Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 3, pp. 641–652
DOI : 10.47974/JDMSC-1814

On graded S-2-absorbing primary submodules

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Abstract

In this paper, we introduce the concept of *gr*-*S*-2^A-PS as a generalization of *gr*-2^A-PS. Some properties of this class and their homogeneous components are investigated.

Subject Classification (2010): *13A02, 16W50.*

Keywords: *Graded S-2-absorbing submodules, Graded S-2-absorbing primary submodule, Graded S-prime submodule.*

1. Introduction

In this article, we assume that $\mathcal{A}$ is a commutative $\mathfrak{I}$-graded ring with identity and $\mathcal{D}$ is a unitary graded $\mathcal{A}$-Module. Also, $S \subseteq h(\mathcal{A})$ will denote a multiplicatively closed subset of $\mathcal{A}$.

S.E. Atani in [1] extended graded prime ideals to graded prime submodules. The notion of graded 2-absorbing ideals as a generalization of graded prime ideals was introduced and studied in [2]. K. Al-Zoubi and R. Abu-Dawwas in [3] extended graded 2-absorbing ideals to graded 2-absorbing submodules. The notion of graded primary ideals of a comutative graded rings was introduced and studied by Refai and K. Al-Zoubi in [4]. In [5], K. Al-Zoubi and N. sharafat introduced the concept of

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graded 2-absorbing primary ideal which is a generalization of graded primary ideal. The notion of graded 2-absorbing primary submodules as a generalization of graded 2-absorbing primary ideals of a comutative graded rings was introduced and studied by Y. Celikel in [6].

Here, we introduce the concept of *gr*-*S*-2^A-PS and then study some important properties of this class. As in [3], we will continue to use the same notations here and going forward. The basic properties of graded rings and graded modules can be found in [7, 8, 9, 10].

Let $\mathcal{A}$ be a $\Im$-graded ring, $\mathcal{D}$ a graded $\mathcal{A}$-module and L a graded ideal of $\mathcal{A}$. The *graded radical* of L, denoted by $Gr(L)$, is the set of all $x = \sum_{g\in\Im} x_g \in \mathcal{A}$ such that for each $g \in \Im$ there exists $n_g > 0$ with $x_g^{n_g} \in L$. Note that, if r is a homogeneous element, then $r \in Gr(L)$ if and only if $r^n \in L$ for some $n \in \mathbb{N}$, (see [11, 12]).

A nonempty subset S of a $\Im$-graded ring $\mathcal{A}$ is said to be a multiplicatively closed subset (Abbreviated, *m.c.s.*) of $\mathcal{A}$ if $0 \notin S$, $1 \in S$ and $ab \in S$ for each $a, b \in B$, see [13, 14]. By $\mathcal{U} \leq_{\Im} \mathcal{D}$, we mean that $\mathcal{U}$ is a $\Im$-graded submodule of $\mathcal{D}$.

2. Results

Definition 2.1: Let $S \subseteq h(\mathcal{A})$ be a m.c.s of $\mathcal{A}$. A graded submodule $\mathcal{U}$ of $\mathcal{D}$ is said to be a graded S-2-absorbing primary submodule (Abbreviated, *gr*-*S*-2^A-PS) if $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$ and there exists a fixed element $s_t \in S$ such that whenever $c_i d_h m_j \in \mathcal{U}$ where $c_i,\ d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \Im$, then $s_t c_i m_j \in \mathcal{U}$ or $s_t d_h m_j \in \mathcal{U}$ or $s_t c_i d_h \in Gr((\mathcal{U} :_{\mathcal{A}} \mathcal{D}))$.

Definition 2.2: Let $S \subseteq h(\mathcal{A})$ be a m.c.s of $\mathcal{A}$, and P be a graded ideal of $\mathcal{A}$ with $P \cap S = \varnothing$. Then P is called a graded S-2-absorbing primary ideal (Abbreviated, *gr*-*S*-2^A-PI) of $\mathcal{A}$ if there exists a fixed element $s_t \in S$ such that whenever $c_i d_h m_j \in P$ where $c_i,\ d_h,\ m_j \in h(\mathcal{A})$, and $i,\ j,\ h \in \Im$, then $s_t c_i m_j \in P$ or $s_t d_h m_j \in P$ or $s_t c_i d_h \in Gr(P)$.

In particular, A graded ideal P of $\mathcal{A}$ is said to be a *gr*-*S*-2^A-PI of $\mathcal{A}$ if it is a *gr*-*S*-2^A-PS of the graded $\mathcal{A}$-module $\mathcal{A}$.

Let $\mathcal{U} \leq_{\Im} \mathcal{D}$. Then $\mathcal{U}$ is said to be a graded 2-absorbing primary submodule (Abbreviated, gr-2^A-PS of $\mathcal{D}$ if whenever $a,b \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$ with $abm \in \mathcal{U}$, then $ab \in Gr\ (\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ or $am \in \mathcal{U}$ or $bm \in \mathcal{U}$ (see [6]).

The following result can be easily obtain. (We omit the proof)

Theorem 2.3: *Let $S_1 \subseteq S_2 \subseteq h(\mathcal{A})$ be two m.c.s of $\mathcal{A}$. If $\mathcal{U}$ is a gr-S_1-2^A-PS of $\mathcal{D}$ such that $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S_2 = \varnothing$, then $\mathcal{U}$ is also a gr-S_2-2^A-PS of $\mathcal{D}$. In particular, every gr-2^A-PS $\mathcal{U}$ with $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$ is a gr-S-2^A-PS.*

Now, the following example shows that the converse of Theorem 2.3 is not true in general

Example 2.4: Let $\Im = (\mathbb{Z},+)$ and $\mathcal{A} = (\mathbb{Z},+,.)$. Define $\mathcal{A}_g = \begin{Bmatrix} \mathbb{Z} & \text{if } g=0 \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\mathcal{A}$ is $\Im$-graded ring. Take $\mathcal{D} = \mathbb{Z}_{72}$, $\mathcal{D}$ is $\Im$-graded $\mathcal{A}$-module with $\mathcal{D}_g = \begin{Bmatrix} \mathbb{Z}_{72} & \text{if } g=0 \\ \{0\} & \text{otherwise} \end{Bmatrix}$, and $S = \mathbb{Z} - \{0\} \subseteq \mathbb{Z}$ be a m.c.s of $h(\mathcal{A})$. Let $\mathcal{U} = \{0,36\}$. Then it is clear that $\mathcal{U} \leq_{\Im} \mathcal{D}$. Since $2\cdot 2\cdot 9 \in \mathcal{U}$ and neither $2\cdot 9 \in \mathcal{U}$, nor $2\cdot 2 \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) = 6\mathbb{Z}$, then $\mathcal{U}$ is not gr-S-2^A-PS of $\mathcal{D}$. At the same time, $\mathcal{U}$ is a gr-2^A-PS. This can be easily shown by taking $s = 36$ and whenever $c_i d_h m_j \in \mathcal{U}$, then $sc_i m_j \in \mathcal{U}$, $sd_h m_j \in \mathcal{U}$ or $sc_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, for any $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$ and $i,j,h \in \Im$.

Now, the following theorem shows that the converse of Theorem 2.3 can be true under some

Theorem 2.5: *Let $\mathcal{U} \leq_{\Im} \mathcal{D}$ such that $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$. If $S \subseteq u(\mathcal{A}) \cap h(\mathcal{A})$, then $\mathcal{U}$ is gr-S-2^A-PS of $\mathcal{D}$ if and only if $\mathcal{U}$ is gr-2^A-PS of $\mathcal{D}$, where $u(\mathcal{A})$ is the set of all units in $\mathcal{A}$.*

Proof: Assume that $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$ and $s_t \in S$ is the element which satisfies the graded gr-S-2^A-PS condition. Let $c_i d_h m_j \in \mathcal{U}$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i,j,h \in \Im$. So we have either $s_t c_i m_j \in \mathcal{U}$ or $s_t d_h m_j \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Now s_t is unit, so it has an inverse $s_t^{-1} \in h(\mathcal{A})$. Since $\mathcal{U} \leq_{\Im} \mathcal{D}$, we get $c_i m_j \in \mathcal{U}$ or $d_h m_j \in \mathcal{U}$. Moreover, the third case give $c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Thus, $\mathcal{U}$ is gr-S-2^A-PS of $\mathcal{D}$. For the converse, see Theorem 2.3

The set $W_{\mathcal{D}}(\mathcal{A}) = \{c \in \mathcal{A} : c\mathcal{D} = \mathcal{D}\}$ is a saturated multiplication subset of $\mathcal{A}$. Moreover, it is clear that $u(\mathcal{A}) \subseteq W_{\mathcal{D}}(\mathcal{A})$.

Theorem 2.6: *Let $\mathcal{D} = Rv_g$, where $v_g \in h(\mathcal{A})$ be a graded cyclic $\mathcal{A}$-module, $S \subseteq h(\mathcal{A})$ be a m.c.s of $\mathcal{A}$ and $\mathcal{U} \leq_{\mathfrak{I}} \mathcal{D}$ such that $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$. If $S \subseteq W_{\mathcal{D}}(\mathcal{A}) \cap h(\mathcal{A})$, then $\mathcal{U}$ is gr-S-2^A-PS of $\mathcal{D}$ if and only if $\mathcal{U}$ is gr-2^A-PS of $\mathcal{D}$.*

Proof: Suppose that $\mathcal{U}$ is a *gr-S-*2^A-PS of $\mathcal{D}$ and $s_t \in S$ be the element which satisfies the *gr-S-*2^A-PS condition. Let $c_i d_h m_j \in \mathcal{U}$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \mathfrak{I}$. So we have either $s_t c_i m_j \in \mathcal{U}$ or $s_t d_h m_j \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Now $m_j = z_l v_g$ for some $l \in \mathfrak{I}$. If the first two cases satisfies then we have $s_t c_i z_l R v_g = c_i z_l R v_g \subseteq \mathcal{U}$ or $s_t d_h z_l R v_g = d_h z_l R v_g \subseteq \mathcal{U}$, which implies $c_i m_j = c_i z_l v_g \in \mathcal{U}$ or $d_h m_j = c_i z_l v_g \in \mathcal{U}$. If $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, then $(s_t c_i d_h)^n \in (\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ for some $n \in \mathbb{N}$. Since $s_t \in W_{\mathcal{D}}(\mathcal{A})$ then $s_t^n \in W_{\mathcal{D}}(\mathcal{A})$ which gives $(c_i d_h)^n \mathcal{D} = (s_t c_i d_h)^n \mathcal{D} \in (\mathcal{U} :_{\mathcal{A}} \mathcal{D})\, \mathcal{D} = \mathcal{U}$ because $\mathcal{D}$ is a graded multiplication module. Thus, $\mathcal{U}$ is *gr*-2^A-PS. For the converse, see Theorem 2.3.

Theorem 2.7: *Let $S_1 \subseteq S_2 \subseteq h(\mathcal{A})$ be two m.c.ss of $\mathcal{A}$ and for every $s_t \in S_2$, there exists $z_l \in h(\mathcal{A})$ such that $s_t z_l \in S_1$, where $t, l \in \mathfrak{I}$. If $\mathcal{U}$ is a gr-S_2-2^A-PS of $\mathcal{D}$, then $\mathcal{U}$ is gr-S_1-2^A-PS of $\mathcal{D}$.*

Proof: Suppose that $\mathcal{U}$ is a *gr-*S_2-2^A-PS of $\mathcal{D}$, then $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S_2 = \varnothing$, which implies $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S_1 = \varnothing$. Let $c_i d_h m_j \in \mathcal{U}$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$ and $i, h, j \in \mathfrak{I}$. Then there exists $s_t \in S_2$ such that either $s_t c_i m_j \in \mathcal{U}$, $s_t d_h m_j \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ as $\mathcal{U}$ is a *gr-*S_2-2^A-PS of $\mathcal{D}$. Since $s_t \in S_2$, there exists $z_l \in h(\mathcal{A})$ such that $s_t z_l \in S_1$, where $l \in \mathfrak{I}$. Let $s' = s_t z_l \in S_1$, then either $s' c_i m_j = (s_t z_l)\, c_i m_j \in \mathcal{U}$, $s' d_h m_j = (s_t z_l)\, d_h m_j \in \mathcal{U}$ or $s' c_i d_h m_j = (s_t z_l)\, c_i d_h m_j \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Thus $\mathcal{U}$ is *gr-*S_1-2^A-PS of $\mathcal{D}$.

The saturation of S is defined by $S^* = \{s_g \in h(\mathcal{A}) : \frac{s_g}{1}$ is unit in $S^{-1}\mathcal{A}\}$. Then S^* is m.c.s of $\mathcal{A}$ containing S. Also, a m.c.s $S \subseteq h(\mathcal{A})$ is saturated set if $S = S^*$.

Theorem 2.8: *Suppose that S^* is the saturation of S. Then a graded submodule $\mathcal{U}$ of $\mathcal{D}$ is gr-S-2^A-PS if and only if it is graded gr- S^* -2^A-PS of $\mathcal{D}$.*

Proof: Suppose that $\mathcal{U}$ is a *gr-S*-2^A-PS of $\mathcal{D}$, so $(\mathcal{U}:_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$. If $(\mathcal{U}:_{\mathcal{A}} \mathcal{D}) \cap S^* \neq \varnothing$, then we have $r_g \in (\mathcal{U}:_{\mathcal{A}} \mathcal{D}) \cap S^*$. Hence, $\frac{r_g}{1}$ is a unit in $S^{-1}\mathcal{A}$, so there exist $t_h \in h(\mathcal{A})$ and $s_i \in S$ such that $\frac{r_g}{1}\frac{t_h}{s_i} = \frac{r_g t_h}{s_i} = 1$. Thus, $u_j s_i = u_j r_g t_h$ for some $u_j \in S$. Since $r_g \in (\mathcal{U}:_{\mathcal{A}} \mathcal{D})$, $u_j r_g t_h \in (\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. This gives that $u_j r_g t_h \in (\mathcal{U}:_{\mathcal{A}} \mathcal{D}) \cap S$, which is a contradiction. Since $S \subseteq S^*$, then by Theorem 2.3, $\mathcal{U}$ is *gr-* S^* -2^A-PS of $\mathcal{D}$. For the converse, suppose that $\mathcal{U}$ is a *gr-* S^* -2^A-PS of $\mathcal{D}$, so $(\mathcal{U}:_{\mathcal{A}} \mathcal{D}) \cap S^* = \varnothing$. Since $S \subseteq S^*$, then $(\mathcal{U}:_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$. Now, let $c_i d_h m_j \in \mathcal{U}$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \Im$. So, there exists $s_t \in S^*$ with $t \in \Im$ such that either $s_t c_i m_j \in \mathcal{U}$ or $s_t d_h m_j \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. As $s_t \in S^*$, then $\frac{s_t}{1}$ is unit in $S^{-1}\mathcal{A}$. This leads to there exists $r_y \in h(\mathcal{A})$ and $t_z \in S$ with $\frac{s_t}{1}\frac{r_y}{t_z} = 1$, where $y, z \in \Im$. Hence, $v_w s_t r_y = v_w t_z$, for some $v_w \in S$ and $w \in \Im$. Now take $s' = v_w t_z = v_w r_y s_t \in S$. Then either $s' c_i m_j = v_w r_y (s_t c_i m_j) \in \mathcal{U}$, $s' d_h m_j = v_w r_y (s_t d_h m_j) \in \mathcal{U}$ or $s' c_i d_h = v_w r_y (s_t c_i d_h) \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. Therefore, $\mathcal{U}$ is *gr-S*-2^A-PS of $\mathcal{D}$.

Theorem 2.9: *If $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$, then $S^{-1}\mathcal{U}$ is a gr-2^A-PS of $S^{-1}\mathcal{D}$.*

Proof: Suppose that $\mathcal{U}$ is a *gr-S*-2^A-PS of $\mathcal{D}$. Let $\frac{c}{s}\frac{d}{t}\frac{m}{w} \in S^{-1}\mathcal{U}$ with $\frac{c}{s}, \frac{d}{t} \in h(S^{-1}\mathcal{A})$ and $\frac{m}{w} \in h\,(S^{-1}\mathcal{D})$. Then $(uc)dm \in \mathcal{U}$ for some $u \in S$. Since $\mathcal{U}$ is *gr-S*-2^A-PS of $\mathcal{D}$, there exists $s' \in S$ such that $s'(uc)\, m \in \mathcal{U}$, $s'dm \in \mathcal{U}$ or $s'(uc)d \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. This implies that $\frac{c}{s}\frac{m}{w} = \frac{s'ucm}{s'usw} \in S^{-1}\mathcal{U}$ $\frac{d}{t}\frac{m}{w} = \frac{s'udm}{s'utw} \in S^{-1}\mathcal{U}$ or $\frac{c}{s}\frac{d}{t} = \frac{s'ucd}{s'ust} \in Gr(S^{-1}\mathcal{U} :_{S^{-1}\mathcal{A}} S^{-1}\mathcal{D})$. Therefore, $S^{-1}\mathcal{U}$ is *gr*-2^A-PS of $S^{-1}\mathcal{D}$.

In [14], we can see that a m.c.s $S \subseteq h(\mathcal{A})$ satisfies the maximal multiple condition if there exists $u \in S$ where *r* divides *u* for all $r \in S$.

Theorem 2.10: *Let $S \subseteq h(\mathcal{A})$ be a m.c.s of $\mathcal{A}$ satisfying the maximal multiple condition. If $S^{-1}\mathcal{U}$ is a gr-2^A-PS of $S^{-1}\mathcal{D}$, then $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$.*

Proof: Assume that $S^{-1}\mathcal{U}$ is a *gr*-2^A-PS of $S^{-1}\mathcal{D}$. Since *S* satisfying the maximal multiple condition, then there exists $u \in S$ where *r* divides *u* for all $r \in S$.

Let $c_i d_h m_j \in \mathcal{U}$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \Im$. Hence, $\frac{c_i}{1}\frac{d_h}{1}\frac{m_j}{1} \in S^{-1}\mathcal{U}$. Since $\frac{c_i}{1}, \frac{d_h}{1} \in h(S^{-1}\mathcal{A})$, $\frac{m_j}{1} \in h(S^{-1}\mathcal{D})$ and $S^{-1}\mathcal{U}$ is $gr-2^A$ -primary submodule of $S^{-1}\mathcal{D}$, we get either $\frac{c_i}{1}\frac{m_j}{1} \in S^{-1}\mathcal{U}$, $\frac{d_h}{1}\frac{m_j}{1} \in S^{-1}\mathcal{U}$ or

$\frac{c_i}{1}\frac{d_h}{1} \in Gr\ (S^{-1}\mathcal{U} :_{S^{-1}\mathcal{A}} S^{-1}\mathcal{D})$. If $\frac{c_i}{1}\frac{m_j}{1} \in S^{-1}\mathcal{U}$, then there exists $s_1 \in S$ such that $s_1 c_i m_j \in \mathcal{U}$. But s_1 divides u, so we get that $uc_i m_j \in \mathcal{U}$. Similarly, if $\frac{d_h}{1}\frac{m_j}{1} \in S^{-1}\mathcal{U}$, we get that $ud_h m_j \in \mathcal{U}$. Now, If $\frac{c_i}{1}\frac{d_h}{1} \in Gr\ (S^{-1}\mathcal{U} :_{S^{-1}\mathcal{A}} S^{-1}\mathcal{D})$, then there exists $n \in$ such that $\frac{c_i^n}{1}\frac{d_h^n}{1} \in (S^{-1}\mathcal{U} :_{S^{-1}\mathcal{A}} S^{-1}\mathcal{D})$, which gives that $u^n c_i^n d_h^n \mathcal{D} = (uc_i d_h)^n\ \mathcal{D} \subseteq \mathcal{U}$ so $ucd \in Gr\ (\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Therefore, $\mathcal{U}$ is gr-S-2^A\$-PS of $\mathcal{D}$.

Lemma 2.11: *Let $\mathcal{U}$ be a gr-S-2^A-PS of $\mathcal{D}$. If $Id_h m_j \subseteq \mathcal{U}$ where I is an ideal of $\mathcal{A}_e, d_h \in h(\mathcal{A})$, and $m_j \in h(\mathcal{D})$. Then there exists $s_t \in S$ such that either $s_t I m_j \subseteq \mathcal{U}$ or $s_t d_h m_j \in \mathcal{U}$ or $s_t I d_h \subseteq Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$.*

Proof: Let $Idm_j \subseteq \mathcal{U}$ where I is an ideal of $\mathcal{A}_e, d_h \in h(\mathcal{A})$, and $m_j \in h(\mathcal{D})$. Let $s_t \in S$ be the element that satisfies the condition of the *gr-S*-2^A-PS of $\mathcal{U}$. Assume that $s_t d_h m_j \notin \mathcal{U}$ and $s_t I d_h \nsubseteq Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, which means that there exists an element $w \in I$ such that $s_t w d_h \notin Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Now, we have that $Id_h m_j \subseteq \mathcal{U}$ which means $wd_h m_j \in \mathcal{U}$ which gives that $s_t w m_j \in \mathcal{U}$ because $s_t d_h m_j \notin \mathcal{U}$ and $s_t w d_h \notin Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Let $x \in I$, then $(w+x) \in I$ which means $(w+x)\ d_h m_j \in \mathcal{U}$. Since $\mathcal{U}$ is a *gr-S*-2^A-PS of $\mathcal{D}$, then $s_t(w+x)\ m_j \in \mathcal{U}$ or $s_t(w+x)\ d_h \in Gr\ (\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ because $s_t d_h m_j \notin \mathcal{U}$. If $s_t(w+x)\ m_j \in \mathcal{U}$, then $s_t x m_j \in \mathcal{U}$. If $s_t(w+x) d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, then $s_t x d_h \notin Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ because $s_t w d_h \notin Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Since $xd_h m_j \in \mathcal{U}$, then $s_t x m_j \in \mathcal{U}$ because $s_t d_h m_j \notin \mathcal{U}$ and $s_t x d_h \notin Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. So, in both cases we have $s_t x m_j \in \mathcal{U}$ which means $s_t I m_j \subseteq \mathcal{U}$.

Theorem 2.12: *Let $\mathcal{U}$ be a gr-S-2^A-PS of $\mathcal{D}$. If $IJm_j \subseteq \mathcal{U}$ where I, J are ideals of $\mathcal{A}_e$, and $m_j \in h(\mathcal{D})$. Then there exists $s_t \in S$ such that either $s_t I m_j \subseteq \mathcal{U}$ or $s_t J m_j \subseteq \mathcal{U}$ or $s_t IJ \subseteq Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$.*

Proof: Assume that $IJm_j \subseteq \mathcal{U}$ where *I,J* are ideals of $\mathcal{A}_e$, and $m_j \in h(\mathcal{D})$. Let $s_t \in S$ be the element that satisfies the condition of the *gr-S*-2^A-PS for $\mathcal{U}$. Assume that $s_t I m_j \nsubseteq \mathcal{U}$ and $s_t J m_j \nsubseteq \mathcal{U}$, so $s_t I \nsubseteq (\mathcal{U} :_{\mathcal{A}} m_j)$ and $s_t J \nsubseteq (\mathcal{U} :_{\mathcal{A}} m_j)$. Let $w \in I$ and $x \in J$. Since $s_t I \nsubseteq (\mathcal{U} :_{\mathcal{A}} m_j)$ and $s_t J \nsubseteq (\mathcal{U} :_{\mathcal{A}} m_j)$, then there exist $y \in I$ and $z \in J$ such that $y \in I - (\mathcal{U} :_{\mathcal{A}} m_j)$ with $s_t y m_j \notin \mathcal{U}$ and $z \in J - (\mathcal{U} :_{\mathcal{A}} m_j)$ with $s_t z m_j \notin \mathcal{U}$. By $IJm_j \subseteq \mathcal{U}$, we have that $Izm_j \subseteq \mathcal{U}$, so by Lemma 2.11 there exists $s_t \in S$ such that $s_t I z \subseteq Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$

because $s_t I m_j \nsubseteq \mathcal{U}$ and $s_t z m_j \notin \mathcal{U}$. So, $s_t I\,[J-(\mathcal{U}:_{\mathcal{A}} m_j)] \subseteq Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. Similarly, $s_t[I-(\mathcal{U}:_{\mathcal{A}} m_j)]\,J \subseteq Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. Thus, we have $s_t wz \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$, $s_t yz \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$, and $s_t yx \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. Now, we have that $w+y \in I$ and $x+z \in J$, so $(w+y)\,(x+z)\,m_j \in \mathcal{U}$ (as $IJm \subseteq \mathcal{U}$) which gives that $s_t(w+y)\,m_j \in \mathcal{U}$ or $s_t(x+z)\,m_j \in \mathcal{U}$ or $s_t(w+y)\,(x+z) \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. If $s_t(w+y)\,m_j \in \mathcal{U}$, then $s_t w m_j \notin \mathcal{U}$ because $s_t y m_j \notin \mathcal{U}$, so $w \in I-(\mathcal{U}:_{\mathcal{A}} s_t m_j)$. Since $s_t[I-(\mathcal{U}:_{\mathcal{A}} m_j)]\,J \subseteq Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$, then $s_t wx \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. Similarly, if $s_t(x+z)\,m_j \in \mathcal{U}$, then $s_t wx \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. If $s_t(w+y)(x+z) \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$, then $s_t wx + s_t wz + s_t yx + s_t yz \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$ which gives $s_t wx \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$ because $s_t wz + s_t yx + s_t yz \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. Thus, $s_t IJ \subseteq Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$.

Theorem 2.13: *Let $x_g \in h(\mathcal{D})-\mathcal{U}$ with $(\mathcal{U}:_{\mathcal{A}} x_g)\cap S=\varnothing$ where $g\in\Im$. If $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$, then $(\mathcal{U}:_{\mathcal{A}} x_g)$ is a gr-S-2^A\$-PI of $\mathcal{A}$.*

Proof: Suppose that $\mathcal{U}$ is a *gr-S*-2^A-PS of $\mathcal{D}$ where $s_t \in S$ is the element that satisfy the *gr-S*-2^A-PS condition. Let $c_i d_h v_j \in (\mathcal{U}:_{\mathcal{A}} x_g)$ where $c_i, d_h, v_j \in h(\mathcal{A})$, and $i, j, h \in \Im$, so we have that $c_i d_h\,(v_j x_g) \in \mathcal{U}$ which gives that $s_t c_i\,(v_j x_g) \in \mathcal{U}$ or $s_t d_h\,(v_j x_g) \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$. The first two cases give that $s_t c_i v_j \in (\mathcal{U}:_{\mathcal{A}} x_g)$ or $s_t d_h v_j \in (\mathcal{U}:_{\mathcal{A}} x_g)$. If $s_t c_i d_h \in Gr(\mathcal{U}:_{\mathcal{A}} \mathcal{D})$, then for every $m \in h(\mathcal{D})$ we have that $(s_t c_i d_h)^n\, m \in \mathcal{U}$ where $n \in \mathbb{N}$, which gives that $(s_t c_i d_h)^n\, x_g \in \mathcal{U}$ so $s_t c_i d_h \in Gr(\mathcal{U}:_{\mathcal{A}} x_g)$. Thus, $(\mathcal{U}:_{\mathcal{A}} x_g)$ is a *gr-S*-2^A-PI of $\mathcal{A}$.

Theorem 2.14: *If J is a gr-S-2^A-PI of $\mathcal{A}$, then $Gr(J)$ is a gr-S-2^A-PI of $\mathcal{A}$.*

Proof: Suppose that *J* is a *gr-S*-2^A-PI of $\mathcal{A}$ with $s_t \in S$ as the element that satisfies the condition of the *gr-S*-2^A-PI. Let $r_g z_h m_j \in Gr(J)$, where $r_g, z_h, m_j \in h(\mathcal{A})$. Assume that $s_t r_g m_j \notin Gr(J)$ and $s_t z_h m_j \notin Gr(J)$. Since $r_g z_h m_j \in Gr(J)$, then $(r_g z_h m_j)^n \in J$ where $n \in \mathbb{N}$. Since *J* is a *gr-S*-2^A-PI of $\mathcal{A}$, then we have that $(s_t r_g z_h)^n \in J$ which gives that $s_t r_g z_h \in Gr(J)$. Thus, $Gr(J)$ is a *gr-S*-2^A-PI of $\mathcal{A}$.

Theorem 2.15: *Let $\mathcal{U} \leq_{\Im} \mathcal{D}$, and $q_g \in h(\mathcal{A}) - (\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ with $((\mathcal{U} :_{\mathcal{D}} q_g) :_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$ where $g \in \Im$. If $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$, then $(\mathcal{U} :_{\mathcal{D}} q_g)$ is a gr-S-2^A-PS of $\mathcal{D}$.*

Proof: Suppose that $\mathcal{U}$ is a *gr-S-*2^A-PS of $\mathcal{D}$ where $s_t \in S$ is the element that satisfies the *gr-S-*2^A-PS condition. Let $c_i d_h m_j \in (\mathcal{U} :_{\mathcal{D}} q_g)$ where c_i, $d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{A})$, and $i, j, h \in \Im$, so we have that $c_i d_h\ (m_j q_g) \in \mathcal{U}$ which gives that $s_t c_i\ (m_j q_g) \in \mathcal{U}$ or $s_t d_h\ (m_j q_g) \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. The first two cases holds lead us to have that $s_t c_i m_j \in (\mathcal{U} :_{\mathcal{D}} q_g)$ or $s_t d_h m_j \in (\mathcal{U} :_{\mathcal{D}} q_g)$. If $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, then we have that $(s_t c_i d_h)^n\ \mathcal{D} \subseteq \mathcal{U} \subseteq (\mathcal{U} :_{\mathcal{D}} q_g)$. So, $s_t c_i d_h \in Gr((\mathcal{U} : \mathcal{D} q_g) :_{\mathcal{A}} \mathcal{D})$. Thus, $(\mathcal{U} :_{\mathcal{D}} q_g)$ is a %graded *gr-S-*2^A-PS of $\mathcal{D}$.

Lemma 2.16: *Let $\mathcal{U} \leq_{\Im} \mathcal{D}$. If $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$, then there exists $s_t \in S$ such that $(\mathcal{U} :_{\mathcal{D}} s_t^3) = (\mathcal{U} :_{\mathcal{D}} s^a)$ for all $a \geq 3$.*

Proof: Assume that $\mathcal{U}$ is a *gr-S-*2^A-PS of $\mathcal{D}$ with the element s_t as the element which make the condition of the *gr-S-*2^A-PS satisfies. Let $m = \Sigma_{j \in \Im} m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^4)$. Since $(\mathcal{U} :_{\mathcal{D}} s_t^4) \leq_{\Im} \mathcal{D}$, we have that $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^4)$ for all $j \in \Im$. Let $j \in \Im$, then $s_t^4 m_j = s_t^2 s_t^2 m_j \in \mathcal{U}$, where $s_t^2 \in h(\mathcal{A})$ and $m_j \in h(\mathcal{D})$, hence $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^3)$ or $s_t^5 \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. If $s_t^5 \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, then $(s_t^5)^b = s_t^{5b} \in \in$ where $b \in \mathbb{N}$, since $S \subseteq h(\mathcal{A})$ be a m.c.s of $\mathcal{A}$, then $s_t^{5b} \in (\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S$ which is contradiction. Since we get that $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^3)$, then $(\mathcal{U} :_{\mathcal{D}} s_t^4) \subseteq (\mathcal{U} :_{\mathcal{D}} s_t^3)$. Now, since $(\mathcal{U} :_{\mathcal{D}} s_t^3) \subseteq (\mathcal{U} :_{\mathcal{D}} s_t^4)$, then $(\mathcal{U} :_{\mathcal{D}} s_t^3) = (\mathcal{U} :_{\mathcal{D}} s_t^4)$. Now, Assume that $(\mathcal{U} :_{\mathcal{D}} s_t^3) = (\mathcal{U} :_{\mathcal{D}} s_t^n)$ for all $n < a$. Let $m = \Sigma_{j \in \Im} m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^a)$, then $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^a)$ for all $j \in \Im$. Let $j \in \Im$, then then $s_t^a m_j = s_t^2 s_t^{a-2} m_j \in \mathcal{U}$, where s_t^2, $s_t^{a-2} \in h(\mathcal{A})$ and $m_j \in h(\mathcal{D})$, hence $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^3)$ or $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^{a-1})$ or $s_t^{a+1} \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Similarly, $s_t^{a+1} \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ leads us to a contradiction. Now, we have $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^3) \cup (\mathcal{U} :_{\mathcal{D}} s_t^{a-1}) = (\mathcal{U} :_{\mathcal{D}} s_t^3)$. Since $m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^3)$, then $(\mathcal{U} :_{\mathcal{D}} s_t^a) \subseteq (\mathcal{U} :_{\mathcal{D}} s_t^3)$. Since $(\mathcal{U} :_{\mathcal{D}} s_t^3) \subseteq (\mathcal{U} :_{\mathcal{D}} s_t^a)$, then $(\mathcal{U} :_{\mathcal{D}} s_t^3) = (\mathcal{U} :_{\mathcal{D}} s_t^a)$. Therefore, $(\mathcal{U} :_{\mathcal{D}} s_t^3) = (\mathcal{U} :_{\mathcal{D}} s^a)$ for all $a \geq 3$.

Lemma 2.17: *Let $\mathcal{U} \leq_{\Im} \mathcal{D}$. If $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$, then there exists $s_t \in S$ such that $(\mathcal{U} :_{\mathcal{A}} s_t^3 \mathcal{D}) = (\mathcal{U} :_{\mathcal{A}} s_t^a \mathcal{D})$ for all $a \geq 3$.*

Proof: Follows directly from (i)

Theorem 2.19: *Let $\mathcal{U} \leq_{\Im} \mathcal{D}$ with $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap \varnothing$. $\mathcal{U}$ is a gr-S-2^A-PS if and only if $(\mathcal{U} :_{\mathcal{D}} s_t)$ is a gr-2^A-PS of $\mathcal{D}$ for some $s_t \in S$.*

Proof: Let $\mathcal{U}$ be a *gr-S-*2^A-PS of $\mathcal{D}$ and $s_t \in S$ is the element that satisfies the condition of the *gr-S-*2^A-PS. By Lemma 2.16, we have $(\mathcal{U} :_{\mathcal{D}} s_t^3) = (\mathcal{U} :_{\mathcal{D}} s_t^a)$ for all $a \geq 3$. Now our target is to show that $L = (\mathcal{U} :_{\mathcal{D}} s_t^6) = (\mathcal{U} :_{\mathcal{D}} s_t^3)$ is gr-2^A-PS of $\mathcal{D}$. Let $c_i d_h m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^6)$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \Im$. Then $s_t^6 c_i d_h m_j = (s_t^2 c_i)\ (s_t^2 d_h)\ (s_t^2 m_j) \in \mathcal{U}$, where $(s_t^2 c_i)$, $(s_t^2 d_h) \in h(\mathcal{A})$ and $(s_t^2 m_j) \in h(\mathcal{D})$. Since $\mathcal{U}$ is *gr-S-*2^A-PS of $\mathcal{D}$, then either $s_t^5 c_i m_j \in \mathcal{U}$, $s_t^5 d_h m_j \in \mathcal{U}$ or $s_t^5 c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. The first two cases give that $c_i m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^5)$ or $d_h m_j \in (\mathcal{U} :_{\mathcal{D}} s_t^5)$. If $s_t^5 c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, then $(s_t^5 c_i d_h)^n \in (\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, where $n \in \mathbb{N}$. Now, we have $(c_i d_h)^n \in (\mathcal{U} :_{\mathcal{A}} s_t^{5n}\mathcal{D})$, by Lemma 2.17, $(\mathcal{U} :_{\mathcal{A}} s_t^{5n}\mathcal{D}) = (\mathcal{U} :_{\mathcal{A}} s_t^3\mathcal{D}) = (\mathcal{U} :_{\mathcal{A}} s_t^6\mathcal{D}) = ((\mathcal{U} :_{\mathcal{D}} s_t^6) :_{\mathcal{A}} \mathcal{D})$, so we have $(c_i d_h)^n \in ((\mathcal{U} :_{\mathcal{D}} s_t^6) :_{\mathcal{A}} \mathcal{D})$ which means $c_i d_h \in Gr(((\mathcal{U} :_{\mathcal{D}} s_t^6) :_{\mathcal{A}} \mathcal{D}))$. Thus, $(\mathcal{U} :_{\mathcal{D}} s_t^6)$ is a *gr-*2^A-PS of $\mathcal{D}$. Now suppose that $(\mathcal{U} :_{\mathcal{D}} s_t)$ is a *gr-*2^A-PS for some $s_t \in S$ with $t \in \Im$. Let $c_i d_h m_j \in \mathcal{U}$ where c_i, $d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \Im$. Since $(\mathcal{U} :_{\mathcal{D}} s_t)$ is *gr-*2^A-PS of $\mathcal{D}$, then either $c_i m_j \in (\mathcal{U} :_{\mathcal{D}} s_t)$, $d_h m_j \in (\mathcal{U} :_{\mathcal{D}} s_t)$ or $c_i d_h \in Gr(\mathcal{U} :_{\mathcal{D}} s_t)$. So $s_t c_i m_j \in \mathcal{U}$, $s_t d_h m_j \in \mathcal{U}$ or $s_t c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Thus, $\mathcal{U}$ is *gr-S-*2^A-PS of $\mathcal{D}$.

Theorem 2.19: *Let $\mathcal{U} \leq_{\Im} \mathcal{D}$, and $(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ be a graded prime ideal of $\mathcal{A}$ with $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S = \varnothing$. $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$ if there exists $s_t \in S$ such that for all l_z, $v_g \in h(\mathcal{D})$, if $(\mathcal{U} :_{\mathcal{A}} l_z) - ((\mathcal{U} :_{\mathcal{A}} s_t^2 v_g) \cup Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})) \neq \varnothing$, then $\mathcal{U} = (\mathcal{U} + Rl_z) \cap (\mathcal{U} + Rs_t v_g)$.*

Proof: Assume that exists $s_t \in S$ such that for all l_z, $v_g \in h(\mathcal{D})$, if $(\mathcal{U} :_{\mathcal{A}} l_z) - ((\mathcal{U} :_{\mathcal{A}} s_t^2 v_g) \cup Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})) \neq \phi$, then $\mathcal{U} = (\mathcal{U} + Rl_z) \cap (\mathcal{U} + Rs_t v_g)$. Let $c_i d_h m_j \in \mathcal{U}$ where c_i, $d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \Im$. Assume that $s_t^2 d_h m_j \notin \mathcal{U}$ and $s_t^2 c_i d_h \notin Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$, our goal is to show that $s_t^2 c_i m_j \in \mathcal{U}$. Now, $d_h \in (\mathcal{U} :_{\mathcal{A}} c_i m_j) - ((\mathcal{U} :_{\mathcal{A}} s_t^2 m_j) \cup Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D}))$. By the assumption we have that $\mathcal{U} = (\mathcal{U} + Rc_i m_j) \cap (\mathcal{U} + Rs_t m_j)$ so $\in \mathcal{U}$. Thus, $\mathcal{U}$ is a *gr-S-*2^A-PS of $\mathcal{D}$.

Definition 2.20: $\mathcal{D}$ is called graded S-concellative module if there exists $s_t \in S$ such that whenever if $c_i m_j = c_i v_g$ where m_j, $v_g \in h(\mathcal{D})$ and

$c_i \in h(\mathcal{A})$, then $m_j = s_t v_g$ and $v_g = s_t m_j$. Moreover, A graded submodule J of $\mathcal{D}$ is said to be graded pure if for all $r_q \in h(\mathcal{A})$, $r_q h(\mathcal{D}) \cap J = r_q J$.

Theorem 2.21: *Let $\mathcal{U}$ be a proper graded submodule of $\mathcal{D}$ with $(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S = \emptyset$. If $\mathcal{D}$ is a graded S-cancellative module and $\mathcal{U}$ is a graded pure submodule, then $\mathcal{U}$ is a gr-S-2^A-PS with $Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) = \{0\}$.*

Proof: Let $\mathcal{U}$ be a graded pure submodule, and $\mathcal{D}$ be a graded S-cancellative module and $s_t \in S$ be the element that satisfies the cancellative condition. Let $c_i d_h m_j \in \mathcal{U}$ where $c_i, d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, j, h \in \mathfrak{I}$. But $\mathcal{U}$ is a graded pure submodule, so $c_i d_h m_j \in \mathcal{U} \cap c_i d_h \mathcal{D} = c_i d_h \mathcal{U}$. Now, for some $v_g \in \mathcal{U} \cap h(\mathcal{D})$, $c_i d_h m_j = c_i d_h v_g$, so $d_h m_j = s_t d_h v_g$ and $d_h v_g = s_t d_h m_j$\$, which gives that $s_t d_h m_j \in \mathcal{U}$, so $\mathcal{U}$ is a *gr*-*S*-2^A-PS.

Now, let $z_\alpha \neq 0 \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$ where $z_\alpha \in h(\mathcal{A})$ and $\alpha \in \mathfrak{I}$. But $\mathcal{U}$ is a proper graded submodule of $\mathcal{D}$, so there is $m = \Sigma_{g \in \mathfrak{I}} m_g \in \mathcal{D} - \mathcal{U}$ which means that there exists $j \in \mathfrak{I}$ such that $m_j \in h(\mathcal{D}) - \mathcal{U} \subseteq \mathcal{D} - \mathcal{U}$. Then, for some $n \in \mathbb{N}$, $z_\alpha^n m_j \in z_\alpha^n \mathcal{D} \cap \mathcal{U} \subseteq z_\alpha^n \mathcal{D} \cap \mathcal{U} = z_\alpha^n \mathcal{U}$, which means that there is $c_i \in \mathcal{U}$ such that $z_\alpha^n m_j = z_\alpha^n c_i$. For some $s_t \in S$, we have that $m_j = s_t c_i$ and $c_i = s_t m_j$. Since $c_i \in \mathcal{U}$, $m_j = s_t c_i$, and $\mathcal{U}$ is a graded submodule of $\mathcal{D}$, then $m_j \in \mathcal{U}$ which is a contradiction. Thus, $Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) = \{0\}$.

Theorem 2.22: Let $\mathcal{D}, \mathcal{D}'$ be graded $\mathcal{A}$-modules and $L : \mathcal{D} \to \mathcal{D}'$ be a graded $\mathcal{A}$-homomorphism. If $\mathcal{U}'$ is a *gr*-2^A-PS of $\mathcal{D}'$ and $(L^{-1}(\mathcal{U}') :_{\mathcal{A}} \mathcal{D}) \cap S = \emptyset$, then $L^{-1}(\mathcal{U}')$ is a *gr*-*S*-2^A-PS of $\mathcal{D}$.

Proof: Assume that $\mathcal{U}'$ is a *gr*-*S*-2^A-PS of $\mathcal{D}'$ and $(L^{-1}(\mathcal{U}') :_{\mathcal{A}} \mathcal{D}) \cap S = \emptyset$. Let $c_i d_h m_j \in L^{-1}(\mathcal{U}')$ where c_i, $d_h \in h(\mathcal{A})$, $m_j \in h(\mathcal{D})$, and $i, h, j \in \mathfrak{I}$. That gives us $L(c_i d_h m_j) = c_i d_h L(m_j) \in \mathcal{U}'$, so there is $s_t \in S$ where $t \in \mathfrak{I}$ such that $s_t c_i L(m_j) = L(s_t c_i m_j) \in \mathcal{U}'$ or $s_t d_h L(m_j) = L(s_t d_h m_j) \in \mathcal{U}'$ or $s_t c_i d_h \in Gr(\mathcal{U}' :_{\mathcal{A}} \mathcal{D}')$. By the first two cases we have $s_t c_i m_j \in L^{-1}(\mathcal{U}')$ or $s_t d_h m_j \in L^{-1}(\mathcal{U}')$. If $s_t c_i d_h \in Gr(\mathcal{U}' :_{\mathcal{A}} \mathcal{D}')$, then $s_t c_i d_h \in Gr(L^{-1}(\mathcal{U}') :_{\mathcal{A}} \mathcal{D})$. Therefore, $L^{-1}(\mathcal{U}')$ is a *gr*-*S*-2^A-PS of $\mathcal{D}$.

Theorem 2.23: *Let $\mathcal{D}, \mathcal{D}'$ be graded $\mathcal{A}$-modules and $L : \mathcal{D} \to \mathcal{D}'$ be a graded $\mathcal{A}$-epimorphism. If $\mathcal{U}$ is a gr-S-2^A-PS of $\mathcal{D}$ and $ker(L) \subseteq \mathcal{U}$, then $L(\mathcal{U})$ is a gr-S-2^A-PS of $\mathcal{D}'$.*

Proof: Assume that $\mathcal{U}$ is a *gr-S*-2^A-PS of $\mathcal{D}$ and $ker(L) \subseteq \mathcal{U}$. Firstly, we want to show that $(L(\mathcal{U}) :_{\mathcal{A}} \mathcal{D}') \cap S = \emptyset$. Suppose not, then there exists $s_t \in (L(\mathcal{U}) :_{\mathcal{A}} \mathcal{D}') \cap S$ which means that $s_t\mathcal{D}' \subseteq L(\mathcal{U})$. Since L is a graded $\mathcal{A}$-epimorphism, then $s_t\mathcal{D}' = s_tL(\mathcal{D}) = L(s_t\mathcal{D}) \subseteq L(k)$, so $s_t\mathcal{D} \subseteq s_t\mathcal{D} + ker(L) \subseteq \mathcal{U}$, so $s_t \in (\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \cap S$ which is a contradiction. Let $c_i d_h m'_j \in L(\mathcal{U})$ where c_i, $d_h \in h(\mathcal{A})$, $m'_j \in h(\mathcal{D}')$, and $i, h, j \in \Im$. Since L is a graded $\mathcal{A}$-epimorphism, then there exists $m_j \in h(\mathcal{D})$ and $k \in \mathcal{U}$ such that $m'_j = L(m_j)$ and $c_i d_h m'_j = L(k)$. Now, we have that $L(k) = c_i d_h L(m_j) = L(c_i d_h m_j)$, so $L(k - c_i d_h m_j) = 0$ which means that $k - c_i d_h m_j \in ker(L) \subseteq \mathcal{U}$. Since $\mathcal{U} \leq_{\Im} \mathcal{D}$, then $c_i d_h m_j \in \mathcal{U}$. Now, as $\mathcal{U}$ is a *gr-S*-2^A-PS, there exists $a_g \in S$ where $g \in \Im$ such that either $a_g c_i m_j \in \mathcal{U}$ or $a_g d_h m_j \in \mathcal{U}$ or $a_g c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D})$. Now, we have that either $L(a_g c_i m_j) = a_g c_i L(m_j) = a_g c_i m'_j \in L(\mathcal{U})$ or $L(a_g d_h m_j) = a_g d_h L(m_j) = a_g d_h m'_j \in L(\mathcal{U})$ or $a_g c_i d_h \in Gr(\mathcal{U} :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(L(\mathcal{U}) :_{\mathcal{A}} \mathcal{D}')$. Thus, $L(\mathcal{U})$ is a *gr-S*-2^A-PS of $\mathcal{D}'$.

As am immediate consequence of Theorem 2.22 and Theorem 2.23, we have the following corollaries.

Corollary 3.3: If $\mathcal{U}'$ is a *gr-S*-2^A-PS of $\mathcal{D}$ with $(\mathcal{U}' :_{\mathcal{A}} A) \cap S = \emptyset$, then $A \cap \mathcal{U}'$ is a *gr-S*-2^A-PS of A.

Corollary 3.3: Let $A \subseteq \mathcal{U}$ be a graded submodules of $\mathcal{D}$. Then $\mathcal{U}$ is a *gr-S*-2^A-PS of $\mathcal{D}$ if and only if $\mathcal{U} / A$ is a graded *gr-S*-2^A-PS of $\mathcal{D} / A$.

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Received November, 2022

Revised February, 2023