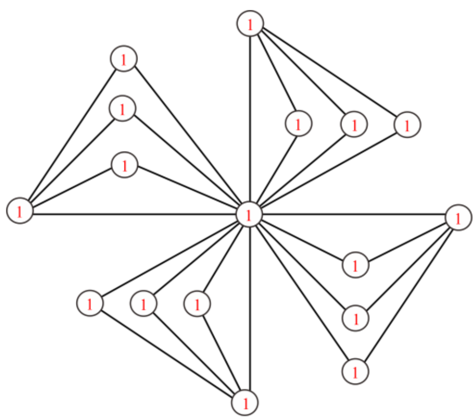

Figure 2
The Illustration Rainbow Vertex Coloring of $\boldsymbol{Amal(Tb_3, v, 4)}$

Furthermore, we determine the rainbow vertex anti-magic coloring of Dutch windmill graph D_n^3, diamond graph D_n, Octopus graph O_n, and Vertex amalgamation product of triangular book with vertex $Amal(Tb_m, v, n)$.

Theorem 2.5: $For\ n \geq 3,\ rvac(D_n^3) = 4.$

Proof: Let D_n^3 be a dutch windmill graph with $V(D_n^3) = \{a\} \cup \{x_{i,j};\ 1 \leq i \leq n, 1 \leq j \leq 3\}$ and $E(D_n^3) = \{ax_{i,1},\ 1 \leq i \leq n\} \cup \{ax_{i,3}, 1 \leq i \leq n\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq n, 1 \leq j \leq 2\}$. The cardinality of the vertex set and edge set of D_n^3, respectively are $3n+1$ and $4n$. To prove $rvac(D_n^3) = 4$, we prove that $rvac(D_n^3) \geq 4$ and $rvac(D_n^3) \leq 4$. First, it will be proved that $rvac \geq 4$. Based on **Lemma 2.1** and **Remark 1.4** we can see that $rvac(D_n^3) \geq 3$. If in D_n^3 graph, there are 3 vertex rainbow connection numbers, then the same internal vertex weights are obtained at $x_{i,1}$ and $x_{j,3}$ such that the vertex rainbow connection number is $rvac(D_n^3) \geq 4$.

Furthermore, it will be proved that $rvac\ (D_n^3) \leq 4$. To prove $rvac(D_n^3) \leq 4$, by defining the label edge $f: E(D_n^3) \rightarrow \{1,2,3,\ldots,4n\}$ as follows.

$$f(x_{i,1}x_{i,2}) = i, \quad for\ 1 \leq i \leq n$$
$$f(x_{i,2}x_{i,3}) = 2n - i + 1, for\ 1 \leq i \leq n$$
$$f(x_{i,1}a) = 3n - i + 1, \quad for\ 1 \leq i \leq n$$
$$f(x_{\{i,3\}a}) = 3n + i, for\ 1 \leq i \leq n$$

Based on the edge label function above, the vertex weights are obtained as follows:

$$w(a) = 6n^2 + n$$
$$w(x_{1,i}) = 3n + 1$$
$$w(x_{2,i}) = 2n + 1$$
$$w(x_{3,i} = 5n + 1$$

Based on the edge label function and vertex weights obtained, $x - y$ rainbow vertex of D_n^3 on Table 1 as follows.

Table 1
The rainbow vertex $x - y$ of D_n^3

Case	Vertex 1	Vertex 2	Rainbow Vertex
1	$x_{1,i}$	$x_{3,i}$	$x_{2,i}$
2	$x_{2,i}$	$x_{2,i}$	$x_{1,i}, a, x_{3,i}$
3	$x_{1,i}$	$x_{1,i}$	a
4	$x_{3,i}$	$x_{3,i}$	a
5	$x_{1,i}$	$x_{3,i}$	a

From the weight function above, we can see that the dutch windmill graph has 4 colors, so the rainbow vertex connection number is $rvac\ (D_n^3) \leq 4$.

Therefore, it can be concluded that $rvac(D_n^3) = 4$. The illustration rainbow vertex anti-magic coloring of D_7^3 can be seen in Figure 3.

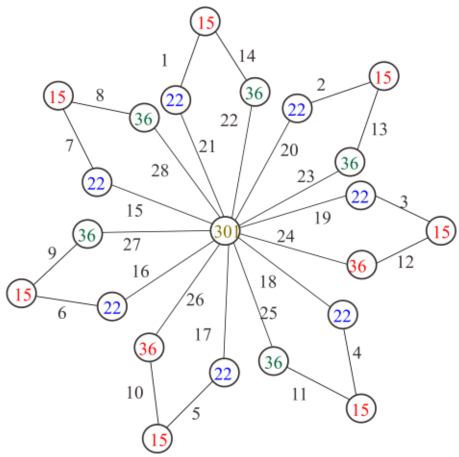

Figure 3
The illustration rainbow vertex anti-magic coloring of D_7^3

Theorem 2.6: *For every integer* n *even and* $n \geq 2, rvac(D_n) = 2$.

Proof: Let D_n be a diamond graph with $V(D_n) = \{p\} \cup \{q\} \cup \{x_i;\ 1 \leq i \leq n\} \cup \{y_i;\ 1 \leq i \leq n\}$ and $E(D_n) = \{pq\} \cup \{px_i, 1 \leq i \leq n\} \cup \{qx_i, 1 \leq i \leq n\} \cup \{py_i, 1 \leq i \leq n\} \cup \{qy_i, 1 \leq i \leq n\}$. The cardinality of the vertex set and edge set of D_n, respectively are $2n + 2$ and $4n + 1$. To prove $rvac(D_n) = 2$, we prove that $rvac(D_n) \leq 2$ and $rvac(D_n) \geq 2$. Based on **Lemma 2.2** and **Remark 1.4**, we can see that $rvac\ (D_n) \geq 1$. In addition, the degrees of the vertices x, y and p, q are different, where $x = y$ and $p = q$. so that the vertex weights obtained are different. Hence, the vertex rainbow connection number is $rvac(D_n) \geq 2$.

Next, it will be proved that $rvac(D_n) \leq 2$. To prove $rvac(D_n) \leq 2$, by defining the label edge:

$$f(pq) = 4n + 1$$

$$f(px_i) = \begin{cases} i, & if\ i \equiv 1\ (mod\ 2) \\ 4n - i + 1, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(qx_i) = \begin{cases} 4n - i + 1, & if\ i \equiv 1\ (mod\ 2) \\ i, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(py_i) = \begin{cases} 3n - i + 1, & if\ i \equiv 1\ (mod\ 2) \\ n + i, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(qy_i) = \begin{cases} n + i, & if\ i \equiv 1\ (mod\ 2) \\ 3n - i + 1, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

Based on the edge label function above, the vertex weights are obtained as follows:

$$w(p) = w(q) \;=\; 4n^2 + 5n + 1$$
$$w(x_i) = w(y_i) \;=\; 4n + 1$$

Based on the edge label function and vertex weights obtained, $x - y$ rainbow vertex on D_n, where x is vertex 1 and y is vertex 2, on Table 2 as follows.

Table 2
The rainbow vertex $x - y$ of D_n

Case	Vertex 1	Vertex 2	Rainbow Vertex
1	x_i	x_j	p
2	x_i	y_i	p
3	x_i	y_j	p
4	p	q	x_i

From the weight function above, we can see that the diamond graph has 2 colors, so the rainbow vertex connection number is $rvac\ (D_n) \leq\ 2$. Therefore, it can be concluded that $rvac(D_n)\ =\ 2$. The illustration rainbow vertex anti-magic coloring of D_6 can be seen in Figure 4.

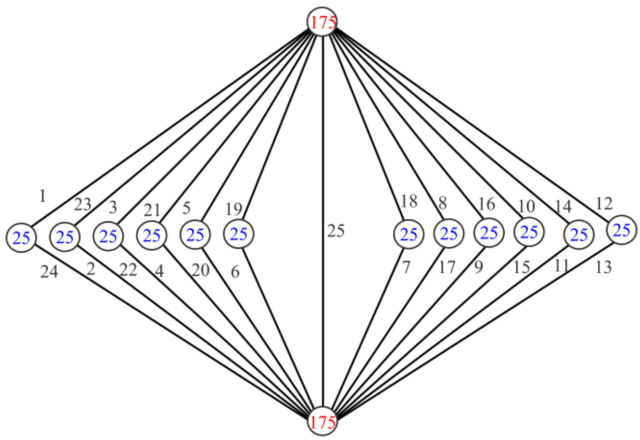

Figure 4
The illustration rainbow vertex anti-magic coloring of D_6

Theorem 2.7: *For every integer* $n \geq\ 3$, $rvac(O_n) =\ n + 1, for\ n \geq\ 3$.

Proof: Let O_n be an octopus graph with $V(O_n) = \{a\} \cup \{x_i;\ 1 \leq\ i \leq\ n\} \cup \{y_i;\ 1 \leq\ i \leq\ n\}$ and $E(O_n) = \{ax_i;\ 1 \leq\ i \leq\ n\} \cup \{x_ix_{i+1};\ 1 \leq\ i \leq\ n - 1\} \cup \{y_i, 1 \leq\ i \leq\ n\}$. The cardinality of the vertex set and edge set of O_n, respectively are $2n + 1$ and $3n - 1$. The weight of the vertex on x_i is equal to a portion of the weight of the vertex on y_i, the vertex a has degree of much greater than the others, it must have a different vertex weight than the others. Besides, the vertex y_i has 1 degree, so each vertex in y_i with $1 \leq i \leq n-1$, has a different weight. So we get the lower bound $rvac(O_n) \geq n+1$.

Next, we will prove the upper bound. This proof divided into two cases to show the upper bound as follows:

Case 1: For $O_n, n \equiv 1(mod\ 2)$

$$f(x_i x_{i+1}) = \begin{cases} \frac{i}{2}, & if\ i \equiv 0\ (mod\ 2) \\ \frac{n+i}{2}, & if\ i \equiv 1\ (mod\ 2) \end{cases}$$

$$f(ax_i) = \begin{cases} 2n-1, & if\ i = n \\ 2n-i-1, & if\ 1 \leq i \leq n-1 \end{cases}$$

$$f(y_i) = 2n+i-1$$

From the edges label above, the vertex weights are obtained as follows:

$$w(x_i) = \frac{5n-3}{2}$$

$$w(a) = 4n^2 - n$$

$$w(y_i) = 2n+i-1$$

For $i = \frac{n-1}{2}$, it gives the vertex weight $w(y_i) = \frac{5n-3}{2}$. The next step is to see that the vertex weights induce rainbow vertex anti-magic coloring on the graph O_n. The sets $w(y_i) = \{2n, 2n+1, 2n+2, \dots, 3n-1\}$ and in $w(a)$ there is one vertex weight which must be different, thus the number of distinct colors of $w(a) \cup w(y_i)\ is\ n+1$. To determine the number of vertex weight based on the set of vertex weight, an arithmetic sequence formula is used. The following is a solution to the arithmetic formula.

$$\begin{aligned} u_z &= a + (z-1)b \\ 3n-1 &= 2n + (z-1)1 \\ 3n-1 &= 2n + z - 1 \\ 3n &= 2n + z \\ z &= n \end{aligned}$$

For $z = n$ summed with $w(y_i)$ gives $n+1$. It shows the vertex weight induces a rainbow vertex anti-magic coloring of $n+1$ colors. Therefore $rvac(O_n) \leq n+1$.

Case 2: For $O_n, n \equiv 0(mod\ 2)$

$$f(x_i x_{i+1}) = \begin{cases} \frac{i}{2}, & if\ i \equiv 0\ (mod\ 2) \\ \frac{n+i-1}{2}, & if\ i \equiv 1\ (mod\ 2) \end{cases}$$

$$f(ax_i) = \begin{cases} 2n-1, & if\ i = n \\ 2n-i-1, & if\ 1 \leq i \leq n-1 \end{cases}$$

$$f(y_i) = 2n + i - 1$$

From the edges label above, the vertex weights are obtained as follows:

$$w(x_i) = \begin{cases} 3n - 2, & if\ i = n \\ \frac{5n}{2} - 2, & if\ 1 \le i \le n-1 \end{cases}$$

$$w(a) = 4n^2 - n$$

$$w(y_i) = 2n + i - 1$$

The next step is to see that the vertex weights induce rainbow anti-magic coloring on the graph O_n The sets $w(y_i) = \{2n, 2n+1, 2n+2, \dots, 3n-1\}$ and in $w(a)$ there is one vertex weight which must be different, thus the number of distinct colors of $w(a) \cup w(y_i)$ is $n+1$. To determine the number of vertex weight based on the set of vertex weight, an

$$\begin{aligned} u_z &= a + (z-1)b \\ 3n - 1 &= 2n + (z-1)1 \\ 3n - 1 &= 2n + z - 1 \\ 3n &= 2n + z \\ z &= n \end{aligned}$$

For $z = n$ summed with $w(y_i)$ gives $n+1$. It shows the vertex weight induces a rainbow vertex anti-magic coloring of $n+1$ colors. Therefore $rvac(O_n) \le n+1$.

Based on that, it is easy to see that the different weight is $n+1$. Based on the edge label function and vertex weights obtained, $x - y$ rainbow vertex on O_n, where x is vertex 1 and y is vertex 2, on Table 3 as follows.

Table 3
The rainbow vertex $x - y$ of O_n

Case	Vertex 1	Vertex 2	Rainbow Vertex
1	x_i	a	x_i, a
2	x_i	y_j	x_i, a, y_j
3	x_i	x_j	$x_i, x_k, x_j; i \le k \le j$
4	y_i	y_j	x_1

From the weight function above, we can see that the Octopus graph has $n+1$ colors. Therefore, it can be concluded that $rvac(O_n) = n+1$.

Theorem 2.8: *For* $m, n \ge 3$, $rvac(Amal(Tb_m, v, n)) = 3$.

Proof: Let $Amal(Tb_m, v, n)$ be a vertex amalgamation product of triangular book with vertex set $V\left(Amal(Tb_m, v, n)\right) = \{z\} \cup \{x_i\ ;\ 1 \le i \le n\} \cup \{y_{i,j};\ 1 \le i \le n\ and\ 1 \le j \le m\}$ and edge set $E\left(Amal(Tb_m, v, n)\right) = \{zx_i;\ 1 \le i \le n\} \cup \left\{zy_{i,j};\ 1 \le i \le n, 1 \le j \le m\right\} \cup \{x_i y_{i,j};\ 1 \le i \le$

$n, 1 \leq j \leq m\}$. So, we get $|V(Amal(Tb_m, v, n))| = nm + n + 1$ and $|E(Amal(Tb_m, v, n))| = 2nm + n$. Based on **Lemma 2.4**, we have $rvac(Amal(Tb_m, v, n) = 1$. Since the vertex z has degree of much greater than the others, it must have a different vertex weight than the others. Besides, There are more degrees on vertex x_i than on vertex $y_{i,j}$, so $w(x_i) \neq w(y_{i,j})$. So, we get the lower bound $rvac(Amal(Tb_m, v, n) \geq 3$.

To show the upper bound of the rainbow vertex anti-magic connection number $Amal(Tb_m, v, n)$, we construct the bijective function of edge labels:

$$f(zy_{i,j}) = \begin{cases} nj + i - n, & if\ i \equiv 1\ (mod\ 2) \\ nj - i + 1, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(x_i y_{i,j}) = \begin{cases} 2nm + n - i - nj + 1, & if\ i \equiv 1\ (mod\ 2) \\ 2nm - nj + i, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(zx_i) = 2nm + i$$

From the edge labels, we have the vertex weight as follows.

$$w(z) = \frac{n}{2}(nm^2 + 4nm + m + n + 1)$$

$$w(x_i) = \frac{3nm^2 + 4nm + n + m + 1}{2}$$

$$w(y_{i,j}) = 2nm + 1$$

From the vertex weight above, it is easy to see that the differen weight is 3. Based on the edge label function and vertex weights obtained, $x - y$ rainbow vertex on $Amal(Tb_m, v, n)$, where x is vertex 1 and y is vertex 2, on Table 4.

Table 4
The rainbow vertex $x - y$ of $Amal(Tb_m, v, n)$

Case	Vertex 1	Vertex 2	Rainbow Vertex
1	x_i	x_j	x_i, z, x_j
2	x_i	$y_{i,j}$	$x_i, y_{i,j}$
3	z	x_i	z, x_i
4	z	$y_{i,j}$	$z, y_{i,j}$
5	$y_{i,j}$	$y_{i,j}$	$y_{i,j}, z, y_{i,j}$

Based on the proof above, we get that $rvac\,(Amal(Tb_m, v, n)) \leq 3$ and $rvac(Amal(Tb_m, v, n)) \geq 3$, then it can be concluded that $rvac(Amal(Tb_m, v, n)) = 3$. The illustration rainbow vertex anti-magic coloring of $Amal(Tb_3, v, 4)$ can see in **Figure 5.**

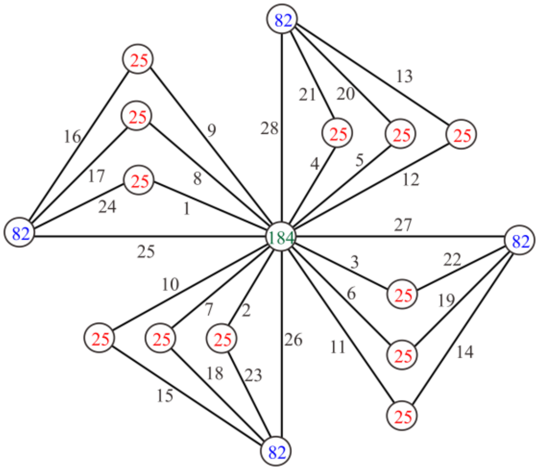

Figure 5
The illustration rainbow vertex anti-magic coloring of $Amal(Tb_3, v, 4)$

3. Conclusion

We get the exact values of rainbow vertex anti-magic connection number from several graphs, including the dutch windmill graph, diamond graph, octopus graph, and vertex amalgamation product of star with vertex. We propose the following open problems.

a. Determine the exact values of rainbow vertex anti-magic connection number of some graph family

b. Determine the exact values of rainbow vertex anti-magic connection number of several graph operations, including comb, amalgamation, and corona.

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