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An investigation of k-ideals in $(3, n)$-semirings with $(3, n)$-ring congruences

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Abstract

Universal algebra commonly holds that if a congruence τ is imposed on an algebra A, the quotient algebra A/τ retains the structure of the such kind of the algebra A. In this paper, we provide a congruence relation τ using a k-ideal on a $(3, n)$ -semiring A that transforms the quotient structure A/τ to be a $(3, n)$ -ring.

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1. Introduction

The introduction of a semiring, which serves like a comprehensive extension encompassing both rings and distributive lattices, can be attributed to Vandiver [29] in 1934. This algebraic structure manifests naturally in various applications within the realms of formal languages, automata theory, and optimization theory. Notably, references to its utilization can be found in (for instance, [3, 5, 6, 7, 14, 15, 17, 18, 20]). From an algebraic structural perspective, an n-ary semiring offers a valuable extension of semirings. Semirings themselves can be regarded as a specific type of n-ary semirings when $n = 2$. As a consequence, any results pertaining to n-ary semirings hold true for semirings, but the reverse fails.

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In abstract algebra, we know that if there is a homomorphism between two rings, then the kernel is an ideal [2] and it is true for the case of semirings as well. On the other hand, it is possible to pair every ring ideal with a kernel. However, it does not hold for the case of semirings [1]. Notwithstanding, it can be held if we change the ideal to be a k-ideal introduced by Henriksen [19].

In order to extend an algebra with binary operations to encompass n-ary operations, Dörnte [11] presented an n-ary group in 1928. Subsequently, Timm conducted a study on n-ary groups possessing commutative properties [28]. The algebraic structure of an n-ary semigroup, serving as a general case of both semigroups and n-ary groups, was introduced by Siosson [26, 27]. Later, Dudek conducted a study on various properties concerning idempotent elements in n-ary semigroups [13]. Crombez and Timm [8, 9] studied various concepts, including homomorphism, quotient structures, and aspects of ideal theory on algebraic structures of n-ary semigroups. Dudek [12] also investigated the divisibility property of (m,n)-rings. In 2008, Rao [22] studied the notion of (m,n)-semirings, which serves as an extension encompassing both semirings and (m,n)-rings.

In 1992, Sen and Adhikari [24] conducted a study on the concept of full k-ideals and used it to establish a congruence relation that enables the quotient semiring to be a ring. Sen and Maity [25] further investigated the properties of full k-ideals in semirings in 2021.

In the field of universal algebra, it is known that when a congruence relation τ is imposed on an n-ary semiring A, the quotient structure A/τ retains the structure of an n-ary semiring. An intriguing question arises as to what type of congruence relation τ would transform A/τ into an n-ary ring. In our previous research [21], we used a full k-ideal of an n-ary semiring to establish a congruence with the aim of transforming the quotient algebra into an n-ary ring.

This work investigates some properties of k-ideals in $(3,n)$-semirings. The algebraic structure of a $(3,n)$-semiring is an extension of an n-ary semiring, constituting an algebraic structure that encompasses both a ternary operation and an n-ary operation. We provide some basic properties of $(3,n)$-semirings accompanied by illustrative examples. Then we present the notion of k-ideals in $(3,n)$-semirings and illustrate its essential role in the structure and properties of the $(3,n)$-semiring. Next, we provide the concepts of additively regular and additively inverse $(3,n)$-semirings, followed by a characterization of a $(3,n)$-ring by an additively inverse $(3,n)$-semiring. Finally, in Section 5, we construct a congruence relation on a $(3,n)$-semiring with respect to a full k-ideal, which forms that the quotient $(3,n)$-semiring becomes a $(3,n)$-ring.

2. Preliminaries

We set the notation $\mathbb{N}$ to be the set that contains all positive integers. Let $A \neq \varnothing$ and $n \in \mathbb{N}$ where $n \geq 2$. An n-ary operation $h : A^n \to A$ on A together with the set A forms an algebraic structure $(A;h)$ called an *n-ary groupoid* [13]. Let $i \in \{1,\ldots,n\}$. The notation b_i^j is the sequence b_i, b_{i+1}, b_{i+2}, $\ldots$, b_j of elements of A. In case $j < i$, b_i^j becomes the empty symbol. If elements a_1, a_2, $\ldots$, $a_{i-1} \in A$ equal to $a \in A$, then we denote $\overset{i-1}{a}$ for a_1^{i-1}.

An n-ary groupoid $(A;h)$ is *commutative* [28] if for any $x_1^n \in A$ and for any permutation β of $\{1,2,\ldots,n\}$,

$$h(x_1^n) = h(x_{\beta(1)}, x_{\beta(2)}, \ldots, x_{\beta(n)}).$$

An *n-ary semigroup* [13] is an n-ary groupoid $(A;h)$ that satisfies the *associativity* [16], i.e., for each $1 \leq i < j \leq n$,

$$h(x_1^{i-1}, h(x_i^{n+i-1}), x_{n+i}^{2n-1}) = h(x_1^{j-1}, h(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

for all $x_1^{2n-1} \in A$.

A 3-ary semigroup $(A;g)$ is usually called a *ternary semigroup* [23]. If $\varepsilon \in A$ satisfies $c = g(\overset{2}{\varepsilon}, c) = g(\varepsilon, c, \varepsilon) = g(c, \overset{2}{\varepsilon})$ for each $c \in A$, then ε is called an *identity relative to the operation g on A* or simply called a *g-identity* of $(A;g)$. A ternary semigroup $(A;g)$ with a g-identity is called a *ternary monoid*.

Let $(A;g)$ be a ternary monoid together with a g-identity ε. For $c \in A$, if there is $z \in A$ with $g(\overset{2}{z}, c) = g(z, c, z) = g(z, \overset{2}{c}) = g(\overset{2}{c}, z) = g(c, z, c) = g(c, \overset{2}{z}) = \varepsilon$ then z is said to be an *inverse of c relative to the operation g* on A or simply called a *g-inverse* of c in $(A;g)$ and usually denoted by c^{-1}. If every element of a ternary monoid $(A;g)$ has a g-inverse, then $(A;g)$ is said to be a *ternary group*.

Definition 2.1: A $(3,n)$-semiring [4] is $(A;g,h)$ including a nonempty set A, a ternary operation g, and an n-ary operation h on A satisfying:

(1) $(A;g)$ is a ternary semigroup;

(2) $(A;h)$ is an n-ary semigroup;

(3) $(A;g,h)$ holds the distributive law, namely,

$$h(c_1^{i-1}, g(z,y,x), c_{i+1}^n) = g(h(c_1^{i-1}, z, c_{i+1}^n), h(c_1^{i-1}, y, c_{i+1}^n), h(c_1^{i-1}, x, c_{i+1}^n))$$

for any $z, y, x, c_1^n \in A$ and $1 \le i \le n$.

A $(3,n)$-semiring $(A;g,h)$ is called *additively commutative* or *g-commutative* if $(A;g)$ is commutative. *Throughout this work, we always assume that every* $(3,n)$*-semiring* $(A;g,h)$ *is g-commutative*.

In order to express the ternary operation g of a $(3,n)$-semiring $(A;g,h)$ in a simplified form, we define the notations g_2 and g_3 using the associativity of g on A by for all $x_1^7 \in A$,

$$g_2(x_1^5) = g(g(x_1^3), x_4^5) \text{ and}$$
$$g_3(x_1^7) = g(g(g(x_1^3), x_4^5), x_6^7).$$

An additively commutative $(3,n)$-semiring $(A;g,h)$ is called a $(3,n)$*-ring* if $(A;g)$ is a ternary group. It is obvious that every $(3,n)$-ring is a $(3,n)$-semiring but not conversely.

Example 2.2: Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be the set of all residue classes modulo 4. We define a ternary operation g and an n-ary operation h on $\mathbb{Z}_4$ where $n \ge 2$ by

$$g(\overline{z}, \overline{y}, \overline{x}) := \overline{z+y+x} \text{ and } h(\overline{c}_1, \ldots, \overline{c}_n) := \overline{c_1 \cdots c_n}$$

for all $\overline{z}, \overline{y}, \overline{x}, \overline{c}_1, \ldots, \overline{c}_n \in \mathbb{Z}_4$. Then $(\mathbb{Z}_4;g,h)$ is a $(3,n)$-semiring which g is commutative and $\overline{0}$ is a g-identity. However, $(\mathbb{Z}_4;g,h)$ is not a $(3,n)$-ring because the equation $g(\overline{1}, x, x) = \overline{0}$ does not have a solution.

It can be observed that a *semiring* can be classified as a $(2,2)$-semiring, while an *n-ary semiring* can be categorized as a $(2,n)$-semiring. It is possible to construct an n-ary semiring from a given semiring, and similarly, a given n-ary semiring can be utilized to construct a $(3,n)$-semiring. Nevertheless, there exists a $(3,n)$-semiring that cannot be categorized as an n-ary semiring. Hence, it is possible to regard the $(3,n)$-semiring as a generalization of the concept of an n-ary semiring. In this instance, we demonstrate it as follows.

Example 2.3: Let $\mathbb{S} = \{\overline{1}, \overline{3}\} \subseteq \mathbb{Z}_4$ where $(\mathbb{Z}_4;g,h)$ is the $(3,n)$-semiring defined in Example 2.2. Then $(\mathbb{S};g,h)$ is also a $(3,n)$-semiring. However, it can not be considered as a $(2,n)$-ary semiring because $\overline{3} + \overline{1} = \overline{0} \notin \mathbb{S}$.

An illustration of a $(3,n)$-ring is provided as follows.

Example 2.4: Let $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ be the set of all residue classes modulo 3. Then $(\mathbb{Z}_3; g, h)$ is a $(3, n)$-ring where the operations g and h are defined in Example 2.2. Since $\overline{a} = \overline{a + 0 + 0}$ and $\overline{0} = \overline{a + a + a}$ for each $\overline{a} \in \mathbb{Z}_3$, we have $\overline{0}$ a g-identity.

The notation C_i^j is the sequence $C_i, C_{i+1}, C_{i+2}, \ldots, C_j$ of nonempty subsets of a $(3, n)$-semiring $(A; g, h)$ for $1 \leq i \leq j \leq n$. The sequence C_i^j becomes an empty symbol if $j < i$. If $\emptyset \neq C_1, C_2, \ldots, C_{i-1} \subseteq A$ equal to $C \subseteq A$ then we write $\overset{i-1}{C}$ instead of C_1^{i-1}.

Let Z, Y, X, and C_1^n be nonempty subsets of a $(3, n)$-semiring $(A; g, h)$. We denote that

$$g(Z, Y, X) = \{g(t, s, r) \in A \mid t \in Z, s \in Y, \text{ and } r \in X\} \text{ and}$$
$$h(C_1^n) = \{h(c_1^n) \in A \mid c_1 \in C_1, c_2 \in C_2, \ldots, c_n \in C_n\}.$$

Let $(A; g, h)$ be a $(3, n)$-semiring and $\emptyset \neq C \subseteq A$. Then C is called an *ideal* of A if $g(\overset{3}{C}) \subseteq C$ and $h(\overset{j-1}{A}, C, \overset{n-j}{A}) \subseteq C$ for any $j \in \{1, \ldots, n\}$.

3. *k*-ideals of (3, *n*) -semirings

Prior to giving the definition of *k*-ideals, we provide some contextual information to illustrate their essential role in the structure and properties of the $(3, n)$-semiring.

The g-identity ε of a $(3, n)$-semiring $(A; g, h)$ is an *absorbing zero* if every $i \in \{1, \ldots, n\}$, $h(c_1^{i-1}, \varepsilon, c_{i+1}^n) = \varepsilon$ for all $c_1^n \in A$.

This study does not require a $(3, n)$-semiring to possess an absorbing zero. Nevertheless, if it becomes necessary, we shall provide a criterion for its inclusion.

Let $(3, n)$-semirings $(A; g, h)$ and $(A^*; g^*, h^*)$ be such that each is equipped with an absorbing zero ε and ε^*, respectively. A *homomorphism* ψ is a function $\psi : A \to A^*$ holding these conditions.

(1) For all $z, y, x \in A$, $\psi(g(z, y, x)) = g^*(\psi(z), \psi(y), \psi(x))$.

(2) For all $c_1^n \in A$, $\psi(h(c_1^n)) = h^*(\psi(c_1), \psi(c_2), \ldots, \psi(c_n))$.

(3) $\psi(\varepsilon) = \varepsilon^*$.

Let ψ be a homomorphism from $(A; g, h)$ to $(A^*; g^*, h^*)$. The set $\ker(\psi) := \{c \in A \mid \psi(c) = \varepsilon^*\}$ is designed to be the *kernel* of ψ where ε^* is an absorbing zero of $(A^*; g^*, h^*)$.

It is readily apparent that $\varepsilon \in \ker(\psi) \neq \emptyset$.

Theorem 3.1: *Let $(3,n)$-semirings $(A;g,h)$ and $(A^*;g^*,h^*)$ be such that each is equipped with an absorbing zero ε and ε^*, respectively. If $\psi : A \to A^*$ is a homomorphism, then ker(ψ) is an ideal in $(A;g,h)$.*

Proof: Let $z,y,x \in \ker(\psi)$. Then $\psi(g(z, y, x)) = g^*(\psi(z), \psi(y), \psi(x)) = g^*(\varepsilon^*) = \varepsilon^*$. It means that $g(z,y,x) \in \ker(\psi)$. Let $c_1^n \in A$ and $1 \le i \le n$. Then

$$\begin{aligned}\psi(h(c_1^{i-1}, z, c_{i+1}^n)) &= h^*(\psi(c_1),\ldots,\psi(c_{i-1}),\psi(z),\psi(c_{i+1}),\ldots,\psi(c_n))\\ &= h^*(\psi(c_1),\ldots,\psi(c_{i-1}),\varepsilon^*,\psi(c_{i+1}),\ldots,\psi(c_n))\\ &= \varepsilon^*.\end{aligned}$$

It turns out $\ker(\psi)$ an ideal of $(A;g,h)$.

Now, we present that the converse of Theorem 3.1 fails.

Example 3.2: Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A ternary operation g and an n-ary operation h on $\mathbb{N}_0$ are defined by letting $z,y,x,c_1^n \in \mathbb{N}_0$,

$$g(z,y,z) := \max\{z,y,x\} \text{ and } h(c_1^n) := c_1 \cdot c_2 \cdots c_n$$

whenever $\cdot$ is the usual multiplication of natural numbers. Thus $(\mathbb{N}_0;g,h)$ is a $(3,n)$-semiring and 0 is an absorbing zero. Obviously, $2\mathbb{N}_0 = \{0,2,4,6,\ldots\}$ is an ideal. Suppose that $(A^*;g^*,h^*)$ is a $(3,n)$-semiring with an absorbing zero ε^* and $\psi : \mathbb{N}_0 \to A^*$ is a homomorphism where $2\mathbb{N}_0 = \ker(\psi)$. Let $\kappa \in \mathbb{N}_0 \setminus 2\mathbb{N}_0$. Then we obtain that

$$\begin{aligned}\psi(\kappa) &= g^*(\psi(\kappa),\varepsilon^*,\varepsilon^*)\\ &= g^*(\psi(\kappa),\psi(\kappa+1),\psi(\kappa+1))\\ &= \psi(g(\kappa,\kappa+1,\kappa+1))\\ &= \psi(\kappa+1)\\ &= \varepsilon^*.\end{aligned}$$

This shows that $\psi(\kappa) = \varepsilon^*$ for all $\kappa \in \mathbb{N}_0$. Hence, $2\mathbb{N}_0 \subsetneq \mathbb{N}_0 = \ker(\psi)$. It is a contradiction. This means that there does not exist a $(3,n)$-semiring $(A^*;g^*,h^*)$ and a homomorphism $\psi : \mathbb{N}_0 \to A^*$ with $2\mathbb{N}_0 = \ker(\psi)$.

To establish the converse of Theorem 3.1, a special type of ideal, known as a *k*-ideal, is required.

Definition 3.3: A *k*-ideal D of a $(3,n)$-semiring $(A;g,h)$ is an ideal satisfying for any $a \in A$, $g(a,z,y) \in D$ for some $z,y \in D$ implies $a \in D$.

Let $(A;g,h)$ be a $(3,n)$-semiring. We denote that the set $K(A)$ contains all k-ideals of A.

Example 3.4: Define a ternary operation g and an n-ary operation h on $\mathbb{N}$ by

$$g(z,y,x):=\max\{z,y,x\} \text{ and } h(c_1^n):=\min\{c_1,c_2,\dots,c_n\}$$

for all $z,y,x,c_1^n \in \mathbb{N}$. Then $(\mathbb{N};g,h)$ is a $(3,n)$-semiring. Clearly, $\{1,\dots,k\} \in K(\mathbb{N})$ for each $k\in\mathbb{N}$.

Next example shows that ideals of a $(3,n)$-semiring need not k-ideals.

Example 3.5: According to the ideal $2\mathbb{N}_0$ of the $(3,n)$-semiring $(\mathbb{N}_0;g,h)$ precisely defined in Example 3.2, it fails to hold the condition of being a k-ideal because $\max\{1,2,4\} = 4\in 2\mathbb{N}_0$, whereas $2\mathbb{N}_0$ does not contain 1.

Remark 3.6: In a $(3,n)$-ring $(A;g,h)$, the ideals and the k-ideals are coincidence.

Proof: Let $(A;g,h)$ be a $(3,n)$-semiring containing a g-identity ε. Let D be an ideal of $(A;g,h)$. Let $x\in A$ be such that $w = g(x,z,y) \in D$ for some $z,y\in D$.

Then $x = g(x,\overset{2}{\varepsilon}) = g(x,\ g(z,\ z^{-1},\ z^{-1}),\ g(y,\ y^{-1},\ y^{-1})) = g(g(g(x,\ z,\ y),\ z^{-1},\ z^{-1}),\ y^{-1},\ y^{-1}) = g(g(w,\ z^{-1},\ z^{-1}),\ y^{-1},\ y^{-1}) \in D$. Now, $D\in K(A)$.

For a $(3,n)$-semiring $(A;g,h)$ and $\varnothing\neq D\subseteq A$, we denote that the set

$$\overline{D}=\{x\in A \mid g(x,z,y)=w \ \text{ for some } z,y,w\in D\}$$

is the *k-closure* of D.

Lemma 3.7: *Let $(A;g,h)$ be a $(3,n)$-semiring and $\varnothing\neq D$, $D_1^n\subseteq A$. These assertions hold.*

(1) If $g(\overset{3}{D})\subseteq D$, then $D\subseteq\overline{D}$ and $\overline{D}=\overline{\overline{D}}$.

(2) If $D_1\subseteq D_2$, then $\overline{D_1}\subseteq\overline{D_2}$.

(3) $g(\overline{D_1},\overline{D_2},\overline{D_3})\subseteq\overline{g(D_1^3)}$.

(4) $h(D_1^{i-1},\overline{D_i},D_{i+1}^n)\subseteq\overline{h(D_1^n)}$ for each $1\leq i\leq n$.

Proof: (1) and (2) are straightforward. We prove only (3) and (4). Let $\varnothing\neq D_1^n\subseteq A$.

(3) Let $x \in g(\overline{D_1}, \overline{D_2}, \overline{D_3})$ Then $x = g(\rho, \beta, \alpha)$ such that $g(\rho, w_1, v_1) = u_1$, $g(\beta, w_2, v_2) = u_2$, and $g(\alpha, w_3, v_3) = u_3$ for some $w_1, v_1, u_1 \in D_1$, $w_2, v_2, u_2 \in D_2$, and $w_3, v_3, u_3 \in D_3$. It turns out that

$$\begin{aligned} g(x, g(w_1^3), g(v_1^3)) &= g(g(\rho, \beta, \alpha), g(w_1^3), g(v_1^3)) \\ &= g(g(\rho, w_1, v_1), g(\beta, w_2, v_2), g(\alpha, w_3, v_3)) \\ &= g(u_1^3). \end{aligned}$$

Since $g(w_1^3)$, $g(v_1^3)$, and $g(u_1^3)$ belong to $g(D_1^3)$, we get that $x \in \overline{g(D_1^3)}$. Hence, $g(\overline{D_1}, \overline{D_2}, \overline{D_3}) \subseteq \overline{g(D_1^3)}$.

(4) Let $1 \le i \le n$ and let $z \in h(D_1^{i-1}, \overline{D_i}, D_{i+1}^n)$. Then $z = h(\delta_1^{i-1}, y, \delta_{i+1}^n)$ for some $y \in \overline{D_i}$ and $\delta_j \in D_j$ where $1 \le j \le n$ and $j \ne i$. So, $g(y, \rho, \beta) = \alpha$ for some $\rho, \beta, \alpha \in D_i$. It turns out that

$$\begin{aligned} g(z, h(\delta_1^{i-1}, \rho, \delta_{i+1}^n), h(\delta_1^{i-1}, \beta, \delta_{i+1}^n)) &= g(h(\delta_1^{i-1}, y, \delta_{i+1}^n), h(\delta_1^{i-1}, \rho, \delta_{i+1}^n), h(\delta_1^{i-1}, \beta, \delta_{i+1}^n)) \\ &= h(\delta_1^{i-1}, g(y, \rho, \beta), \delta_{i+1}^n) \\ &= h(\delta_1^{i-1}, \alpha, \delta_{i+1}^n). \end{aligned}$$

Since $h(\delta_1^{i-1}, \rho, \delta_{i+1}^n)$, $h(\delta_1^{i-1}, \beta, \delta_{i+1}^n)$, and $h(\delta_1^{i-1}, \alpha, \delta_{i+1}^n)$ belong to $h(D_1^n)$, we get that $z \in \overline{h(D_1^n)}$. Hence, $h(D_1^{i-1}, \overline{D_i}, D_{i+1}^n) \subseteq \overline{h(D_1^n)}$.

Lemma 3.8: *Let $(A; g, h)$ be a $(3, n)$-semiring. Then $\overline{D} \in K(A)$ whenever D is an ideal of A.*

Proof: Using Lemma 3.7(3), $g(\overset{3}{\overline{D}}) \subseteq \overline{g(\overset{3}{D})} \subseteq \overline{D}$. Moreover, using Lemma 3.7(4), we get that for each $1 \le i \le n$, $h(\overset{i-1}{A}, \overline{D}, \overset{n-i}{A}) \subseteq \overline{h(\overset{i-1}{A}, D, \overset{n-i}{A})} \subseteq \overline{D}$.

Let $z \in A$ be such that $g(z, w, v) = u$ for some $w, v, u \in \overline{D}$. Using Lemma 3.7(1), $z \in \overline{\overline{D}} = \overline{D}$. Hence, $\overline{D} \in K(A)$.

Corollary 3.9: *Let $(A; g, h)$ be a $(3, n)$-semiring and D be an ideal of A. Then $D \in K(A)$ iff $D = \overline{D}$.*

Analogously to Theorem 3.1, we derive the subsequent theorem in the case of *k*-ideals.

Theorem 3.10: *Let* $(A;g,h)$ *and* $(A^*;g^*,h^*)$ *be* $(3,n)$*-semirings be such that each is equipped with an absorbing zero* ε *and* ε^**, respectively. If* $\psi: A \to A^*$ *is a homomorphism, then ker* $(\psi) \in K(A)$.

Proof: Let $z \in A$ with $g(z,w,v) \in \ker(\psi)$ for some $w,v \in \ker(\psi)$. Then $\psi(z) = g^*(\psi(z), \varepsilon^*, \varepsilon^*) = g^*(\psi(z), \psi(w), \psi(v)) = \psi(g(z, w, v)) = \varepsilon^*$. Hence, $z \in \ker(\psi)$. Therefore, $\ker(\psi) \in K(A)$.

Before setting up the converse of Theorem 3.10, it is imperative to provide contextual information regarding a congruence relation and a quotient $(3,n)$-semiring.

A *congruence relation* [10] τ on a $(3,n)$-semiring is an equivalent relation satisfying the condition; for each w, v, $\delta_1^n \in A$, $(w,v) \in \tau$ implies

$$(g(\delta_1,w,\delta_2), g(\delta_1,v,\delta_2)) \in \tau \text{ and } (h(\delta_1^{i-1},w,\delta_{i+1}^n), g(\delta_1^{i-1},v,\delta_{i+1}^n)) \in \tau$$

for any $i \in \{1,\ldots,n\}$.

We set $C(A)$ the set containing all congruence relations on $(A;g,h)$. Let $\tau \in C(A)$. The algebraic structure $(A/\tau;G,H)$ is also a $(3,n)$-semiring which is called the *quotient* $(3,n)$*-semiring* of $(A;g,h)$ where for all $\delta, \delta_1^n \in A$,

$$[\delta]_\tau := \{\rho \in A \mid (\delta,\rho) \in \tau\},$$

$$A/\tau := \{[\delta]_\tau \mid \delta \in A\},$$

$$G([\delta_1]_\tau,[\delta_2]_\tau,[\delta_3]_\tau) := [g(\delta_1,\delta_2,\delta_3)]_\tau, \text{ and}$$

$$H([\delta_1]_\tau,[\delta_2]_\tau,\ldots,[\delta_n]_\tau) := [h(\delta_1^n)]_\tau.$$

The homomorphism $\psi: A \to A/\tau$ defined by letting $\delta \in A$, $\psi(\delta) := [\delta]_\tau$ is called the *natural homomorphism*.

Theorem 3.11: *Let* $(A;g,h)$ *be a* $(3,n)$*-semiring containing an absorbing zero* ε*. If D is a k-ideal of* $(A;g,h)$*, then* $D = \ker(\psi)$ *for some a homomorphism* ψ.

Proof: Assume that D is a k-ideal. Define a relation with respect to D on A by

$$\sigma := \{(\delta,\rho) \in A \times A \mid g(\delta,w_1,v_1) = g(\rho,w_2,v_2) \text{ for some } w_1,v_1,w_2,v_2 \in D\}.$$

It is easy to obtain that σ is an equivalent relation. So, we are able to define equivalent classes on A by for each $\delta \in A$,

$$[\delta]_\sigma := \{\rho \in A \mid (\delta,\rho) \in \sigma\}.$$

It obtains that $\sigma \in C(A)$. Let $A/\sigma := \{[\delta]_\sigma \mid \delta \in A\}$. Define a ternary operation G_σ and H_σ on A/σ by

$$G_\sigma([\delta_1]_\sigma, [\delta_2]_\sigma, [\delta_3]_\sigma) := [g(\delta_1, \delta_2, \delta_3)]_\sigma \text{ and } H_\sigma([\delta_1]_\sigma, [\delta_2]_\sigma, \ldots, [\delta_p]_\sigma) := [h(\delta_1^n)]_\sigma$$

for any $\delta_1^n \in A$. It can be immediately obtained that $(A/\sigma; G_\sigma, H_\sigma)$ is the quotient $(3,n)$-semiring of $(A; g, h)$. Let $\psi : A \to A/\sigma$ be the natural homomorphism defined as $\psi(\delta) := [\delta]_\sigma$ for all $\delta \in A$.

Now, we show that $D = \ker(\psi)$. For any $\delta, \delta_1^n \in A$, we have that $G_\sigma([\delta]_\sigma, [\varepsilon]_\sigma) = [g(\delta, \varepsilon)]_\sigma = [\delta]_\sigma$ and

$$H_\sigma([\delta_1]_\sigma, \ldots, [\delta_{i-1}]_\sigma, [\varepsilon]_\sigma, [\delta_{i+1}]_\sigma, \ldots, [\delta_p]_\sigma) = [h(\delta_1^{i-1}, \varepsilon, \delta_{i+1}^n)]_\sigma = [\varepsilon]_\sigma.$$

Hence, $[\varepsilon]_\sigma$ is an absorbing zero of $(A/\sigma; G_\sigma, H_\sigma)$ and $\ker(\psi) = \{\delta \in A \mid \psi(\delta) = [\varepsilon]_\sigma\}$. Consider the following assertion.

$$\begin{aligned}
\delta \in D &\Leftrightarrow \delta \in D = \overline{D} \\
&\Leftrightarrow g(\delta, w, v) = u \text{ for some } w, v, u \in D \\
&\Leftrightarrow g(\delta, w, v) = g(\varepsilon, \varepsilon, u) \text{ for some } w, v, u \in D \\
&\Leftrightarrow (\delta, \varepsilon) \in \sigma \\
&\Leftrightarrow \psi(\delta) = [\delta]_\sigma = [\varepsilon]_\sigma \\
&\Leftrightarrow \delta \in \ker(\psi).
\end{aligned}$$

Therefore, $D = \ker(\psi)$.

Next result is obtained by Theorem 3.10 and 3.11.

Corollary 3.12: *An ideal D of a $(3,n)$-semiring $(A; g, h)$ is the kernel of a homomorphism if and only if $D \in K(A)$.*

4. Additively inverse $(3,n)$-semirings

This section introduces the concepts of additively regular and additively inverse $(3,n)$-semirings, followed by a characterization of a $(3,n)$-ring by an additively inverse $(3,n)$-semiring.

Section 2 presents the definition of a g-inverse of an element in a $(3,n)$-semiring using the g-identity. In the subsequent discussion, we define the notion of an additively inverse of an element in a $(3,n)$-semiring utilizing the concept of an additively regular element, as described below.

Definition 4.1: An *additively regular element* α of a $(3,n)$-semiring $(A; g, h)$ is an element of A satisfying

$$\alpha = g_3(\alpha, \overset{2}{\delta}, \alpha, \overset{2}{\delta}, \alpha) \tag{4.1}$$

for some $\delta \in A$. If all elements of A are additively regular then $(A; g, h)$ said to be *additively regular*. If the element δ in the equation (4.1) is unique and satisfies the condition:

$$\delta = g_3(\delta, \overset{2}{\alpha}, \delta, \overset{2}{\alpha}, \delta), \tag{4.2}$$

then δ is said to be *additively inverse* to α in $(A; g, h)$ and usually given by α'. We call a $(3, n)$-semiring $(A; g, h)$ to be *additively inverse* if each of its element has an additively inverse.

Throughout this section and the next section, we will always denote $(A; g, h)$ *as an additively inverse* $(3, n)$*-semiring.*

One can prove that $t = (t')'$ and $g(t, v, u)' = g(t', v', u')$ for all $t, v, u \in (A; g, h)$.

Lemma 4.2: *Let* $\alpha_1^n \in (A; h, g)$. *Then* $(h(\alpha_1^n))' = h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n)$ *for each* $1 \le i \le n$.

Proof: Consider

$$\begin{aligned}
&g_3(h(\alpha_1^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n), h(\alpha_1^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n), h(\alpha_1^n))\\
&= g(h(\alpha_1^{i-1}, g(\alpha_i, \overset{2}{\alpha_i'}), \alpha_{i+1}^n), h(\alpha_1^{i-1}, g(\alpha_i, \overset{2}{\alpha_i'}), \alpha_{i+1}^n), h(\alpha_1^{i-1}, \alpha_i, \alpha_{i+1}^n))\\
&= h(\alpha_1^{i-1}, g_3(\alpha_i, \overset{2}{\alpha_i'}, \alpha_i, \overset{2}{\alpha_i'}, \alpha_i), \alpha_{i+1}^n)\\
&= h(\alpha_1^{i-1}, \alpha_i, \alpha_{i+1}^n)\\
&= h(\alpha_1^n) \text{ and}\\
&g_3(h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n), h(\alpha_1^n), h(\alpha_1^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n), h(\alpha_1^n), h(\alpha_1^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n))\\
&= g(h(\alpha_1^{i-1}, g(\alpha_i', \overset{2}{\alpha_i}), \alpha_{i+1}^n), h(\alpha_1^{i-1}, g(\alpha_i', \overset{2}{\alpha_i}), \alpha_{i+1}^n), h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n))\\
&= h(\alpha_1^{i-1}, g_3(\alpha_i', \overset{2}{\alpha_i}, \alpha_i', \overset{2}{\alpha_i}, \alpha_i'), \alpha_{i+1}^n)\\
&= h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n).
\end{aligned}$$

Hence, $(h(\alpha_1^n))' = h(\alpha_1^{i-1}, \alpha_i', \alpha_{i+1}^n)$.

An element ρ of $(A; g, h)$ is said to be *additively idempotent* if $\rho = g(\overset{3}{\rho})$ and we denote $E(A) = \{\rho \in A \mid \rho = g(\overset{3}{\rho})\}$. Obviously, $E(A)$ is an ideal of $(A; g, h)$.

Remark 4.3: Let $(A; g, h)$ be a $(3, n)$-ring with a g-identity ε. Then ε is the only one additively idempotent of $(A; g, h)$.

Proof: Clearly, $\varepsilon = g(\overset{3}{\varepsilon})$ is an additively idempotent. Assume that $\rho \in E(A)$. Then $g(\overset{3}{\rho}) = \rho = g(g(\overset{3}{\rho}), \overset{2}{\rho})$ and

$$
\begin{aligned}
g(\rho^{-1}, \rho, \rho^{-1}) &= g(\rho^{-1}, g(g(\overset{3}{\rho}), \overset{2}{\rho})), \rho^{-1}) \\
\varepsilon &= g(g(\rho^{-1}, \overset{2}{\rho}), \rho, g(\overset{2}{\rho}, \rho^{-1})) \\
\varepsilon &= g(\varepsilon, \rho, \varepsilon) \\
\varepsilon &= \rho.
\end{aligned}
$$

This shows that ε is the only one additively idempotent of $(A; g, h)$.

Remark 4.4: Let $\delta \in (A; g, h)$. Then $g(\delta, \overset{2}{\delta'})$ and $g(\delta', \overset{2}{\delta})$ are additively idempotent.

Proof: Let $\delta \in (A; g, h)$. Then $\delta = g_3(\delta, \overset{2}{\delta'}, \delta, \overset{2}{\delta'}, \delta) = g(g(\delta, \overset{2}{\delta'}), g(\delta, \overset{2}{\delta'}), \delta)$ and $g(\delta, \overset{2}{\delta'}) = g(g(\delta, \overset{2}{\delta'}), g(\delta, \overset{2}{\delta'}), g(\delta, \overset{2}{\delta'}))$. Hence, $g(\delta, \overset{2}{\delta'})$ is additively idempotent. The another case can be proved similarly.

Theorem 4.5: *A $(3, n)$-semiring $(A; g, h)$ is additively inverse and contains exactly one additively idempotent element iff $(A; g, h)$ is a $(3, n)$-ring.*

Proof: Let $\delta \in A$. Assume that $(A; g, h)$ is additively inverse and contains exactly one additively idempotent element. By assumption, we obtain that $\delta = g_3(\delta, \overset{2}{\delta'}, \delta, \overset{2}{\delta'}, \delta)$. Using assumption and Remark 4.4, we set

$$
g(\delta, \overset{2}{\delta'}) = \varepsilon = g(\delta', \overset{2}{\delta}). \tag{4.3}
$$

Then, we get that $g(\delta, \overset{2}{\varepsilon}) = g(\delta, g(\delta, \overset{2}{\delta'}), g(\delta, \overset{2}{\delta'})) = g_3(\delta, \overset{2}{\delta'}, \delta, \overset{2}{\delta'}, \delta) = \delta$. Now, ε is a g-identity of $(A; g, h)$. Using Equation (4.3), we already have that δ' is a g-inverse of δ in $(A; g, h)$. Therefore, $(A; g, h)$ is a $(3, n)$-ring.

Conversely, assume that $(A; g, h)$ is a $(3, n)$-ring together with a g-identity ε. Let $\rho \in A$. Then $\rho = g(\overset{2}{\varepsilon}, \rho) = g(g(\rho, \rho^{-1}, \rho^{-1}), g(\rho, \rho^{-1}, \rho^{-1}), \rho) = g_3(\rho, \rho^{-1}, \rho^{-1}, \rho, \rho^{-1}, \rho^{-1}, \rho)$ and $\rho^{-1} = g(\overset{2}{\varepsilon}, \rho^{-1}) = g(g(\rho^{-1},$

$\overset{2}{\rho}),\ g(\rho^{-1},\ \overset{2}{\rho}),\ \rho^{-1}) = g_3(\rho^{-1},\ \overset{2}{\rho},\ \rho^{-1},\ \overset{2}{\rho},\ \rho^{-1})$. If there exists $\beta \in A$ such that $\rho = g_3(\rho,\ \overset{2}{\beta},\ \rho,\ \overset{2}{\beta},\ \rho) = g(g(\rho,\ \overset{2}{\beta}),\ g(\rho,\ \overset{2}{\beta}),\ \rho)$ and $\beta = g_3(\beta,\ \overset{2}{\rho},\ \beta,\ \overset{2}{\rho},\ \beta)$ $= g(g(\beta,\ \overset{2}{\rho}),\ g(\beta,\ \overset{2}{\rho}),\ \beta)$, then we obtain that $g(\beta, \overset{2}{\rho}) = g(g(\beta,\ \overset{2}{\rho}),\ g(\beta,\ \overset{2}{\rho}),\ g(\beta,\ \overset{2}{\rho}))$ and $g(\overset{2}{\rho},\ \beta) = g(g(\overset{2}{\rho},\ \beta),\ g(\overset{2}{\rho},\ \beta),\ g(\overset{2}{\rho},\ \beta))$ are additively idempotent. By Remark 4.3, we get that $g(\beta, \overset{2}{\rho}) = \varepsilon = g(\overset{2}{\rho}, \beta)$.. It follows that

$$
\begin{aligned}
\beta &= g(g(\beta, \overset{2}{\rho}), g(\beta, \overset{2}{\rho}), \beta) \\
&= g(\overset{2}{\varepsilon}, \beta) \\
&= g(g(\rho^{-1}, \overset{2}{\rho}), g(\rho^{-1}, \overset{2}{\rho}), \beta) \\
&= g(g(\rho^{-1}, \overset{2}{\rho}), \rho^{-1}, g(\overset{2}{\rho}, \beta)) \\
&= g(g(\rho^{-1}, \overset{2}{\rho}), \rho^{-1}, \varepsilon) \\
&= g(g(\rho^{-1}, \overset{2}{\rho}), \rho^{-1}, g(\overset{2}{\rho}, \rho^{-1})) \\
&= g(g(\rho^{-1}, \overset{2}{\rho}), (\rho^{-1}, \overset{2}{\rho}), \rho^{-1}) \\
&= \rho^{-1}.
\end{aligned}
$$

Hence, $(A; g, h)$ is additively inverse. Using Remark 4.3, we immediately obtain that $(A; g, h)$ contains exactly one additively idempotent element.

5. Full *k*-ideals and (3, *n*) -ring congruences of (3, *n*) -semirings

In the field of universal algebra, it is widely recognized that if A is algebra and $\tau \in C(A)$ then algebra A/τ serves as the quotient algebra of A [10]. Hence, given a congruence τ on a $(3, n)$-semiring A, it follows that A/τ is the corresponding quotient $(3, n)$-semiring of A.

This section presents the construction of a congruence relation on $(A; g, h)$ with respect to a special type of *k*-ideals, specifically full *k*-ideals, which guarantees that the resulting quotient $(3, n)$-semiring constitutes a $(3, n)$-ring.

Definition 5.1: A *full k-ideal D* of $(A; g, h)$ is a *k*-ideal such that $E(A) \subseteq D$.

We denote $FK(A) = \{D \in K(A) \mid \mathrm{E}(A) \subseteq D\}$.

Example 5.2: The k-ideal $\{1,2,\ldots,k\}$ of the $(3,n)$-semiring $(\mathbb{N};g,h)$ defined in Example 3.4 is not full, since $\mathbb{N} = E(\mathbb{N}) \not\subseteq \{1,2,\ldots,k\}$.

Example 5.3: Define a ternary operation g on $\mathbb{N}_0$ by $g(w,v,u) := w+v+u$ for all $w,v,u \in \mathbb{N}_0$. Then $(\mathbb{N}_0;g,h)$ is a $(3,n)$-semiring where h is the n-ary operation as Example 3.2 and $2\mathbb{N}_0$ is a full k-ideal.

Remark 5.4: If $D \in FK(A)$ then D is also additively inverse.

Proof: Let $\eta \in D$. We have $\eta = g_3(\eta, \overset{2}{\eta'}, \eta, \overset{2}{\eta'}, \eta) = g(\eta', g(\eta', \overset{2}{\eta}), g(\eta, \overset{2}{\eta'}))$ $\in D$. Since $g(\eta', \overset{2}{\eta}), g(\eta, \overset{2}{\eta'}) \in E(A) \subseteq D,$ we get that $\eta' \in \overline{D} = D.$ Hence, D is additively inverse.

If the quotient $(3,n)$-semiring derived from a given $(3,n)$-semiring is a $(3,n)$-ring, then the subsequent notion is introduced and defined.

Definition 5.5: A relation $\tau \in C(A)$ is said to be a $(3,n)$-*ring congruence* if the quotient $(3,n)$-semiring $(A/\tau;G_\tau,H_\tau)$ is a $(3,n)$-ring.

At this point, we provide a $(3,n)$-ring congruence of a $(3,n)$-semiring using a full k-ideal.

Theorem 5.6: *If* $D \in FK(A)$ *then*

$$\tau_D := \{(\eta,\iota) \in A \times A \mid g(\eta, \overset{2}{\iota'}) \text{ and } g(\iota', \overset{2}{\eta}) \text{ belong to } D\}$$

is a $(3,n)$*-ring congruence.*

Proof: We prove τ_D an equivalent relation on A. Let $z,\ w,\ d \in A$. By Remark 4.4, we get $g(z, \overset{2}{z'}),\ g(z', \overset{2}{z}) \in E(A) \subseteq D$. This shows that $(z,z) \in \tau_D$, i.e., τ_D is reflexive. If $(z,w) \in \tau_D$, then $g(z', \overset{2}{w}) = (g(z, \overset{2}{w'}))' \in D$ and $g(w, \overset{2}{z'}) = (g(w', \overset{2}{z}))' \in D$. So, $(w,z) \in \tau_D$, i.e., τ_D is symmetric. Assume that $(z,w),\ (w,d) \in \tau_D$. Then $g(z, \overset{2}{w'}),\ g(w', \overset{2}{z}),\ g(w, \overset{2}{d'}),\ g(d', \overset{2}{w}) \in D$. By Remark 4.4, we get $g(w, \overset{2}{w'}),\ g(w', \overset{2}{w}) \in E(A) \subseteq D$. It follows that

$$g(g(z, \overset{2}{d'}), g(w, \overset{2}{w'}), g(w, \overset{2}{w'})) = g(g(z, \overset{2}{w'}), g(w, \overset{2}{w'}), g(w, \overset{2}{d'})) \in D \text{ and}$$
$$g(g(d', \overset{2}{z}), g(w', \overset{2}{w}), g(w', \overset{2}{w})) = g(g(w', \overset{2}{z}), g(w', \overset{2}{w}), g(d', \overset{2}{w})) \in D.$$

These imply $g(z,\overset{2}{d'})\in D$ and $g(d',\overset{2}{z})\in D$. Thus, $(z,d)\in\tau_D$, i.e., τ_D is transitive. Hence, τ_D is an equivalent relation on A.

Next, we show that $\tau_D\in C(A)$. Let $\eta,\iota,\delta_1^n\in A$ with $(\eta,\iota)\in\tau_D$. It turns out $g(\eta,\overset{2}{\iota'})$, $g(\iota',\overset{2}{\eta})\in D$. Moreover, we get $g(\delta_1,\overset{2}{\delta_1'})$, $g(\delta_1',\overset{2}{\delta_1})$, $g(\delta_2,\overset{2}{\delta_2'})$, $g(\delta_2',\overset{2}{\delta_2})\in E(A)\subseteq D$. It serves that

$$\begin{aligned}
g(g(\eta,\delta_1^2),g(\iota,\delta_1^2)',g(\iota,\delta_1^2)') &= g(g(\eta,\delta_1^2),g(\iota',\delta_1',\delta_2'),g(\iota',\delta_1',\delta_2'))\\
&= g(g(\eta,\overset{2}{\iota'}),g(\delta_1,\overset{2}{\delta_1'}),g(\delta_2,\overset{2}{\delta_2'}))\\
&\in D \text{ and}\\
g(g(\iota,\delta_1^2)',g(\eta,\delta_1^2),g(\eta,\delta_1^2)) &= g(g(\iota',\delta_1',\delta_2'),g(\eta,\delta_1^2),g(\eta,\delta_1^2))\\
&= g(g(\iota',\overset{2}{\eta}),g(\delta_1',\overset{2}{\delta_1}),g(\delta_2',\overset{2}{\delta_2}))\\
&\in D.
\end{aligned}$$

This shows that $(g(\eta,\delta_1^2),g(\iota,\delta_1^2))\in\tau_D$. Using the fact that D is an ideal, it turns out that for each $1\le i\le n$,

$$\begin{aligned}
&g(h(\delta_1^{i-1},\eta,\delta_{i+1}^n),h(\delta_1^{i-1},\iota,\delta_{i+1}^n)',h(\delta_1^{i-1},\iota,\delta_{i+1}^n)')\\
&= g(h(\delta_1^{i-1},\eta,\delta_{i+1}^n),h(\delta_1^{i-1},\iota',\delta_{i+1}^n),h(\delta_1^{i-1},\iota',\delta_{i+1}^n))\\
&= h(\delta_1^{i-1},g(\eta,\overset{2}{\iota'}),\delta_{i+1}^n)\in D \text{ and}\\
&g(h(\delta_1^{i-1},\iota,\delta_{i+1}^n)',h(\delta_1^{i-1},\eta,\delta_{i+1}^n),h(\delta_1^{i-1},\eta,\delta_{i+1}^n))\\
&= g(h(\delta_1^{i-1},\iota',\delta_{i+1}^n),h(\delta_1^{i-1},\eta,\delta_{i+1}^n),h(\delta_1^{i-1},\eta,\delta_{i+1}^n))\\
&= h(\delta_1^{i-1},g(\iota',\overset{2}{\eta}),\delta_{i+1}^n)\in D.
\end{aligned}$$

This shows that $(h(\delta_1^{i-1},\eta,\delta_{i+1}^n),h(\delta_1^{i-1},\iota,\delta_{i+1}^n))\in\tau_D$. Hence, $\tau_D\in C(A)$.

Now, we define the equivalent class A/τ_D, the ternary operation G, and the n-ary operation H on A/τ_D by

$$\begin{aligned}
[\rho]_{\tau_D} &:= \{\alpha\in A\mid(\rho,\alpha)\in\tau_D\},\\
A/\tau_D &:= \{[\rho]_{\tau_D}\mid\rho\in A\},\\
G([\rho_1]_{\tau_D},[\rho_2]_{\tau_D},[\rho_3]_{\tau_D}) &:= [g(\rho_1^3)]_{\tau_D}, \text{ and}\\
H([\rho_1]_{\tau_D},[\rho_2]_{\tau_D},\dots,[\rho_n]_{\tau_D}) &:= [h(\rho_1^n)]_{\tau_D}.
\end{aligned}$$

for all $\rho, \rho_1^n \in A$. Now, $(A/\tau_D;\, G, H)$ is the quotient $(3, n)$-semiring of $(A; g, h)$.

We prove that $(A/\tau_D;\, G, H)$ is a $(3, n)$-ring. Let $\rho \in A$ and $\varepsilon \in E(A)$. Then $g(\rho, \overset{2}{\rho'}),\ g(\rho', \overset{2}{\rho}) \in E(A) \subseteq D$. It follows that

$$g(g(\rho, \overset{2}{\varepsilon}), \overset{2}{\rho'}) = g(g(\rho, \overset{2}{\rho'}), \overset{2}{\varepsilon}) \in D \text{ and}$$

$$\begin{aligned} g(\rho', g(\rho, \overset{2}{\varepsilon}), g(\rho, \overset{2}{\varepsilon})) &= g(g(\rho', \overset{2}{\rho}), g(\overset{3}{\varepsilon}), \varepsilon) \\ &= g(g(\rho', \overset{2}{\rho}), \overset{2}{\varepsilon}) \in D. \end{aligned}$$

Hence, $(g(\rho, \overset{2}{\varepsilon}), \varepsilon) \in \tau_D$. We obtain that

$$G([\rho]_{\tau_D}, [\overset{2}{\varepsilon}]_{\tau_D}) = [g(\rho, \overset{2}{\varepsilon})]_{\tau_D} = [\rho]_{\tau_D}.$$

Thus, $[\varepsilon]_{\tau_D}$ is a G-identity of $(A/\tau_D; G, H)$. We obtain that

$$g(\varepsilon, (g(\rho, \overset{2}{\rho'}))', (g(\rho, \overset{2}{\rho'}))') = g(\varepsilon, g(\rho', \overset{2}{\rho}), g(\rho', \overset{2}{\rho})) \in D,$$

$$g((g(\rho, \overset{2}{\rho'}))', \overset{2}{\varepsilon}) = g(g(\rho', \overset{2}{\rho}), \overset{2}{\varepsilon}) \in D,$$

$$g(\varepsilon, (g(\rho', \overset{2}{\rho}))', (g(\rho', \overset{2}{\rho}))') = g(\varepsilon, g(\rho, \overset{2}{\rho'}), g(\rho, \overset{2}{\rho'})) \in D, \text{ and}$$

$$g((g(\rho', \overset{2}{\rho}))', \overset{2}{\varepsilon}) = g(g(\rho, \overset{2}{\rho'}), \overset{2}{\varepsilon}) \in D.$$

This shows that $(\varepsilon, g(\rho, \overset{2}{\rho'})), (\varepsilon, g(\rho', \overset{2}{\rho})) \in \tau_D$. We obtain that

$$G([\rho]_{\tau_D}, [\overset{2}{\rho'}]_{\tau_D}) = [g(\rho, \overset{2}{\rho'})]_{\tau_D} = [\varepsilon]_{\tau_D} \text{ and}$$

$$G([\rho']_{\tau_D}, [\overset{2}{\rho}]_{\tau_D}) = [g(\rho', \overset{2}{\rho})]_{\tau_D} = [\varepsilon]_{\tau_D}.$$

This means that $[\rho']_{\tau_D}$ is a G-inverse of $[\rho]_{\tau_D}$ in $(A/\tau_D; G, H)$. Consequently, $(A/\tau_D; G, H)$ is a $(3, n)$-ring.

Theorem 5.7: *Let $\rho \in C(A)$. If $(A/\rho; G, H)$ is a $(3, n)$-ring, then there exists $D \in FK(A)$ such that $\rho = \tau_D$.*

Proof: Let $D = \{\delta \in A \mid (\delta, \varepsilon) \in \rho$ for some $\varepsilon \in E(A)\}$. Clearly, $(\varepsilon, \varepsilon) \in \rho$ for each $\varepsilon \in E(A)$ and so $E(A) \subseteq D \neq \varnothing$. Let $y, v, c \in D$. Then (y, e_1), (v, e_2), $(c, e_3) \in \rho$ for some $e_1, e_2, e_3 \in E(A)$. Now, we get that $(g(y, v, c), g(e_1^3)) \in \rho$. Since $E(A)$ is an ideal of $(A; g, h)$, we get $g(e_1^3) \in E(A)$. Hence,

$g(y,v,c) \in D$ and so $g(\overset{2}{D}) \subseteq D$. Let $z \in h(\overset{i-1}{A}, D, \overset{n-i}{A})$ for each $1 \leq i \leq n$. So, $z = h(\delta_1^{i-1}, b, \delta_{i+1}^n)$ for some $\delta_1^n \in A$ and $b \in D$ such that $(b, e_4) \in \rho$ for some $e_4 \in E(A)$. Since ρ is a congruence relation on A, we get $(z, h(\delta_1^{i-1}, e_4, \delta_{i+1}^n)) = (h(\delta_1^{i-1}, b, \delta_{i+1}^n), h(\delta_1^{i-1}, e_4, \delta_{i+1}^n)) \in \rho$. Then $h(\delta_1^{i-1}, e_4, \delta_{i+1}^n) \in E(A)$, since $E(A)$ is an ideal of $(A; g, h)$. Hence, $z = h(\delta_1^{i-1}, b, \delta_{i+1}^n) \in D$ and so $h(\overset{i-1}{A}, D, \overset{n-i}{A}) \subseteq D$. Now, D is an ideal of $(A; g, h)$.

Let $x \in \overline{D}$. We get $g(x, w, u) = a$ for some $w, u, a \in D$ such that (w, e_5), (u, e_6), $(a, e_7) \in \rho$ for some $e_5, e_6, e_7 \in E(A)$. Since $[e_5]_\rho$, $[e_6]_\rho$, $[e_7]_\rho \in E(A/\rho)$, we have $[e_5]_\rho = [e_6]_\rho = [e_7]_\rho$ the G-identity of $(A/\rho; G, H)$. Then

$$[e_7]_\rho = [a]_\rho = [g(x,w,u)]_\rho = G([x]_\rho, [w]_\rho, [u]_\rho) = G([x]_\rho, [e_5]_\rho, [e_6]_\rho) = [x]_\rho.$$

Thus, $(x, e_7) \in \rho$ and hence $x \in D$. Hence, $\overline{D} = D$. By Corollary 3.9, $D \in FK(A)$.

Now, we prove that $\rho = \tau_D$. Let $(\eta, \iota) \in \rho$. Using the fact that $\rho \in C(A)$, we get that $(g(\eta, \overset{2}{\iota'}), g(\iota, \overset{2}{\iota'})) \in \rho$. Since $g(\iota, \overset{2}{\iota'}) \in E(A)$, $g(\eta, \overset{2}{\iota'}) \in D$. Using the fact that $\rho \in C(A)$ again, we obtain that $(g(\iota', \overset{2}{\eta}), g(\iota', \iota, \eta))$, $(g(\iota', \iota, \eta), g(\iota', \overset{2}{\iota})) \in \rho$. Using the transitivity of ρ, we get that $(g(\iota', \overset{2}{\eta}), g(\iota', \overset{2}{\iota})) \in \rho$. Since $g(\iota', \overset{2}{\iota}) \in E(A)$, $g(\iota', \overset{2}{\eta}) \in D$. Now, $(\eta, \iota) \in \tau_D$. So, $\rho \subseteq \tau_D$. If $(\eta, \iota) \in \tau_D$, then $g(\eta, \overset{2}{\iota'})$, $g(\iota', \overset{2}{\eta}) \in D$. Thus, $(g(\eta, \overset{2}{\iota'}), e_8)$, $(g(\iota', \overset{2}{\eta}), e_9) \in \rho$ for some $e_8, e_9 \in E(A)$. We obtain that

$$\begin{aligned}
[\iota]_\rho &= G([\iota]_\rho, [e_8]_\rho, [e_9]_\rho) \\
&= G([\iota]_\rho, [g(\eta, \overset{2}{\iota'})]_\rho, [e_9]_\rho) \\
&= G([\iota]_\rho, G([\eta]_\rho, [\overset{2}{\iota'}]_\rho), [e_9]_\rho) \\
&= G([\eta]_\rho, G([\iota]_\rho, [\overset{2}{\iota'}]_\rho), [e_9]_\rho) \\
&= G([\eta]_\rho, [g(\iota, \overset{2}{\iota'})]_\rho, [e_9]_\rho)
\end{aligned}$$

Since $g(\iota,\overset{2}{\iota'}) \in E(A)$, $[g(\iota,\overset{2}{\iota'})]_{\rho} = [\varepsilon]_{\rho}$ is the G-identity of $(A/\rho;G,H)$. Thus,

$$[\iota]_{\rho} = G([\eta]_{\rho},[e_9]_{\rho},[e_9]_{\rho}) = [\eta]_{\rho}$$

and so $(\eta,\iota) \in \rho$. Hence, $\tau_D \subseteq \rho$. Therefore, $\rho = \tau_D$.

Conclusion and Discussion

As an extension of a semiring and an n-ary semiring, we provide the concept of a $(3,n)$-semiring. A semiring and an n-ary semiring can be classified as a $(2,2)$-semiring and a $(2,n)$-semiring, respectively. We are able to construct a $(3,n)$-semiring from a given semiring or a given n-ary semiring. However, there is a $(3,n)$-semiring that can not be considered as an n-ary semiring as shown by Example 2.3.

The kernel of a homomorphism between $(3,n)$-semiring is an ideal but not conversly as shown by Example 3.2. This statement is true if we change an ideal to be a k-ideal. So, we obtain that an ideal of a $(3,n)$-semiring is the kernel of a homomorphism if and only if it is a k-ideal.

A $(3,n)$-semiring $(A;g,h)$ becomes a $(3,n)$-ring if $(A;g)$ is a ternary group, in other words, $(A;g,h)$ has a g-identity and every element of A has a g-inverse. In Section 4, we present the notion of an additively regular and additively inverse $(3,n)$-semiring and prove that an additively inverse $(3,n)$-semiring is a $(3,n)$-ring if and only if it contains exactly one additively idempotent element.

This work answers the question that what kind of a congruence relation τ on a $(3,n)$-semiring would transform the quotient $(3,n)$-semiring into a $(3,n)$-ring as Theorem 5.6. Moreover, Theorem 5.7 provides the converse of this statement that if A/ρ is a $(3,n)$-ring, then $\rho = \tau_D$ for some a full k-ideal D of A.

To generalize all results investigated in this work, it is interesting to research on a (m,n)-semiring which is a general structure of a $(3,n)$-semiring whenever $m \geq 2$.

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