Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 3, pp. 765–781
DOI : 10.47974/JDMSC-1966

Double Roman domination number of the zero-divisor graphs of commutative rings

Ravindra Kumar [§]
Department of Mathematics
Ram Sewak Singh Mahila College Sitamarhi
Babasaheb Bhimrao Ambedkar Bihar University
Muzaffarpur 842001
Bihar
India

and

Department of Mathematics
Indian Institute of Technology Patna
Patna 801106
Bihar
India

Ashutosh Singh [†]
Om Prakash *
Department of Mathematics
Indian Institute of Technology Patna
Patna 801106
Bihar
India

Abstract

In 2016, Beeler et al. [9] introduced the concept of the double Roman dominating function and the double Roman domination number(DRDN) of a graph. Following the same concept, we extend the study and calculate the double Roman domination number of the

[§] *E-mail:* **ravindra.pma15@iitp.ac.in**
[†] *E-mail:* **ashutosh_1921ma05@iitp.ac.in**
* *E-mail:* **om@iitp.ac.in (Corresponding Author)**

zero-divisor graph of a commutative ring $\mathcal{N}$ and its complement graph. Also, we find the DRDN of the total graph of the ring $\mathcal{N}$.

Subject Classification: *05C22, 05C25, 13A70.*

Keywords: *Commutative ring, Roman domination number, Double roman domination number, Total graph, Zero-divisor graph of a ring.*

1. Introduction

In 1988, Beck [8] introduced the concept of the zero divisor graph(ZDG) over a commutative ring and studied the coloring of this graph. The zero divisor graph of a commutative ring $\mathcal{N}$ is denoted by $\Gamma(\mathcal{N})$ and defined as the graph with the set of vertices $\Phi(\Gamma(\mathcal{N})) = \mathcal{N}$ and distinct vertices x and y are adjacent if and only if $xy = 0$. Later, in 1999, the above definition was modified by Anderson and Livingston [6] by restricting the set of vertices to the zero divisor elements of the ring. In 2003, the planarity of a ZDG of a finite commutative ring was studied by Akbari et al. [2]. They also determined the rings whose ZDGs are complete r–partite graphs. Later, many authors studied various properties of the graph over commutative rings, refer [3, 5, 15]. Recently, we have extended the property of divisor graph over the complement of $\Gamma(\mathcal{N})$ in [12], and pancyclic property of $\Gamma(\mathbb{Z}_n[i])$ in [13].

Throughout the paper, $\mathcal{G}$ represents a simple graph with vertex set $\Phi = \Phi(\mathcal{G})$ and edge set $E = E(\mathcal{G})$. We use $N(v)$ to denote the open neighbourhood of vertex v, defined by the set $\{u \in \Phi(\mathcal{G}) : uv \in E(\mathcal{G})\}$, whereas the set $N[v] = N(v) \bigcup \{v\}$ represents the closed neighbourhood of a vertex v. The minimal and maximum degrees of the graph $\mathcal{G}$ are denoted by $\delta(\mathcal{G})$ and $\Delta(\mathcal{G})$, respectively. A dominating set $S \subseteq \Phi(\mathcal{G})$ is a set defined as if every vertex in $\Phi(\mathcal{G})$ is either adjacent to a vertex in S or a member of S. The domination number, denoted by $\gamma(\mathcal{G})$, is the least cardinality among all dominating sets of $\mathcal{G}$. For basic definitions and results on graphs, we refer [7].

On the other hand, the Roman domination was initially defined and studied by Stewart [18], Revelle and Rosing [16], and later a mathematical definition of this problem was given by Cockayne et al. [10]. Let $f : \Phi(\mathcal{G}) \to \{0,1,2\}$ be a function such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. It is known as the Roman dominating function (RDF) for the function f. The minimal weight among all weights corresponding to the RDFs on a graph $\mathcal{G}$ is the Roman domination number (RDN), represented by $\gamma_R(\mathcal{G})$.

In 2016, Beeler et al. [9] introduced the idea of double Roman domination number (DRDN) of a graph. Let $f : \Phi \to \{0,1,2,3\}$ be a function and the ordered partition of Φ induced by f is $\Phi_0, \Phi_1, \Phi_2, \Phi_3$ where $|\Phi_i| = n_i$ and $\Phi_i = \{v \in \Phi \mid f(v) = i\}$, for $i = 0,1,2,3$. The ordered partition $\Phi_0, \Phi_1, \Phi_2, \Phi_3$ of Φ and the functions $f : \Phi \to \{0,1,2,3\}$ have a one-to-one correlation. Therefore, we can write $f = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$. Now, the function $f : \Phi(G) \to \{0,1,2,3\}$ is called a double Roman dominating function (DRDF) on $\mathcal{G}$ if the following properties hold:

(1) Vertex v needs at least two neighbours in Φ_2 or one neighbour in Φ_3 if $f(v) = 0$.

(2) Vertex v must have at least one neighbour in $\Phi_2 \cup \Phi_3$ if $f(v) = 1$.

The double Roman domination number $\gamma_{dR}(\mathcal{G})$ of the graph $(\mathcal{G})$ is equal to the minimum weight among all weights corresponding to the DRDFs on $\mathcal{G}$. In 2019, to extend the study of DRDF for the graph $\overline{\mathcal{G}}$, Shao et al. [17] introduced the global double Roman dominating function (GDRDF) of $\mathcal{G}$. They have investigated the basic properties of GDRDF and also presented its some sharp bounds. Recently, Kumar and Prakash [14] studied the RDN of some ZDGs of the commutative ring and established some of its properties.

The manuscript is organized as follows: Section 2 recalls some basic definitions and results on the DRDN. In Section 3, we present the DRDN of the zero divisor graphs over various rings and derive some bounds on the DRDN of $\Gamma(\mathcal{N})$ while Section 4 discusses the DRDN of the complement of $\Gamma(\mathcal{N})$. Bounds of the DRDN of total graph is discuss in Section 5.

2. Preliminary results

Throughout the paper, $\Gamma(\mathcal{N})$ represents the ZDG of commutative ring $\mathcal{N}$ and ID stands for integral domain. This section starts with some exact values of the DRDN for paths, cycles, and complete bipartite graphs. Also, we provide some basic results on DRDN that were observed by Beeler et al. [9].

For a graph K_n, $\gamma_{dR}(K_n) = 3$. For a partite set $\Phi_1, \Phi_2, \ldots, \Phi_r$ such that $|\Phi_i| > 2$, $1 \le i \le r$, let $\mathcal{G}$ be a complete $r-$ partite graph $(r \ge 2)$. Then $\gamma_{dR}(\mathcal{G}) = 6$. If $|\Phi_i| = 2$ for some i, then $\gamma_{dR}(\mathcal{G}) = 4$. If $|\Phi_i| = 1$ for some i, then $\gamma_{dR}(\mathcal{G}) = 3$. Hence, we can say that the DRDN of any star graph is 3 and the bistar graph is 4.

The result of DRDN of Paths and cycles is given by Ahangar et al. [1].

Proposition 2.1: ([1], Proposition 1). For $n \geq 1$,

$$\gamma_{dR}(P_n) = \begin{cases} n & \text{if } n \equiv 0\,(mod\;\;3), \\ n+1 & \text{if } n \equiv 1\,or\;\;2\,(mod\;\;3). \end{cases} \tag{1}$$

Proposition 2.2: ([1], Proposition 2). For $n \geq 3$,

$$\gamma_{dR}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 2, 3, 4\,(mod\;\;6), \\ n+1 & \text{if } n \equiv 1,\;5\,(mod\;\;6). \end{cases} \tag{2}$$

Also, we list some well-known results on the DRDN and some bounds with the RDN of a graph.

Proposition 2.3: ([9], Proposition 1). In a DRDF of weight $\gamma_{dR}(\mathcal{G})$, no vertex needs to be assigned the value 1.

Proposition 2.4: ([9], Proposition 2). "Let $\mathcal{G}$ be a graph and $f = (\Phi_0, \Phi_1, \Phi_2)$ a γ_R-function of $\mathcal{G}$. Then $\gamma_{dR}(\mathcal{G}) \leq 2\,|\Phi_1| + 3\,|\Phi_2|$."

Proposition 2.5: ([9], Proposition 5). For every graph $\mathcal{G}$, $\gamma_R(\mathcal{G}) < \gamma_{dR}(\mathcal{G})$.

3. DRDN of $\Gamma(\mathcal{N})$

In this section, we study the DRDN of the ZDG of the product of rings.

Lemma 3.1: *For two rings $\mathcal{N}_1$ and $\mathcal{N}_2$, $(a,b) \in Z(\mathcal{N}_1 \times \mathcal{N}_2)$ if and only if $a \in Z(\mathcal{N}_1)$ or $b \in Z(\mathcal{N}_2)$.*

Proposition 3.2: Let $\mathcal{N}$ be an integral domain. Then $\gamma_{dR}(\Gamma(\mathbb{Z}_2 \times \mathcal{N})) = 3$.

Proof: It is noted that $\Gamma(\mathbb{Z}_2 \times \mathcal{N})$ is a star graph. Therefore, by assigning the value 3 to the center of the graph and 0 to all other vertices, the function $f = (\Phi_0, \Phi_2, \Phi_3)$ becomes DRDF. Hence, the DRDN of $\Gamma(\mathbb{Z}_2 \times \mathcal{N})$ is 3.

Proposition 3.3: Let $\mathcal{N}$ be not an ID. Then $\gamma_{dR}(\Gamma(\mathbb{Z}_2 \times \mathcal{N})) \leq 5\gamma(\Gamma(\mathcal{N}))$ +3.

Proof: Let $D = \{x_1, x_2, \ldots, x_n\}$ be the $\gamma(\Gamma(\mathcal{N}))-$set. Then $\{(0, x_i),\; x_i \in D\} \cup \{(1,0),\; (1, x_i)\}$ is a dominating set of $\Gamma(\mathbb{Z}_2 \times \mathcal{N})$. Now, let $f = (\Phi_0, \Phi_2, \Phi_3)$ be a $\gamma_{dR}-$function and let $\Phi_2 = \{(1,\; x_i),\; x_i \in D\}$, $\Phi_3 = \{(0,\; x_i),\; x_i \in D\} \cup \{(1,0)\}$ and rest of the elements be assigned 0. As *f* is the double Roman dominating function, we have $\gamma_{dR}(\Gamma(\mathbb{Z}_2 \times \mathcal{N})) \leq 2|\Phi_2| + 3|\Phi_3| = 2n + 3(n+1)$, as $|D| = n$. Therefore, $\gamma_{dR}(\Gamma(\mathbb{Z}_2 \times \mathcal{N})) \leq 5n+3$ and hence $\gamma_{dR}(\Gamma(\mathbb{Z}_2 \times \mathcal{N})) \leq 5\gamma(\Gamma(\mathcal{N})) + 3$, as desired.

For a ring $\mathcal{N}$ with unity, Jafari-Rad et al. [11] assigned a parameter $a(\mathcal{N})$ as follows:

$$a(\mathcal{N}) = \begin{cases} 1 & \text{if } Z(\mathcal{N}) = 0, \\ \gamma_{st}(\Gamma(\mathcal{N})) & \text{if } Z(\mathcal{N}) \neq 0. \end{cases}$$

Theorem 3.4: *Given two commutative rings $\mathcal{N}_1$ and $\mathcal{N}_2$ with unity, and $\mathbb{Z}_2 \notin \{\mathcal{N}_1, \mathcal{N}_2\}$, $\gamma_{dR}(\Gamma(\mathcal{N}_1 \times \mathcal{N}_2)) = 3(a(\mathcal{N}_1) + a(\mathcal{N}_2))$.*

Proof: Let $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ be a ring with $\mathbb{Z}_2 \notin \{\mathcal{N}_1, \mathcal{N}_2\}$. Now, we examine the following cases.

Case 1: When $Z(\mathcal{N}_1) = Z(\mathcal{N}_2) = 0$, then we can see that $\{(1,0), (0,1)\}$ is a dominating set. Let $f = (\Phi_0, \Phi_2, \Phi_3)$ be a γ_{dR}-function on the zero divisor graph. Since $|\mathcal{N}_1| > 2$ and $|\mathcal{N}_2| > 2$, we put the value 3 to the vertices $(1,0)$, $(0,1)$ and 0 to all other vertices. Clearly, $\Phi_0 = \Phi \setminus \{(1,0), (0,1)\}$, $\Phi_2 = \emptyset$ and $\Phi_3 = \{(1,0), (0,1)\}$. Since $|\Phi_3|$ is minimum, $\gamma_{dR}(\Gamma(\mathcal{N})) = 3 \times 2 = 6$.

Case 2: When $Z(\mathcal{N}_1) \neq 0$ and $Z(\mathcal{N}_2) = 0$, let S be the $\gamma_{st}(\Gamma(\mathcal{N}_1))$ set and $f = (\Phi_0, \Phi_2, \Phi_3)$ be a DRDF. We define a set $T = \{\{(x,0) : x \in S\} \cup (0,1)\}$. Since any vertex of $\Gamma(\mathcal{N})$ is of the form (x,y) is either in T or adjacent to T. Now, put $\Phi_0 = \Phi(\Gamma(\mathcal{N})) \setminus T$, $\Phi_2 = \emptyset$ and $\Phi_3 = T$, then $f(\Phi) = 3(a(\mathcal{N}_1) + 1)$. To complete the proof, we need to show that there is no other way to define a DRDF g in $\Gamma(\mathcal{N})$ such that $g(\Phi) < f(\Phi)$. If we remove the vertex $(0,1)$ from the set T, then the adjacent vertices $(r,0)$ will also be removed from the set Φ_0 where $r \in Reg(\mathcal{N}_1)$. Then $\Phi_0' = \Phi(\Gamma(\mathcal{N})) \setminus \{\Phi_3' \cup \Phi_2'\}$, $\Phi_2' = \{(r,0) \cup (0,1)\}$ and $\Phi_3' = \{(x,0) : x \in S\}$. Clearly, $g = (\Phi_0', \Phi_2', \Phi_3')$ is a DRDF with $g(\Phi) = 3a(\mathcal{N}_1) + 2 + 2n$, $|Reg(\mathcal{N}_1)| = n \geq 1$, this implies $g(\Phi) > f(\Phi)$. Again, if we remove a vertex, say $(x_i, 0)$ for a fixed i, from the set T, then by definition of semi-total domination number, there is a vertex $x_j \in S$ for some j such that $x_i x_j = 0$. This implies the removal of $(x_j, 0)$ from the set T. Let Y be the set of all vertices which is adjacent to either $(x_i, 0)$ or $(x_j, 0)$, for fixed i and j with $Y > 2$. Then $\Phi_0'' = \Phi(\Gamma(\mathcal{N})) \setminus \{T \cup Y\}$, $\Phi_2'' = Y \cup \{(x_i, 0), (x_j, 0)\}$ and $\Phi_3'' = T - \{(x_i, 0), (x_j, 0)\}$. Here, $g = (\Phi_0'', \Phi_2'', \Phi_3'')$ is also a DRDF of $\Gamma(\mathcal{N})$ with $g(\Phi) = 2|\Phi_2''| + 3|\Phi_3''| =$

$8+3\ (a(\mathcal{N}_1)\ -\ 2+1)\ =\ 3a(\mathcal{N}_1)\ +\ 5 > 3a\ (\mathcal{N}_1)\ +\ 3\ =\ f(\Phi)$. Thus, from both cases, double Roman domination number $\gamma_{dR}(\Gamma(\mathcal{N}))$ is $3(a(\mathcal{N}_1)+1)$.

Case 3: When $Z(\mathcal{N}_1) \neq 0,\ Z(\mathcal{N}_2) \neq 0$. First we assume that S_1 is $\gamma_{st}(\Gamma(\mathcal{N}_1))$ set and S_2 is the $\gamma_{st}(\Gamma(\mathcal{N}_2))$ set. Now , we define $T_1 = \{\{(x,0) : x \in S_1\}$ and $T_2 = \{\{(0,y) : y \in S_2\}$. Similarly, by the case 2, we can obtain $T_1 \cup T_2$ as the dominating set of $\Gamma(\mathcal{N})$. Therefore, we can define a DRDF *f* on a graph $\Gamma(\mathcal{N})$ so that vertices of $T_1 \cup T_2$ are assigned a value 3, and the rest of the vertices have assigned a value 0. Hence, $\gamma_{dR}(\Gamma(\mathcal{N})) = 3|T_1| + 3|T_2| = 3(a(\mathcal{N}_1) + a(\mathcal{N}_2))$.

Theorem 3.5: *For every $i = 1,2,\ldots,n$, let $\mathcal{N}_i$ be the integral domain, and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_n$ be the product ring for a fixed integer $n > 2$. Then $\gamma_{dR}(\Gamma(\mathcal{N})) = 3n$.*

Proof: Let $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_n,\ n > 2$ be a ring where $\mathcal{N}_i$ is an integral domain for each $i = 1,2,\ldots,n$. Let $f = (\Phi_0,\ \Phi_1,\ \Phi_2,\ \Phi_3)$ be a DRDF of $\Gamma(\mathcal{N})$ such that $\Phi_1 = \varnothing$. Since $\Phi_2 \cup \Phi_3$ dominates the graph $\Gamma(\mathcal{N})$, it is noted that the set $S = \{(y_1,\ 0,0,\ \ldots,0),\ (0,y_2,\ 0,\ldots,0),\ \ldots,(0,\ 0,0,\ \ldots,y_n)\}$ is a dominating set of $\Gamma(\mathcal{N})$, for each $i \in \{1,\ldots,n\}$ and y_i is fixed non-zero element. Now, we take $\Phi_2 = \varnothing,\ \Phi_3 = S$ and $\Phi_0 = \Phi(\Gamma(\mathcal{N})) \setminus \Phi_3$, then *f* is a DRDF with weight $w(f) = 2|\Phi_2| + 3|\Phi_3| = 3n$, *i.e.*, $\gamma_{dR}(\Gamma(\mathcal{N})) \leq 3n$. To prove this weight is minimum, let $\Phi_2 \neq \varnothing$. Since $\Phi_2 \cup \Phi_3$ dominates $\Gamma(\mathcal{N})$, we reassigned at least one vertex of Φ_3 to a value 2. Without loss of generality, we suppose $(y_1,0,\ \ldots,0) \in S$ such that $f(y_1,\ 0,\ldots,0) = 2$ for a fixed non-zero element y_1. Since the vertices of type $(0,y_2,\ldots,y_n)$ are adjacent to $(y_1,0,\ldots,0)$ in Φ_0. Then by definition of DRDF, it has at least two adjacent elements in Φ_2, so we take another element $(y_1',0,\ldots,0)$ in Φ_2. Again, *f* is a DRDF of $\Gamma(\mathcal{N})$ with weight $f(\Phi) = 2|\Phi_2| + 3|\Phi_3| = 2 \times 2\ + 3\ (n-1) = 3n+1 \geq 3n$. Hence, we conclude that $\gamma_{dR}(\Gamma(\mathcal{N})) = 3n$.

Example 3.6: A graph $\mathcal{G}$ with its double Roman domination number is equal to 2 if it contains only a single vertex, and a graph $\mathcal{G}$ has a double Roman domination number 3 if it has either a complete graph or a vertex adjacent to every vertex in $\mathcal{G}$. Therefore, for an integral domain *P*, if $\mathcal{N} \cong \mathbb{Z}_2 \times P$, then $\gamma_{dR}(\Gamma(\mathcal{N})) = 3$.

Corollary 3.7: *In [3, Corollary 1], Let $P > 3$ be prime. The graph $\Gamma(\mathbb{Z}_n)$ is Hamiltonian if and only if $n = p^2$. In this case $\Gamma(\mathbb{Z}_n) \cong K_{p-1}$ and hence, DRDN of $\Gamma(\mathbb{Z}_n)$ is 3.*

Theorem 3.8: *For each $i \in \{1,2,\ldots,n\}$, let $\mathcal{N}_i$ be a local ring and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_n$ be a commutative Artinian ring. Then $\gamma_{dR}(\Gamma(\mathcal{N})) = 3n$ for $n \geq 3$.*

Proof: For a local ring $\mathcal{N}_i$ with maximal ideal M_i, let $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_n$ be a ring. Consider the set $T = \{(y_1, 0, 0, \ldots, 0), (0, y_2, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, y_n)\}$ with $ann(y_i) = M_i$ for each *i*. Since any non-zero element of $Z(\mathcal{N})$, say $(a_1, a_2, \ldots, a_n)$, either in *T* or adjacent to a member in *T* as one of its non-zero components, say a_j, is in the member of the corresponding maximal ideal M_j. Then, the set *T* is a dominating set for $\Gamma(\mathcal{N})$. Let $g = (\Phi_0, \Phi_2, \Phi_3)$ be a DRDF of $\Gamma(\mathcal{N})$ with $\Phi_2 \bigcup \Phi_3 = T$. Following the same procedure as in Theorem 3.5, we get $g(\Phi) = 3n$. Hence, $\gamma_{dR}(\Gamma(\mathcal{N})) = 3n$.

Theorem 3.9: *Let $\mathcal{N}$ be a commutative ring with unity that contains at least one prime ideal. Then $\gamma_{dR}(\Gamma(\mathcal{N})) \leq 3(|P|-1)$ where P is a prime ideal.*

Proof: Let $\mathcal{N}$ be a commutative ring with unity and *P* be one of its prime ideals. Then every edge in $\Gamma(\mathcal{N})$ has at least one end vertex in *P*. Let $S = P \setminus \{0\}$. Then *S* becomes a dominating set of $\Gamma(\mathcal{N})$. Since every edge in the graph has at least one vertex in *S*, we can define a function $g = (\Phi_0, \Phi_2, \Phi_3)$ such that $\Phi_0 = \Phi(\Gamma(\mathcal{N})) \setminus S$, $\Phi_2 = \emptyset$, $\Phi_3 = S$. Also, *P* is a prime ideal with minimum cardinality. Therefore, we have $g(\Phi) \leq 3|S| = 3(|P|-1)$. Thus, $\gamma_{dR}(\Gamma(\mathcal{N})) \leq 3(|P|-1)$.

Proposition 3.10: Let $n = p^t$ where *p* is a prime, and $t \geq 2$ is an integer. Then $\gamma_{dR}(\Gamma(\mathbb{Z}_n)) = 3$.

Proof: If $n = p^t$ and *p* is prime with $t = 2$, then $\Gamma(\mathbb{Z}_n)$ will be a complete graph with $p-1$ vertices. Therefore, DRDN is 3. Now, let $t > 2$. Then the domination number is one as the vertex p^{t-1} dominates the graph $\Gamma(\mathbb{Z}_n)$. Therefore, assigning the weight 0 to all the vertices except p^{t-1} and 3 to the vertex p^{t-1}. Hence, we get $\gamma_{dR}(\Gamma(\mathbb{Z}_n)) = 3$.

Now, we will discuss the DRDN of a zero-divisor graph of the ring of Gaussian integers modulo *n*. In 2008, Osba et al. [15] discuss the graph-theoretic properties of $\Gamma(\mathbb{Z}_n[i])$. They have considered three cases:

(1) When $n = 2^m$, $m > 1$ and $m \in \mathbb{Z}^+$.
(2) When $n = q^m$, $q \in \mathbb{Z}$ is a prime and $q \equiv 3\, mod(4)$.
(3) When $n = p^m$, $p \in \mathbb{Z}$ is a prime with $p \equiv 1\, mod(4)$ and $p = x^2 + y^2$ where $x, y \in \mathbb{Z}$.

Proposition 3.11: Let $n = 2^k$, $k > 1$. Then $\gamma_{dR}(\Gamma(\mathbb{Z}_n[i])) = 3$.

Proof: It is clear from Osba et al. [15] that the vertex $(2^{k-1}+i2^{k-1})$ is adjacent to all other vertices of $\Gamma(\mathbb{Z}_n[i])$. Also, there are 2^{2k-2} number of pendent vertices. Now, we define a function $f=(\Phi_0, \Phi_2, \Phi_3)$ such that $\Phi_3 = (2^{k-1}+i2^{k-1})$, $\Phi_2=\varnothing$, $\Phi_0=\Phi-\Phi_3$. Clearly, f is a DRDF with weight 3. Hence, $\gamma_{dR}(\Gamma(\mathbb{Z}_n[i]))=3$.

Proposition 3.12: For $n=q^m$, $m\geq 2$, $\gamma_{dR}(\Gamma(\mathbb{Z}_n[i]))=3$.

Proof: If $m=2$, then $\Gamma(\mathbb{Z}_n[i])$ is complete graph with q^2-1 vertices as shown in Figure 1, for $q = 3$. Hence, γ_{dR} $(\Gamma(\mathbb{Z}_n[i]))=3$. Let $m>2$. To find the DRDN, we need to find the maximum degree of a vertex in $\Gamma(\mathbb{Z}_n[i])$. It is seen that the vertex $u=q^{m-1}$ is adjacent to every other vertices in $\Gamma(\mathbb{Z}_n[i])$. Here, the vertex $u=q^{m-1}$ must be assigned the value 3 under $\gamma_{dR}-$ function f of graph $\Gamma(\mathbb{Z}_n[i])$. Then the function $f=(\Phi-u,\varnothing,u)$ is a DRDF with the fact that the weight is minimum. Therefore, $\gamma_{dR}(\Gamma(\mathbb{Z}_n[i]))=3$.

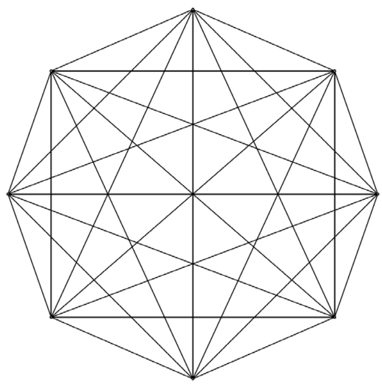

Figure 1
$\Gamma(\mathbb{Z}_9[i])$

Proposition 3.13: If $n=p^m$, then $\gamma_{dR}(\Gamma(\mathbb{Z}_n[i]))=6$ for $m\geq 1$.

Proof: Let $n=p^m$ where p is a prime and $p=x^2+y^2$ for some $x,y\in\mathbb{Z}$. If $m=1$, then by Osba et al. [15], $\Gamma(\mathbb{Z}_n[i])\cong K_{p-1,p-1}$. DRDN of complete bipartite is 6 and hence γ_{dR} $(\Gamma(\mathbb{Z}_n[i]))=6$. If $m>1$, in this case the domination number will be 2 and the dominating set $S=\{(x+iy)^m(x-iy)^{m-1}, (x+iy)^{m-1}(x-iy)^m\}$. The graph has two maximal ideals, namely, $<x+iy>$ and $<x-iy>$ and every vertex in $\Gamma(\mathbb{Z}_n[i])$ is adjacent to only two vertices $(x+iy)^m(x-iy)^{m-1}$ and $(x+iy)^{m-1}(x-iy)^m$. Thus, its DRDN is 6.

4. DRDN of the complement of $\Gamma(\mathcal{N})$

To find the DRDN of the complement of $\Gamma(\mathcal{N})$, we start this section with some basic results on DRDN of the complement of graph $\mathcal{G}$. Let $\mathcal{G}$ be a complete graph of order n. Then $\gamma_{dR}(\overline{\mathcal{G}}) = 2n$.

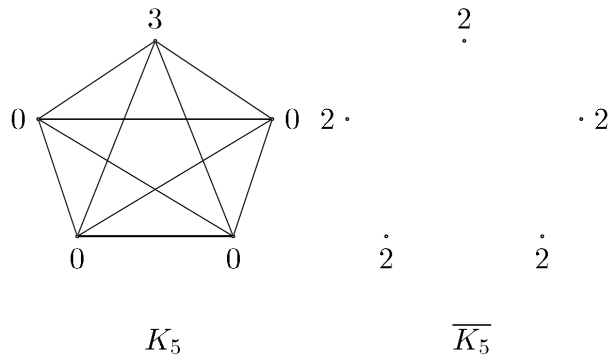

Figure 2

Graph K_5 and its Complement graph

From the Figure 2, we observe that $\gamma_{dR}(K_5) = 3$ and $\gamma_{dR}(\overline{K_5}) = 2 \times 5 = 10$. Thus, it is clear that $\gamma_{dR}(K_n) = 3 < \gamma_{dR}(\overline{K_n}) = 2n$, for $n \geq 2$. Now, consider a complete $r-$ partite graph $\mathcal{H}$ with $(r \geq 2)$, and partite sets Φ_1, $\Phi_2, \ldots, \Phi_r$. If $|\Phi_i| > 2$ for $1 \leq i \leq r$, then $\gamma_{dR}(\mathcal{H}) = 6$. If there exists an i such that $|\Phi_i| = 2$, then $\gamma_{dR}(\mathcal{H}) = 5$ and if $|\Phi_i| = 1$ for an i, then $\gamma_{dR}(\mathcal{H}) = 3$, but $\gamma_{dR}(\overline{\mathcal{H}}) = 3r$, $r \geq 2$ and $|\Phi_i| > 2$ for $1 \leq i \leq r$. Therefore, the DRDN of the graph $\mathcal{H}$ is always less than the DRDN of the complement of the graph $\mathcal{H}$ for $r \geq 2$. For a star graph $K_{1,n}$, if we take its complement, it has two components. One component is a complete graph, and the other is a single vertex. Therefore, $\gamma_{dR}(K_{1,n}) = 3$ but $\gamma_{dR}(\overline{K_{1,n}}) = 5$, $n \geq 2$.

Now, we will find the DRDN to the complement of paths and cycles.

Proposition 4.1: For $n > 2$, $\gamma_{dR}(\overline{P_n}) = 5$.

Proof: Let $P = v_1 v_2 v_3 \ldots v_n$ be a path of order n. Since v_1 is adjacent to only v_2 in P. Therefore, every vertex is adjacent to v_1 except v_2 in the complement of P. Now, define $h : \Phi(\overline{P_n}) \rightarrow \{0,1,2,3\}$ by $h(v_1) = 3$, $h(v_2) = 2$ and for the rest of the vertices we assign 0, *i.e.*, $h(x) = 0$ for $x \in \Phi(\overline{P_n}) \setminus$

$\{v_1, v_2\}$. Clearly, h is a DRDF of $\overline{P_n}$ with weight 5 for $n > 2$. To prove this weight is minimum, on the contrary, we assume that there exists a weight less than 5. Since this graph has more than 2 vertices, the weight must be greater than 2. Also, the graph is neither complete nor star, so its weight is greater than 3. Suppose we assign a value 2 to any two vertices with higher degrees and the rest of the vertices to 0. In this case, we find a vertex assigned 0 is adjacent to at least one vertex assigned 2, which does not satisfy the definition of DRDF of $\overline{P_n}$. Hence, $\gamma_{dR}(\overline{P_n}) = 5$.

Proposition 4.2: For $n > 2$, $\gamma_{dR}(\overline{C_n}) = 6$.

Proof: Let $C = v_1 v_2 v_3 \ldots v_n v_1$ be a cycle of length n, $n \geq 4$. Since v_1 is adjacent to all the vertices in $\overline{C_n}$ except two vertices v_2 and v_n, we define a function $h : \Phi(\overline{C_n}) \to \{0,1,2,3\}$ by $h(v_1) = 3$, $h(v_2) = 3$ and $h(x) = 0$ where $x \in \Phi(\overline{C_n}) \setminus \{v_1, v_2\}$. Hence, we can easily verify that h is a DRDF with weight 6. We follow the same argument as in Proposition 4.1 to prove this weight is minimum. If $n = 3$, then the complement of C_n is a null graph with 3 vertices. In this case, the weight is also 6. Thus, $\gamma_{dR}(\overline{C_n}) = 6$.

In the following theorem, we found the double Roman domination number for the complement graph of a direct product of the integral domains.

Theorem 4.3: *Let k be an integer greater than* 2 *and* $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_k$ *be a ring where for each* $i \in \{1, 2, \ldots, k\}$, $\mathcal{N}_i$*'s are integral domain. Then*

(a) $\gamma_{dR}(\overline{\Gamma(\mathcal{N})}) = 6$, if $k \geq 3$.

(b) $\gamma_{dR}(\overline{\Gamma(\mathcal{N})}) = 6$, if $k = 2$ and $min\{|\mathcal{N}_1|, |\mathcal{N}_2|\} \geq 3$.

(c) $\gamma_{dR}(\overline{\Gamma(\mathcal{N})}) = 5$, if $k = 2$ and $min\{|\mathcal{N}_1|, |\mathcal{N}_2|\} = 2$.

(d) $\gamma_{dR}(\overline{\Gamma(\mathcal{N})}) = 4$, if $k = 2$ and $|\mathcal{N}_1| = |\mathcal{N}_2| = 2$.

Proof: Suppose $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \ldots \times \mathcal{N}_k$ is a ring, and $\mathcal{N}_i$'s are integral domains for each i. If $k = 2$ and $|\mathcal{N}_1| = |\mathcal{N}_2| = 2$, then $\Gamma(\mathcal{N})$ is a single edge, and its complement graph is a null graph with two vertices. Hence, $\gamma_{dR}(\overline{\Gamma(\mathcal{N})}) = 4$. Now, if $min\{|\mathcal{N}_1|, |\mathcal{N}_2|\} = 2$, then $\Gamma(\mathcal{N})$ is a star graph, and its complement has two components, one being a single vertex, and the other a complete graph. Thus, $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})=5.$ In case (b), if $min\{|\mathcal{N}_1|, |\mathcal{N}_2|\}\geq 3,$ then graph is a complete bipartite graph. Hence, the DRDN of its complement is always 6. For the case (a), if we take $k\geq 3,$ then it is clear that the set $T=\{(y_1,y_2,\ldots,y_{k-1},0),(0,0,\ldots,0,y_k)\}$ is a dominating set of $\overline{\Gamma(\mathcal{N})}$ where y_i is a fixed non-zero element of $\mathcal{N}_i$ for each $1\leq i\leq k.$ Precisely, we can choose $y_i=1$ for each $1\leq i\leq k.$ Then the set T will be $\{(1,1,\ldots,1,0),(0,0,\ldots,0,1)\}.$ From [9, Proposition 8], $4\leq\gamma_{dR}(\overline{\Gamma(\mathcal{N})})\leq 6.$ If possible, let $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})=4.$ Then it is necessary that degree of $(1,1,\ldots,1,0)$ and $(0,0,\ldots,0,1)$ are same, which is not possible in $\overline{\Gamma(\mathcal{N})}.$ Thus, $5\leq\gamma_{dR}(\overline{\Gamma(\mathcal{N})}).$

Suppose $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})=5,$ then either $(1,1,\ldots,1,0)$ should be assigned 3 and $(0,0,\ldots,0,1)$ to 2 or $(1,1,\ldots,1,0)$ should be assigned 2 and $(0,0,\ldots,0,1)$ to 3. Therefore, the vertex assigned a value 2 is either an isolated vertex or another vertex with the value 2, which is not possible in both cases. Thus, $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})\neq 5.$ It means that $6\leq\gamma_{dR}(\overline{\Gamma(\mathcal{N})})$ and hence, $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})=6.$

Proposition 4.4: If $\mathcal{N}$ is an ID and $\mathcal{N}\neq\mathbb{Z}_2,$ then $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_2\times\mathcal{N})})=5.$

Proposition 4.5: *If $\mathcal{N}$ is not an ID, then $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_2\times\mathcal{N})})=5.$*

Proof: Let $\mathcal{N}$ be not an ID and let f be a γ_{dR} − function of $\overline{\Gamma(\mathbb{Z}_2\times\mathcal{N})}.$ It is known that the set $D=\{(1,0),(0,1)\}$ forms a dominating set for $\overline{\Gamma(\mathbb{Z}_2\times\mathcal{N})}.$ Since all vertices in $\overline{\Gamma(\mathbb{Z}_2\times\mathcal{N})}$ are adjacent to $(0,1)$ except $(1,0).$ After assigning the vertex $(0,1)$ to 3 and $(1,0)$ to 2, we can see that its DRDN is 5.

Lemma 4.6: *Let $(\mathcal{N},M)$ be a local commutative Artinian ring. Then $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})\geq 5,$ provided $\mathcal{N}$ is not a field.*

Proof: Since $\mathcal{N}$ is an Artinian ring, it is Noetherian and $\text{Ass}(\mathcal{N})\neq\emptyset.$ Therefore, there exists $y\in\mathcal{N}$ such that $\text{ann}(y)=M.$ Hence, M is an associated prime ideal of $\mathcal{N}.$ Also, there are at least two components in $\overline{\Gamma(\mathcal{N})}$ in which y is an isolated vertex. Hence, we must assign a value 2 to y, and the DRDN of the other component is greater than 3. Thus, $\gamma_{dR}(\overline{\Gamma(\mathcal{N})})\geq 5.$

Proposition 4.7: ([9], Proposition 8). "For any graph $\mathcal{G}$, $2\gamma(\mathcal{G})\leq\gamma_{dR}(\mathcal{G})\leq 3\gamma(\mathcal{G})$."

Theorem 4.8: *For distinct primes $p_1, p_2, \ldots, p_k$ and positive integers $t_1, t_2, \ldots, t_k$, suppose $n = p_1^{t_1} p_2^{t_2} \ldots p_k^{t_k}$ for a fixed integer $k \geq 1$. Then the following results hold:*

(a) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 3k$, if $n = p_1^{t_1} \ldots p_k^{t_k}$, $k \geq 3$.

(b) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 6$, if $n = p_1^{t_1} p_2^{t_2}$, where $p_1, p_2 \geq 3$.

(c) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 2(p_1 - 1) + 3$, if $n = p_1^{t_1}$ and $t_1 > 2$.

(d) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 2(p_1 - 1)$, if $n = p_1^2$.

Proof: Suppose $n = p_1^{t_1} p_2^{t_2} \ldots p_k^{t_k}$, $k \geq 3$ and $p_1, p_2, \ldots, p_k$ are distinct primes with positive integers $t_1, t_2, \ldots, t_k$. Then by fundamental theorem of abelian groups, $\mathbb{Z}_n \cong \mathbb{Z}_{p_1}^{t_1} \times \ldots \times \mathbb{Z}_{p_k}^{t_k}$. Also, the set $S = \{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,0,\ldots,1)\}$ is the dominating set of graph $\overline{\Gamma(\mathbb{Z}_n)}$. Hence, by Proposition 4.7, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) \leq 3\gamma(\overline{\Gamma(\mathbb{Z}_n)})$, i.e., $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) \leq 3k$. Now, we define DRDF $f = (\Phi_0, \Phi_1, \Phi_2)$ such that $\Phi_0 = \Phi(\overline{\Gamma(\mathbb{Z}_n)}) \setminus S$, $\Phi_2 = \emptyset$, $\Phi_3 = S$. Since S is the minimum, removing any element from the set S violates the definition of DRDF. Therefore, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 3k$. Let $n = p_1^{t_1} p_2^{t_2}$, $p_1, p_2 \geq 3$. If $t_1 = t_2 = 1$, then $\overline{\Gamma(\mathbb{Z}_n)}$ has two components and both are complete with order $p_1 - 1$ and $p_2 - 1$, respectively. Then its DRDN is 6. If $t_1 = t_2 > 1$, then all the vertices of $\overline{\Gamma(\mathbb{Z}_n)}$ are adjacent to either $(1,0)$ or $(0,1)$ or both. Clearly, in this case, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 6$. Let $n = p_1^{t_1}$ and $t_1 > 2$, then the set of vertices $T = \{p_1^{t_1 - 1}, 2p_1^{t_1 - 1}, \ldots, (p_1 - 1)\, p_1^{t_1 - 1}\}$ is isolated in $\overline{\Gamma(\mathbb{Z}_n)}$ and rest of the vertices are adjacent to p_1 itself. Therefore, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 2(p_1 - 1) + 3$. If $t_1 = 2$, then $\overline{\Gamma(\mathbb{Z}_n)}$ is edgeless graph with $p_1 - 1$ vertices. Thus, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n)}) = 2(p_1 - 1)$.

Now, we discuss the DRDN of the complement of $\Gamma(\mathbb{Z}_n[i])$ for $n = 2^k$, $n = q^k$ and $n = p^k$, respectively.

Proposition 4.9: Let $n = 2^k$ be an integer, $k > 1$. Then $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) = 5$.

Proof: We know that the vertex $u = (1 + i1)^{2k-1}$ is adjacent to every other vertex in $\Gamma(\mathbb{Z}_n[i])$. Therefore, the vertex u is isolated in $\overline{\Gamma(\mathbb{Z}_n[i])}$. Furthermore, it is also known that the graph $\Gamma(\mathbb{Z}_n[i])$ has 2^{2k-2} pendent vertices. Hence, the set of pendent vertices are adjacent to all other vertices in $\overline{\Gamma(\mathbb{Z}_n[i])}$ except vertex u. Thus, the function $f = (\Phi_0, \Phi_2, \Phi_3)$ where

$\Phi_2 = u,\ \Phi_3 = \overline{(1+i)}$ and $\Phi_0 = \Phi - \Phi_2 - \Phi_3$ is a DRDF with weight 5. Hence, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) = 5$.

Proposition 4.10: Let $n = q^k,\ k > 1$. Then

(a) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) = 2(q^2 - 1)$, if $k = 2$.

(b) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) \leq 2q^2 + 1$, if $k > 2$.

Proof: If $k = 2$, then $\overline{\Gamma(\mathbb{Z}_n[i])}$ becomes an edgeless graph with $q^2 - 1$ vertices. Therefore, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) = 2(q^2 - 1)$. Now, suppose $k > 2$. In this case, the graph $\Gamma(\mathbb{Z}_n[i])$ has a maximum clique of order $(q-1)(q+1)$ and the vertex $u = q$ has minimum degree with $N(u) = (q-1)(q+1)$. Hence, in the complement of $\Gamma(\mathbb{Z}_n[i])$, $(q-1)(q+1)$ vertices are isolated, and the vertex u is adjacent to all other vertices except the isolated vertices. Thus, the function $f = (\Phi - N[u],\ N(u), u)$ becomes DRDF with weight $2((q-1)(q+1)) + 3 = 2q^2 + 1$. Hence the result.

Proposition 4.11: Let $n = p^k,\ k \geq 1$. Then

(a) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) = 6$, if $k = 1$.

(b) $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) \leq 9$, if $k > 1$.

Proof: If $k = 1$, then by [15, Theorem 17], $\Gamma(\mathbb{Z}_n[i])$ is a $K_{p-1,p-1}$ graph and its complement is the component of two complete graphs each of order $p-1$. Now, define a function $f = (\Phi_0, \Phi_2, \Phi_3)$ such that each component has weight 3, as the DRDN of the complete graph is 3. Then the graph has only two component, therefore, $\gamma_{dR}(\overline{\Gamma(\mathbb{Z}_n[i])}) = 6$. If $k > 1$, then $\Gamma(\mathbb{Z}_n[i])$ has domination number 2 and dominating set $S = \{(a+ib)^k (a-ib)^{k-1}, (a+ib)^{k-1}(a-ib)^k\}$. Here, we suppose that $u = (a+ib)^k (a-ib)^{k-1}$ and $v = (a+ib)^{k-1}(a-ib)^k$. The vertices u and v contain the maximum degree of the graph $\Gamma(\mathbb{Z}_n[i])$. Therefore, in the complement of graph $\Gamma(\mathbb{Z}_n[i])$, u and v have a minimum degree. Hence, we put the value 3 to vertex u and 0 to its neighbour in the complement of the graph. Also, all those vertices that are adjacent to both u and v are in $\Gamma(\mathbb{Z}_n[i])$ will make an induced complete graph in the complement. Therefore, the DRDN of that induced complete graph is 3. The vertex v will be assigned 3, and all other vertices will be 0.

Finally, this way, we see that the function $f=(\Phi_0,\Phi_2,\Phi_3)$ is DRDF with weight 9.

5. DRDN of the total graph

The total graph of a ring $\mathcal{N}$, represented by $T(\Gamma(\mathcal{N}))$, is the simple graph such that every element of $\mathcal{N}$ represents the set of nodes and two nodes *i* and *j* are connected by an edge if $i+j\in Z(\mathcal{N})$. The subgraph of $T(\Gamma(\mathcal{N}))$ induced by the regular elements of $\mathcal{N}$, zero-divisor elements of $\mathcal{N}$ is denoted by $Reg(\Gamma(\mathcal{N}))$, $Z(\Gamma(\mathcal{N}))$ respectively. In this section, we focus on exploring the DRDN of the total graph of $\Gamma(\mathcal{N})$. We aim to establish an upper bound for the DRDN of $T(\Gamma(\mathcal{N}))$. Additionally, we determine the DRDN values for the graph $\Gamma(\mathbb{Z}_n)$ in the case where $Z(\mathbb{Z}_n)$ is not an ideal.

For an upper bound of the DRDN of $T(\Gamma(\mathcal{N}))$, we state the following result of Anderson and Badawi [4].

Lemma 5.1: ([4] Theorem 2.1). "*Let $\mathcal{N}$ be a commutative ring such that $Z(\mathcal{N})$ is an ideal of $\mathcal{N}$. Then $Z(\Gamma(\mathcal{N}))$ is a complete (induced) subgraph of $T(\Gamma(\mathcal{N}))$ and $Z(\Gamma(\mathcal{N}))$ is disjoint from $Reg(\Gamma(\mathcal{N}))$.*"

Theorem 5.2: *Let $\mathcal{N}$ be a commutative ring (not necessarily finite) with unity and $Z(\mathcal{N})$ be its ideal with $|Z(\mathcal{N})|=\beta$ and $|\mathcal{N}/Z(\mathcal{N})|=\alpha$. Then $\gamma_{dR}(T(\Gamma(\mathcal{N})))\le 3\alpha$.*

Proof: Suppose $Z(\mathcal{N})$ is an ideal of a ring $\mathcal{N}$. Let $|Z(\mathcal{N})|=\beta$. Then, by [4, Theorem 2.2], two cases arise: (i) If $2\in Z(\mathcal{N})$, then each coset $x+Z(\mathcal{N})$ is a complete graph K_β, and (ii) If $2\notin Z(\mathcal{N})$, then cosets $(x+Z(\mathcal{N}))\cup(-x+Z(\mathcal{N}))$ is a complete bipartite graph $K_{\beta,\beta}$. Hence,

$$T(\Gamma(\mathcal{N}))=\begin{cases} K_\beta\cup\underbrace{K_\beta\cup K_\beta\cup\ldots\cup K_\beta}_{(\alpha-1)copies}, & \text{if } 2\in Z(\mathcal{N})\\ K_\beta\cup\underbrace{K_{\beta,\beta}\cup K_{\beta,\beta}\cup\ldots\cup K_{\beta,\beta}}_{\left(\frac{\alpha-1}{2}\right)copies}, & \text{if } 2\notin Z(\mathcal{N}).\end{cases} \tag{3}$$

Now, we can define an DRDF $f=(\Phi,\Phi_2,\Phi_3)$ on the graph $T(\Gamma(\mathcal{N}))$. If $2\in Z(\mathcal{N})$, then from (3), α components of K_β has DRDN 2, and hence, $\gamma_{dR}(T(\Gamma(\mathcal{N})))=3\alpha$. If $2\notin Z(\mathcal{N})$, then the DRDN of each complete bipartite graph has a maximum value 6, and the DRDN of K_β is always 3.

Clearly, *f* will be an DRDF with weight $3+6\left(\frac{\alpha-1}{2}\right)=3\alpha$. Thus, in both cases, $\gamma_{dR}(T(\Gamma(\mathcal{N})))\leq 3\alpha$.

Next, we present the following result concerning the total graph of a finite field.

Theorem 5.3: *Let F be a finite field. Then* $\gamma_{dR}(T(\Gamma(F)))=\frac{3|F|+1}{2}$.

Proof: Let *F* be a finite field and $|F|=p^n$ where *p* is a prime. Also, suppose that $f=(\Phi_0,\Phi_2,\Phi_3)$ is a $\gamma_{dR}-$ function on $T(\Gamma(F))$. For $p=2$, $T(\Gamma(F))$ is an edgeless graph of 2^n vertices i.e $T(\Gamma(F))$ is totally disconnected graph, since $Z(\mathcal{N})=\{0\}$ is an ideal of *F*. Clearly, $\gamma_{dR}(T(\Gamma(F)))$ = 2×2^n = 2^{n+1}. If $p>2$, then the graph $T(\Gamma(F)))$ has one isolated vertex and $\frac{p^n-1}{2}$ copies of K_2. An isolated vertex will always assign the value 2, and the DRDN of K_2 is 3. Thus, $\gamma_{dR}(T(\Gamma(F)))$ = $2+3\times\frac{p^n-1}{2}=\frac{3p^n+1}{2}$.

Remark 1: If *F* is a field (may be finite or infinite), then the RDN of $T(\Gamma(F))$ is always less than the DRDN of $T(\Gamma(F))$.

Theorem 5.4: *Let* $\mathcal{N}=\mathbb{Z}_n$ *be a ring and n be an even integer. If* $Z(\mathcal{N})$ *is not an ideal and* $|Z(\mathcal{N})^*|=r$, *then* $\gamma_{dR}(T(\Gamma(\mathcal{N})))=6$.

Proof: Let $\mathcal{N}=\mathbb{Z}_n$ be a ring and *n*, an even integer such that $Z(\mathcal{N})$ is not an ideal of $\mathcal{N}$. Then the total graph $T(\Gamma(\mathcal{N}))$ is a regular graph, and all the members of set $Z(\mathcal{N})$ are not even. Since, $0+x$ = $x\in Z(\mathcal{N})$, every *x* is adjacent to 0, as $x\in Z(\mathcal{N})$. Now, being $|Z(\mathcal{N})^*|=r$, a vertex 0 has degree *r*. We define a function $f=(\Phi_0,\Phi_2,\Phi_3)$ on $T(\Gamma(\mathcal{N}))$. Then there exists a vertex $y\in Z(\mathcal{N})$ which is an odd number and adjacent to all the vertices which are not in $Z(\mathcal{N})$. Now, we assign the value 3 to the vertices 0 and *y*, i. e., $f=(\Phi(T(\Gamma(\mathcal{N})))-\{0,y\},\varnothing,\{0,y\})$. Clearly, the function *f* is an DRDF and hence, $w(f)=\gamma_{dR}(T(\Gamma(\mathcal{N})))=6$.

Example 5.5: Consider a ring $\mathbb{Z}_6$, then $Z(\mathbb{Z}_6)$ = $\{0,2,3,4\}$. Clearly, $Z(\mathbb{Z}_6)$ is not an ideal and $T(\Gamma(\mathbb{Z}_6))$ is a regular graph of degree 3 as shown in Figure 3. Now, we define DRDF *f* on the graph $T(\Gamma(\mathbb{Z}_6))$ such that $f(1)$ = $f(2)$ = $f(4)$ = $f(5)=0$, $f(0)$ = $f(3)=3$. Then, $w(f)=6$.

Conclusion

We have investigated the DRDN of a ZDG of the direct product of *n* rings. The same concept has also been discussed when $\mathcal{N}$ is a local ring.

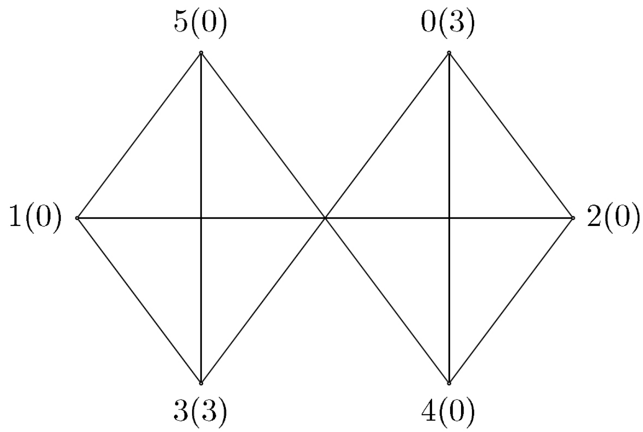

Figure 3
$T(\Gamma(\mathbb{Z}_6))$

Finally, we have calculated the DRDN of the complement of paths, cycles, and $\Gamma(\mathcal{N})$. Moreover, the bound of the total graph of $\Gamma(\mathcal{N})$ has also been discussed.

Acknowledgements

The authors thank to the Indian Institute of Technology Patna for providing research facilities.

References

[1] H. A. Ahangar, M. Chellali, and S. M. Sheikholeslami, "On the double Roman domination in graphs," *Discrete Applied Mathematics*, vol. 232, pp. 1-7 (2017).

[2] S. Akbari, H. R. Maimani and S. Yassemi, "When a zero-divisor graph is planer or a complete *r*-partite graph," *Journal of Algebra*, vol. 270, pp. 169-180 (2003).

[3] S. Akbari and A. Mohammadian, "On the zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 274, pp. 847-855 (2004).

[4] D. F. Anderson, A. Badawi, "The total graph of a commutative ring," *Journal of Algebra*, vol. 320, pp. 2706-2719 (2008).

[5] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, "The zero-divisor graph of a commutative ring II," *Lecture Notes in Pure and Applied Mathematics,* vol. 220, New York: Dekker (2001).

[6] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, pp. 434-447 (1999).

[7] R. Balakrishnan and K. Ranganathan, "A Textbook of Graph Theory," (2nd edition), Springer New York Heidelberg Dordrecht London (2012).

[8] I. Beck, "Coloring of commutative ring," *Journal of Algebra*, vol. 116, pp. 208-226 (1988).

[9] R. A. Beeler, T. W. Haynes and S. T. Hedetniemi, "Double Roman domination," *Discrete Applied Mathematics*, vol. 211, pp. 23-29 (2016).

[10] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, "Roman domination in graphs," *Discrete Mathematics,* vol. 278, pp. 11-22 (2004).

[11] N. Jafari-Rad, S. H. Jafari, and D. A. Mojdeh, "On domination in zero-divisor graph," *Canadian Mathematical Bulletin*, vol. 56, no. 2, pp. 407-411 (2013).

[12] R. Kumar and O. Prakash, "Divisor graph of the complement of $\Gamma(R)$," *Asian-European Journal of Mathematics*, vol. 12, no. 4, pp. 1950057, (2019). DOI: 10.1142/S1793557119500578.

[13] R. Kumar and O. Prakash, "Pancyclic zero divisor graph over the ring $\mathbb{Z}_n i$," *Discrete Mathematics, Algorithms and Applications*, vol. 14, no. 8, pp. 22500495, (2022). doi: 10.1142/S1793830922500495.

[14] R. Kumar and O. Prakash, "Roman domination number of zero-divisor graphs over commutative rings," Preprint arXiv:submit/6059076.

[15] E. A. Osba, S. Al-Addasi and N. A. Jaradeh, "Zero divisor graph for the ring of Gaussian integers modulo n," *Communications in Algebra*, vol. 36, no. 10, pp. 3865-3877 (2008).

[16] C. S. ReVelle and K. E. Rosing, "Defendens imperium Romanum: a classical problem in military strategy," *The American Mathematical Monthly,* vol. 107, no. 7, pp. 585-594 (2000).

[17] Z. Shao, S. M. Sheikholeslami, S. Nazari-Moghaddam and S. Wang, "Global double Roman domination in graphs," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 22, no. 1, pp. 31-44 (2019).

[18] I. Stewart, "Defend the Roman empire!," *Scientific American,* vol. 281, no. 1, pp. 136-139 (1999).

Received September, 2023