Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 3, pp. 797–807
DOI : 10.47974/JDMSC-2000

Minimum cycle basis of the α–product of P_m with P_n

Maref Y. M. Alzoubi *
Talal Ali Al-Hawary †
Department of Mathematics
Yarmouk University
Irbid 21163
Jordan

Abstract

For a graph, its cycle space has a lot of bases, out of which we are interested in finding a basis of minimum length. A cycle basis length is the sum of all cycles lengths. In this work, we give cycle basis of the α–product of different Paths of minimal length.

Subject Classification: *05C38, 15A03.*

Keywords: *α–product, Cycle basis, Minimum cycle basis.*

1. Introduction

In this paper, only finite, connected and simple graphs are considered. For the indefinite terms, see [12] and [18]. Any graph $G=(V(G),E(G))$ corresponds to a vector space of all subsets of $E(G)$ called the edge space W_E. W_E is an $|E(G)|$-dimensional vector pace over $\mathbf{Z}_2$ such that for all $q,w\in W_E$, vector addition $q\oplus w$ is defined as the symmetric difference of q and w and scalar multiplication $1\cdot q=q$ and $0\cdot q=\varnothing$. A cycle space, $\mathfrak{C}(G)$, of G is a subspace of W_E that contains all cycles along with edge disjoint union of them and its dimension is $\dim(\mathfrak{C}(G)) = 1+|E(G)| - |V(G)|$. A cycle basis of G is a basis of $\mathfrak{C}(G)$. A cycle basis with minimal length of G is a cycle basis of minimum length. The basis's length is the sum of all of its cycles's lengths. The number of a cycle edges is called the

* *E-mail:* `maref@yu.edu.jo (Corresponding Author)`
† *E-mail:* `talalhawary@yahoo.com`

length of a cycle. This is an important field of research in the area of independent electric circuits, see for example [19].

In general, cycle bases with minimal length are not very well obedient under the operations of graphs since we might not be able to extend the subgraphs' properties to the original graphs. Cycle spaces bases have varies applications in electrical engineering [19], biology, chemistry and structural analysis [16] and periodic timelabiling [20]. Such applications require cycle bases that has special properties [15]. For more on the preceding notions, the reader is referred to [1], [13], [14], [17], [21], [22].

There are lots of products on graphs such as the Cartesian, strong direct, lexicographic and semi-strong products. In [15], Imrich and Stadler invented cycle bases with minimal length for the Cartesian, strong and direct products of graphs. Alzoubi [11], designed cycle bases with minimal length for the operation of strong product of a special case of paths and cycles. Also in [10], Alzoubi constructed cycle bases with minimal length for the semi-strong product of cycles and paths with distinct kinds of ladders. Alzoubi and Al-Eid [9], constructed cycle bases with minimal length for the semi-strong product of some basic graphs. In [15], Daoud and others introduced the following notion of $\alpha -$product of graphs:

Let $G_1 = (V_1, E_1)$ be a graph with a-vertices and e-edges and $G_2 = (V_2, E_2)$ be a graph with b vertices and d edges. The the $\alpha -$product of G_1 and G_2 denoted by $G_1 \alpha G_2$ is a graph with $n = ab$ vertices and the vertex set of $G_1 \alpha G_2$ is $V(G_1 \alpha G_2) = V_1 \times V_2$ and two vertices (a_1, a_2) and (b_1, b_2) are adjacent if the following two conditions hold:

1. a_1 and b_1 are adjacent in G_1 and a_2 and b_2 are not adjacent in G_2.

2. a_1 is not joint to b_1 in G_1 and a_2 is joint to b_2 in G_2.

In fact, $G_1 \alpha G_2$ is isomorphic to $G_2 \alpha G_1$ and its number of edges is $m(G_1 \alpha G_2) = a^2 d + b^2 e - 4ed$.

These products were also generalized to fuzzy graphs in [2], [3], [4], [5], [6], [7], [8].

Our goal, in this article, is to construct a cycle bases of minimum length for the cycle space of the $\alpha -$product of paths.

2. Cycle basis of minimum length of $P_2 \alpha P_n$

It is clear that $P_2 \alpha P_2$ is a 4-cycle and so its cycle basis of minimum length consists of 4-cycle. For the graph $P_2 \alpha P_3$ it is isomorphic to $K_{3,3}$

which is nonplanar and its cycle basis of minimum length consists of 4 rectangles (4-cycles). Also $P_3\alpha P_3$ is a graph of girth 4 ad its cycle basis of minimum length consists of 4-cycles.

Lemma 1: *The cycle basis with minimum length of the graph* $P_2\alpha P_4$ *consists of rectangles and triangles*

Proof: Let $P_2 = \delta_1\delta_2$ and $P_4 = u_1u_2u_3u_4$. Clearly $\dim\mathfrak{C}(P_2\alpha P_4) = 9$. Define $\mathfrak{B}(P_2\alpha P_4) = S_1 \cup S_2 \cup S_3$ where

$$S_1 = \{(\delta_1,u_1)(\delta_1,u_2)(\delta_1,u_3)(\delta_2,u_1)(\delta_1,u_1),(\delta_2,u_1)(\delta_1,u_3)(\delta_1,u_4)(\delta_2,u_1),\\ (\delta_2,u_1)(\delta_2,u_2)(\delta_1,u_4)(\delta_2,u_1)\},$$

$$S_2 = \{(\delta_1,u_1)(\delta_2,u_1)(\delta_2,u_2)(\delta_2,u_3)(\delta_1,u_1),(\delta_1,u_1)(\delta_2,u_3)(\delta_2,u_4)(\delta_1,u_1),\\ (\delta_1,u_1)(\delta_2,u_4)(\delta_1,u_2)(\delta_1,u_1)\}$$

and

$$S_3 = \{(\delta_1,u_i)(\delta_1,u_{i+1})(\delta_2,u_i)(\delta_i,u_i) : i = 1,2,3\}.$$

Consider the subrapht $G(S_1)$ of $P_2\alpha P_4$ having edge set $\cup E(S_1)$ and vertex set $V(P_2\alpha P_4)$. Then $G(S_1)$ is a planar subgraph with S_1 as its basis because S_1 consists of boundaries of the finite faces of $G(S_1)$. Similarly, $G(S_2)$ is a subgraph of $P_2\alpha P_4$ that has S_2 as its basis. Now S_1 and S_2 are linearly independent (simply, lin. ind.) sets of cycles in $P_2\alpha P_4$. Since every cycle in S_2 contains an edge from the set $E = \{(\delta_1,u_1)\ (\delta_2,u_3),\ (\delta_1,u_1)\ (\delta_2,u_4),\ (\delta_1,u_2)\ (\delta_2,u_4)\}$ which does not occur in any cycle of S_1, then the cycles of S_2 are lin. ind. with the cycles of S_1. Therefore, $S_1 \cup S_2$ is a lin. ind. set of cycles. S_3 is the basis of the Cartesian product of $P_2 \times P_4$ (see [15]), so S_3 is lin. ind. Finally, the cycles in S_3 have two edges $(\delta_1,u_2)\ (\delta_2,u_2)$ and $(\delta_1,u_3)\ (\delta_2,u_3)$ that do not occur in any cycle of $S_1 \cup S_2$ and so $\mathfrak{B}(P_2\alpha P_4)$ is a lin. ind. set of cycles in $\mathfrak{C}(P_2\alpha P_4)$ and since $|\mathfrak{B}(P_2\alpha P_4)| = \dim\mathfrak{C}(P_2\alpha P_4)$, $\mathfrak{B}(P_2\alpha P_4)$ is a basis for $\mathfrak{C}(P_2\alpha P_4)$ that contains triangles and rectangles and since it contains all the 3 cycles of $P_2\alpha P_4$, it is a minimum cycle basis of $P_2\alpha P_4$.

Lemma 2: *The cycle basis of minimum length of tha graph* $P_2\alpha P_5$ *consists of rectangles and triangles*

Proof: Let $P_2 = \delta_1\delta_2$ and $P_5 = u_1u_2u_3u_4u_5$. Clearly $\dim\mathfrak{C}(P_2\alpha P_5) = 16$. Let

$$S_1 = \{(\delta_1,u_1)(\delta_2,u_3)(\delta_2,u_4)(\delta_1,u_1),(\delta_1,u_1)(\delta_2,u_4)(\delta_2,u_5)(\delta_1,u_1),$$
$$(\delta_1,u_1)(\delta_1,u_2)(\delta_2,u_5)(\delta_1,u_1),(\delta_1,u_2)(\delta_1,u_3)(\delta_2,u_5)(\delta_1,u_2)\},$$

$$S_2 = \{(\delta_2,u_1)(\delta_1,u_3)(\delta_1,u_4)(\delta_2,u_1),(\delta_2,u_1)(\delta_1,u_4)(\delta_1,u_5)(\delta_2,u_1),$$
$$(\delta_2,u_1)(\delta_2,u_2)(\delta_1,u_5)(\delta_2,u_1),(\delta_2,u_2)(\delta_2,u_3)(\delta_1,u_5)(\delta_2,u_2)\},$$

S_3 is the set of the cycles

$$S_{31} = (\delta_1,u_1)(\delta_1,u_2)(\delta_2,u_4)(\delta_1,u_1)$$

$$S_{32} = (\delta_1,u_2)(\delta_2,u_4)(\delta_2,u_5)(\delta_1,u_2)$$

$$S_{33} = (\delta_1,u_5)(\delta_2,u_2)(\delta_1,u_4)(\delta_1,u_5)$$

$$S_{34} = (\delta_1,u_4)(\delta_2,u_1)(\delta_2,u_2)(\delta_1,u_4)$$

and

$$S_4 = \{(\delta_1,u_i)(\delta_1,u_{i+1})(\delta_2,u_{i+1})(\delta_2,u_i)(\delta_1,u_i) : i = 1,\dots,4\}.$$

The graph $G(S_1)$ is planar and the cycles of S_1 are the boundaries of its finite faces, so S_1 is a basis for $\mathfrak{C}(G(S_1))$ and thus is a lin. ind. set of cycles in $\mathfrak{C}(P_2\alpha P_5)$. Similarly, $G(S_2)$ is planar and S_2 is a basis for $\mathfrak{C}(G(S_2))$ and so S_2 is a lin. ind. set of cycles in $\mathfrak{C}(P_2\alpha P_5)$. Since every cycle of S_1 contains one or two edges from the set of edges

$$\{(\delta_1,u_1)(\delta_2,u_3),(\delta_1,u_1)(\delta_2,u_4),(\delta_1,u_1)(\delta_2,u_5),(\delta_1,u_2)(\delta_2,u_5),(\delta_1,u_3)(\delta_2,u_5)\}$$

that do not belong to any cycle in S_1, then the cycles of S_1 are lin. ind. with all cycles of S_2 and thus $S_1 \cup S_2$ is a lin. ind. set of cycles. The 3 cycles S_{31} and S_{32} are lin. ind., also S_{33} and S_{34} are lin. ind. and since S_{31} and S_{32} are edge disjoint with S_{33} and S_{34}, we conclude that S_3 is lin. ind.. Since the cycles of S_3 have of the two edges (δ_1,u_2) (δ_2,u_4), (δ_1,u_4) (δ_2,u_2) that can not occur as a linear combination of the cycles in $S_1 \cup S_2$, so S_3 is lin. ind. with $S_1 \cup S_2$, thus $S_1 \cup S_2 \cup S_3$ is lin. ind. S_4 is the basis of $P_2 \times P_5$ (the Cartesian product, see [15]). Every cycle in $S_3 \cup S_4$ contains an edge from the set

$$\{(\delta_1,u_2)(\delta_2,u_4),(\delta_2,u_2)(\delta_1,u_4),(\delta_1,u_2)(\delta_2,u_2),(\delta_1,u_4)(\delta_2,u_4),(\delta_1,u_3)(\delta_2,u_3)\}$$

that does not belong to any cycle in $S_1 \cup S_2$ and so the cycles of $S_1 \cup S_2 \cup S_3 \cup S_4$ are lin. ind. and hence $\mathfrak{B}(P_2\alpha P_5) = S_1 \cup S_2 \cup S_3 \cup S_4$ is a lin. ind. set of cycles of order $16 = \dim\mathfrak{C}$ $(P_2\alpha P_5)$. Therefore, it is a basis for

$\mathfrak{C}(P_2\alpha P_5)$ that contains all number of lin. ind. 3-cycles in $P_2\alpha P_5$ and some 4-cycles which makes it a minimum cycle basis for $\mathfrak{C}(P_2\alpha P_5)$.

Lemma 3: *For $n \geq 4$; the cycle basis of minimum length of the graph $P_2\alpha P_n$ consists of rectangles and triangles*

Proof: We use induction on *n* taking into account that our proof will split into two parts depending on whether *n* is even or odd. We assume that $P_2 = \delta_1\delta_2$.

Case 1: If $n = 2k$ and *k* is at least 2, then $P_2\alpha P_4$ is true by Lemma 1. Assuming that the statement is true for $n = 2k$, for the case $n = 2k+2$, we assume $P_{2k+2} = u_1u_2 \ldots u_{2k+1}u_{2k+2}$ which has $2k+1$ edges and $2k+2$ vertices. The graph $P_2\alpha P_{2k+2}$ has $4k+4$ vertices and $4k+3$ edges. Now deleting the varices (δ_1, u_1), (δ_1, u_{2k+2}), (δ_2, u_1), (δ_2, u_{2k+2}) from the graph $P_2\alpha P_{2k+2}$, we end up with the graph $P_2\alpha u_2u_3 \ldots u_{2k+1}$ which is isomorphic to $P_2\alpha P_{2k}$ and thus by the induction step $P_2\alpha u_2u_3 \ldots u_{2k}$ has a cycle basis of minimum length consisting of rectangles and triangles with dimension $|\mathfrak{B}(P_2\alpha P_{2k})|$ equals $4k^2 - 4k + 1$. It's clear that dim $(\mathfrak{C}(P_2\alpha P_{2k+2})$ = $4k^2 + 4k + 1$ and so dim $(\mathfrak{C}(P_2\alpha P_{2k+2})$ − dim $(\mathfrak{C}(P_2\alpha P_{2k})$ = $8k$. Let $\mathfrak{B} = \mathfrak{B}$ $(P_2\alpha u_2u_3 \ldots u_{2k+1})$. Then our aim is to add $8k$ from $P_2\alpha P_{2k+2}$ with maximum number of 3-cycles to get a minimum cycle basis for $\mathfrak{C}(P_2\alpha P_{2k+2})$. Let

$B_1 = \{(\delta_1, u_1)(\delta_2, u_i)(\delta_2, u_{i+1})(\delta_1, u_1) : 3 \leq i \leq 2k+1\}$
$\cup\{(\delta_1, u_i)(\delta_1, u_{i+1})(\delta_2, u_{2k+1})(\delta_1, u_i) : 1 \leq i \leq 2k-1\}$. Then $|B_1| = 4k-2$. Let
$B_2 = \{(\delta_2, u_1)(\delta_1, u_i)(\delta_1, u_{i+1})(\delta_2, u_1) : 3 \leq i \leq 2k+1\}$
$\cup\{(\delta_1, u_{2k+2})(\delta_2, u_i)(\delta_2, u_{i+1})(\delta_1, u_{2k+2}) : 1 \leq i \leq 2k-1\}$.

Then $|B_2| = 4k-2$. Let $B = B_1 \cup B_2$. Then $|B| = 8k-4$. Now define $\mathfrak{B}(P_2\alpha P_{2k+2})$ as the disjoint union of *B* and $B(P_2\alpha P_{2k})$. Then $G(B_2)$ is a planar subgraph that has the cycles of B_2 as the boundaries of its finite faces, so the cycles of B_2 are lin. ind.. Moreover, every cycle in B_2 has an edge from the set

$$E_2 = \{(\delta_2, u_1)(\delta_1, u_i) : 3 \leq i \leq 2k+2\} \cup \{(\delta_1, u_{2k+2})(\delta_2, u_i) : 2 \leq i \leq 2k\}$$

that does not belong to any cycle in B_1 and so the cycles in B_2 are lin. ind. with all the cycles in B_1 and thus $B_1 \cup B_2$ is a lin. ind. set of cycles. Let

$$B_3 = \{(\delta_1, u_1)(\delta_1, u_2)(\delta_2, u_2)(\delta_2, u_1)(\delta_1, u_1), \\ (\delta_1, u_{2k+1})(\delta_1, u_{2k+2})(\delta_2, u_{2k+2})(\delta_2, u_{2k+1})(\delta_1, u_{2k+1})\},$$

It is clear that the cycles of B_3 are edge disjoint and each of them contains an edge of the form $(\delta_1, u_1)(\delta_2, u_1)$, $(\delta_1, u_{2k+2})(\delta_2, u_{2k+2})$ that does not occur in any cycle of $B_1 \cup B_2$. Then $B_1 \cup B_2 \cup B_3$ is lin. ind. Let

$$B_4 = \{(\delta_1, u_1)(\delta_1, u_2)(\delta_2, u_{2k+1})(\delta_1, u_1), \\ (\delta_2, u_2)(\delta_1, u_{2k+1})(\delta_1, u_{2k+2})(\delta_2, u_2)\}.$$

Each of the 2-cycles in B_4 has only one edge from $P_2\alpha u_2 u_3 ... u_{2k+1}$ and two edges from $P_2\alpha P_{2k+2} \setminus P_2\alpha u_2 u_3 \ ... \ u_{2k+1}$ that makes the cycles of B_4 lin. ind. with all the cycles in $B_1 \cup B_2 \cup B_3$. Therefore, $B_1 \cup B_2 \cup B_3 \cup B_4$ is lin. ind.. Now every cycle in $B_1 \cup B_2 \cup B_3 \cup B_4$ contains an edge from the set

$$E_1 = E_2 \cup \{(\delta_1, u_1)(\delta_2, u_i) : 3 \le i \le 2k+2\} \cup \{(\delta_2, u_{2k+2})(\delta_1, u_i) : 2 \le i \le 2k\} \\ \cup \{(\delta_1, u_1)(\delta_2, u_1), (\delta_1, u_{2k+2})(\delta_2, u_{2k+2})\}$$

that does not belong to any cycle in $\mathfrak{B}$ and thus $\mathfrak{B}(P_2\alpha P_{2k+2}) = \mathfrak{B} \cup B_1 \cup B_2 \cup B_3 \cup B_4$ is a lin. ind. set of cycles and $|B(P_2\alpha P_{2k+2})| = \dim \ \mathfrak{C}(P_2\alpha P_{2k+2})$. Hence $\mathfrak{B}(P_2\alpha P_{2k+2})$ is a minimum cycle basis for $\mathfrak{C}(P_2\alpha P_{2k+2})$ that contains the maximim number of lin. ind. 3-cycles in $P_2\alpha P_n$ in addition to the essential 4-cycles.

Case 2: If $n = 2k+1$ and $k$is at least 2, then $P_2\alpha P_n$ is true by Lemma 2. Assuming that the statement is true for $n = 2k+1$ vertices, for the case $n = 2k+3$, following the same technique in Case 1 and deleting the 4 vertices (δ_1, u_1), (δ_1, u_{2k+3}), (δ_2, u_1), (δ_2, u_{2k+3}) from $P_2\alpha P_{2k+3}$, we end up with the graph $P_2\alpha u_2 u_3 ... u_{2k+2}$ and by the induction step, this graph has a basis consisting of triangles and rectangles. For the rest of the proof, we repeat the arguments in Case 1 by defining the sets B_i where $i = 1, 2, 3, 4$ that have $8k+4$ cycles and their union with $\mathfrak{B}(P_2\alpha u_2 u_3 ... u_{2k+2})$ give the required minimum cycle basis for $\mathfrak{C}(P_2\alpha P_{2k+3})$ that consists of maximal set of triangles in $P_2\alpha P_{2k+3}$ and the essential rectangles.

3. Cycle basis of minimum length of $P_m\alpha P_n$

In this section, we discuss cycle basis of minimum length of $P_m\alpha P_n$ for $m \ge 4$. We begin by discussing cycle basis of minimum length of $P_3\alpha P_n$ and $P_4\alpha P_n$.

Theorem 4: *For $n \ge 4$; the cycle basis of minimum length of the graph $P_3\alpha P_n$ consists of rectangles and triangles.*

Proof: Let $P_3 = \delta_1\delta_2\delta_3$ and $P_n = u_1u_2...u_n$ where $n \geq 4$ and let $B_1 = \mathfrak{B}(\delta_1\delta_2\alpha P_n)$, $B_2 = \mathfrak{B}(\delta_2\delta_3\alpha P_n)$ be the bases of $\delta_1\delta_2\alpha P_n$ and $\delta_2\delta_3\alpha P_n$, respectively. Then clearly $B_1 \cup B_2$ is a lin. ind. set of cycles in $\mathfrak{C}(P_3\alpha P_n)$. Now $E(B_1) \cap E(B_2)$ = $\{(\delta_2, u_i)\ (\delta_2, u_{i+1}) : i = 1, 2, ..., n-1\}$ and $\mathfrak{B}(P_3\alpha P_n)$ = $B_1 \cup B_2 \cup S$ where

$$S = \{(\delta_1,u_i)(\delta_2,u_i)(\delta_3,u_i)(\delta_3,u_{i+1})(\delta_1,u_i) : i = 1,2,...,n-1\} \cup \{(\delta_1,u_i)(\delta_2,u_i)(\delta_3,u_i)(\delta_3,u_{i-1})(\delta_2,u_i) : i = 2,...,n\}.$$

Now $|S|$ = $2(n-1)$ = $\dim\mathfrak{C}(P_3\alpha P_n)$ − $2\dim\mathfrak{C}(P_2\alpha P_n)$ and so $|\mathfrak{B}(P_3\alpha P_n)| = |B_1 \cup B_2 \cup S|$. Moreover, every cycle in S contains an edge of the form $(\delta_1, u_i)\ (\delta_3, u_{i+2})$ or $(\delta_3, u_i)\ (\delta_2, u_i)$ that does not occur in any other cycle. Thus $\mathfrak{B}(P_3\alpha P_n)$ is a lin. ind. set of cycles in $\mathfrak{C}(P_3\alpha P_n)$ and $|\mathfrak{B}(P_3\alpha P_n)|$ = $\dim\mathfrak{C}(P_3\alpha P_n)$. One can easily notice that $\mathfrak{B}(P_3\alpha P_n)$ is a minimum cycle basis of $P_3\alpha P_n$ that contains all 3 cycles of $P_3\alpha P_n$ in addition to rectangles.

Theorem 5: *For $n \geq 4$; the cycle basis of minimum length of the graph $P_4\alpha P_n$ has a consists of rectangles and triangles.*

Proof: Let $P_4 = \delta_1\delta_2\delta_3\delta_4$ and $P_n = u_1u_2...u_n$ where $n \geq 4$. Then $P_4\alpha P_n$ can be considered as a graph with set of edges $E(P_4\alpha P_n)$ decomposed as $E(P_4\alpha P_n)$ = $E(\delta_1\delta_2\alpha P_n) \cup E(\delta_2\delta_3\alpha P_n) \cup E(\delta_3\delta_4\alpha P_n) \cup E$ where $E = E_1 \cup E_2 \cup E_3 \cup E_4$ such that

$$E_1 = \{(\delta_1,u_i)(\delta_3,u_{i+2}),(\delta_2,u_i)(\delta_4,u_{i+1}) : 1 \leq i \leq n-1\},$$
$$E_2 = \{(\delta_2,u_i)(\delta_3,u_{i-1}),(\delta_2,u_i)(\delta_4,u_{i-1}) : 1 \leq i \leq n\},$$
$$E_3 = \{(\delta_1,u_i)(\delta_4,u_{i+1}) : 1 \leq i \leq n-1\},$$
$$E_4 = \{(\delta_2,u_i)(\delta_4,u_{i-1}) : 1 \leq i \leq n\}.$$

Thus $|E| = 6(n-1)$. Since $\delta_i\delta_{i+1}\alpha P_n$ is isomorphic to $P_2\alpha P_n$, dim $(\mathfrak{C}(\delta_i\delta_{i+1}\alpha P_n))$ = $(n-1)^2$ for $i = 1,2,3$ and dim $(\mathfrak{C}(P_4\alpha P_n))$ − dim $(\mathfrak{C}(P_2\alpha P_n))$ = $6(n-1)$, it is easy to see that $\cup_{i=1}^{3}\mathfrak{B}\ (\delta_i\delta_{i+1}\alpha P_n)$ is lin. ind. set of cycles in $\mathfrak{C}(P_4\alpha P_n)$ to meet the $\dim(\mathfrak{C}(P_4\alpha P_n))$, we define the following set of cycles for $1 \leq i \leq n-1$:

$$S_1 = \{(\delta_1,u_i)(\delta_3,u_{i+1})(\delta_4,u_{i+1})(\delta_1,u_i),(\delta_1,u_i)(\delta_2,u_i)(\delta_4,u_{i+1})(\delta_1,u_i)\},$$
$$S_2 = \{(\delta_4,u_i)(\delta_3,u_i)(\delta_1,u_{i+1})(\delta_4,u_i),(\delta_4,u_i)(\delta_1,u_{i+1})(\delta_2,u_{i+1})(\delta_4,u_i)\} \text{ and}$$
$$S_3 = \{(\delta_1,u_i)(\delta_3,u_{i+1})(\delta_3,u_i)(\delta_2,u_i)(\delta_1,u_i), (\delta_2,u_i)(\delta_4,u_{i+1})(\delta_4,u_i)(\delta_3,u_i)(\delta_2,u_i)\}.$$

Then $|S_i| = 2(n-1)$ for $i = 1,2,3$ and $|S_1 \cup S_2 \cup S_3| = 6(n-1)$. Now, anry cycle of S_3 has an edge of the form (δ_1, u_i) (δ_3, u_{i+1}) or (δ_2, u_i) (δ_4, u_{i+1}) that does not belong to any other cycle of S_3 and so the cycles of S_3 are lin. ind.. Moreover, each of these cycles contain an edge of the form (δ_1, u_i) (δ_3, u_i) that does not belong to any cycle of $S_1 \cup S_2$. So the cycles of S_3 are lin. ind. with all the cycles of $S_1 \cup S_2$.

It is easy to see that cycles of S_1 are either edge disjoint or adjacent triangles with edge in common. So, the cycles in S_1 are lin. ind.. Same thing is true for S_2. So $S_1 \cup S_2$ is lin. ind. and thus $S_1 \cup S_2 \cup S_3$ is a lin. ind. set of cycles in $\mathfrak{C}(P_4 \alpha P_n)$. Now, every cycle in $S_1 \cup S_2 \cup S_3$ has an edge from the set E that does not appear in any cycle of $\cup_{i=1}^{3} \mathfrak{B}$ $(\delta_i \delta_{i+1} \alpha P_n)$. So, $S \cup \cup_{i=1}^{3} \mathfrak{B}$ $(\delta_i \delta_{i+1} \alpha P_n)$ is a lin. ind. set of cycles for $\mathfrak{C}(P_4 \alpha P_n)$. Thus $|\mathfrak{B}(P_4 \alpha P_n)| = \dim(\mathfrak{C}$ $(P_4 \alpha P_n))$ and so $\mathfrak{B}(P_4 \alpha P_n)$ is a basis for $\mathfrak{C}(P_4 \alpha P_n)$ that contains all of lin. ind. 3-cycles and the rest are rectangles. Thus, $\mathfrak{B}(P_4 \alpha P_n)$ is a minimum cycle basis for $\mathfrak{C}(P_4 \alpha P_n)$ that contains only triangles and rectangles.

Theorem 6: *For $n \geq 4$ and $m \geq 5$; the cycle basis of minimum length of the graph $P_m \alpha P_n$ consists of rectangles and triangles.*

Proof: Let $P_m = \delta_1 \delta_2 ... \delta_m$ and $P_n = u_1 u_2 ... u_n$ where $n \geq 4$ and $m \geq 5$. For each $i = 1,2,...,m-1$, $\delta_i \delta_{i+1} \alpha P_n$ is a subgraph of $P_m \alpha P_n$ that is isomorphic to $P_2 \alpha P_n$. Define $G' = (V(P_m \alpha P_n), E(G'))$ where $E(G') = \cup_{i=1}^{m-1} E$ $(\delta_i \delta_{i+1} \alpha P_n)$. Let $\mathfrak{B}(\delta_i \delta_{i+1} \alpha P_n)$ be the basis of $\mathfrak{B}(\delta_i \delta_{i+1} \alpha P_n)$ which is equivalent to the basis of $P_2 \alpha P_n$ as in Lemma 3 for each $i = 1,2,...,m-1$. Since the cycles of $\delta_i \delta_{i+1} \alpha P_n$ are either edge disjoint with cycles $\delta_{i+1} \delta_{i+2}$ or have only one edge of the form (δ_{i+1}, u_j) (δ_{i+1}, u_{j+1}). Thus the cycles of $\delta_i \delta_{i+1} \alpha P_n$ are lin. ind. with the cycles of $\delta_{i+1} \delta_{i+2} \alpha P_n$. Hence $\mathfrak{B}(G') = \cup_{i=1}^{m-1} \mathfrak{B}$ $(\delta_i \delta_{i+1} \alpha P_n)$ is a lin. ind. set of cycles in $\mathfrak{C}(G')$. Thus $\cup_{i=1}^{m-1} \mathfrak{B}$ $(\delta_i \delta_{i+1} \alpha P_n)$ is a basis for $\mathfrak{C}(G')$. Now $\dim \mathfrak{C}(G') = \sum_{i=1}^{m-1} \mathfrak{B}$ $(\delta_i \delta_{i+1} \alpha P_n) = \sum_{i=1}^{m-1-}$ $(n-1)^2 = (m-1)$ $(n-1)^2$. Also

$$\begin{aligned} \dim \mathfrak{C}(P_m \alpha P_n) - \dim \mathfrak{C}(G') &= m^2(n-1) + n^2(m-1) - 4(n-1)(m-1) \\ &\quad -mn + 1 - (mn^2 - 2mn + m - n^2 + 2n - 1) \\ &= (n-1)(m-1)(m-2). \end{aligned}$$

Now, we need to add $(n-1)$ $(m-1)$ $(m-2)$ cycles to the cycles in $\cup_{i=1}^{m-1} \mathfrak{B}$ $(\delta_i \delta_{i+1} \alpha P_n)$ to meet the dimension of $\mathfrak{C}(P_n \alpha P_m)$. These cycles will include edges of the form (δ_i, u_j) (δ_k, u_{j+1}) for each $j = 1,2,...,n-1$ and $1 \leq i,k \leq m$, $i \notin \{k, k+1\}$. For each $j = 1,2,...,n-1$, the set of triangles in $u_j u_{j+1} \alpha P_m$ are $(m-1)$ $(m-2)$ and they are lin. ind. because each triangle has a head vertex of the form (δ_i, u_j) and a base edge of the form (δ_k, u_{j+1}) (δ_{k+1}, u_{j+1}) or a triangle with a head vertex (δ_i, u_{j+1}) and a base edge of the form (δ_k, u_j) (δ_{k+1}, u_j) for some $k = 1,2,...,m-1$. Now, cycles with head vertex (δ_i, u_j) or (δ_i, u_{j+1}) are lin. ind. being together form finite boundaries of a planar subgraph of $\delta_j\delta_{j+1}\alpha P_m$. Also, such cycles are lin. ind. with the other cycles with distinct head vertices by varying i from 1 to $m-1$. That is because such cycles are either edge disjoint or have at most an edge in common. So for each $j = 1,2, ..., n-1$, the $(m-1)(m-2)$ cycles in $u_j u_{j+1}\alpha P_m$ are lin. ind. and it is easy to see that the union of all such cycles gives us $(n-1)(m-1)(m-2)$ cycles that are lin. ind.. Let $\mathfrak{B}_T$ be a set of all these triangles. Then $\mathfrak{B}(P_m\alpha P_n)$ is the disjoint union of $\mathfrak{B}_T$ and $\mathfrak{B}(G')$ that forms a basis for $\mathfrak{C}(P_n\alpha P_m)$. Because the cycles of $\mathfrak{B}_T$ consists of edges from $E(P_n\alpha P_m) \setminus E(G')$ that can not occur in any cycle of $\mathfrak{B}(G')$, so $\mathfrak{B}(P_m\alpha P_n)$ is a lin. ind. set of cycles and thus $\mathfrak{B}(P_m\alpha P_n)$ is a basis for $\mathfrak{C}(P_m\alpha P_n)$ that only contains triangles and rectangles. Also from the way that we built $\mathfrak{B}(P_m\alpha P_n)$, we picked the set of all triangles in addition to 4-cycles to meet the dimension of $\mathfrak{C}(P_m\alpha P_n)$. Therefore, $\mathfrak{B}(P_m\alpha P_n)$ is a cycle basis with minimal length for $\mathfrak{C}(P_m\alpha P_n)$.

References

[1] Y. Aboulfath, D. Watel, M. A. Weisser, T. Moutor, and P. Barth, "Maximizing minimum cycle bases intersection," hal-03851365 (2022).

[2] T. A. Al-Hawary, "Anti product fuzzy graphs," *Inter. J. Appl. Math.*, vol. 36, no. 6, pp. 747-756 (2023).

[3] T. A. Al-Hawary, "Maximal Strong Product and Balanced Fuzzy Graphs," *J. Appl. Math. and Informatics*, vol. 41, no. 5, pp. 1145-1155 (2023).

[4] T. A. Al-Hawary, "Strong Modular Product and Complete Fuzzy Graphs," *Ital. J. Pure Appl. Math.*, vol. 50, pp. 98-104 (2023).

[5] T. A. Al-Hawary, "Density results for perfectly regular and perfectly edge-regular fuzzy graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 26, no. 2, pp. 521-528 (2023).

[6] R. V. Jaikumar, R. V., R. Sundareswaran, M. Shanmugapriya, S. Broumi, and T. A. Al-Hawary, "Vulnerability parameters in picture fuzzy

soft graphs and their applications to locate a diagnosis center in cities," *Journal of Fuzzy Extension and Applications,* vol. 5, no. 1, pp. 86-99 (2024).

[7] T. A. Al-Hawary and B. Horani, "On Intuitionistic product fuzzy graphs," *Ital. J. Pure Appl. Math.,* vol. 38, pp. 113-126 (2017).

[8] T. A. Al-Hawary and M. Y. M. Alzoubi, "β–product of product fuzzy graphs," *J. Appl. Math. and Informatics,* vol. 42, no. 2, pp. 283-290 (2024).

[9] M. Y. M. Alzoubi and E. M. Al-Eid, "Minimum cycle bases of the semi-strong product of some basic graphs," *Global J. Pure Appl. Math.,* vol. 7, no. 1, pp. 71-76 (2011).

[10] M. Y. M. Alzoubi, "Minimum cycle bases of the semi-strong product of paths and cycles with different ladders," *J. Appl. Math.,* vol. 24, no. 4, pp. 615-624 (2011).

[11] M. Y. M. Alzoubi, "The bases number and minimum cycle bases of the strong product of paths and cycles with tenets," *Int. J. Pure Appl. Math.,* vol. 62, no. 4, pp. 369-379 (2010).

[12] J. A. Bondy and U. S. R. Marty, Graph Theory with Applications, Elsevier Publishing Co., New York (1982).

[13] H. Changxiang, L. Weilong, W. Baofeng, Y. Zhensheng, and M. Chang, "Fundamental cycle basis of some bipartite graphs," *AKCE Inter. J. Graphs Comb.,* in press.

[14] E. M. El-Kholy, El-Said R. Lashin, and Salama N. Daoud, "New operations on graphs and graph foldings," *Inter. Math. Forum*, vol. 7, no. 46, pp. 2253-2268 (2012).

[15] W. Imrich and P. F. Stadler, "Minimum cycle bases of product graphs," *Aust. J. Comb.,* vol. 26, pp. 233-244 (2002).

[16] A. C. Lassel, J. C. Henderson, and K. Ramachandran, "Minimum cycle bases of minimal measure for the structural analysis of skeletal structures by the flexibility method," *Proc. Royal Society of London Series A,* vol. 350, pp. 61-70 (1976).

[17] P. M. Gleiss, “Short cycles, cycle basis of minimum length of graphs from chemistry and biochemistry,” Ph.D. thesis, Fakultat für Naturwissenschaften und Mathematik der Universität Wien (2001).

[18] J. Gross and J. Yelle, Graph Theory and Its Applications, CRC Press, New York (1999).

[19] Kirchhoff and G. “Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung von Strom geführt wird,” *Poggendrof Ann. Phys. Chem.*, vol. 72, pp. 497-508 (1847).

[20] Ch. Liebnchen, “Periodic timetable optimization in public transport,” Ph.D. thesis, TU Berlin (2006).

[21] Ch. Liebnchen and R. Rizzi, “Classes of cycle bases,” *Disc. Appl. Math.,* vol. 155, no. 5, pp. 337-355 (2007).

[22] E. F. Schemeichl, “The basis number of graphs,” *J. Comb. Theory Series B*, vol. 30, no. 2, pp. 123-129 (1981).

Received January, 2024