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A linear portrayal of the Galois group $\mathcal{G}$ by the ring of irreducible characters of the hyperoctahedral group

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Abstract

The purpose of this paper is to first present the hyperoctahedral group, B_n, as the wreath product of $\mathbb{Z}_2$ associated with $\mathfrak{S}_n$, the symmetric group and then from the ring, A, generated by the irreducible characters of B_n define a linear portrayal of Artin attached to L/K; K is the field associated with the corresponding valuation ring A_K issued from A.

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1. Introduction

Representation theory was born in 1968 in the work of the German mathematician F. G. Frobenius [8]. Representation theory studies symmetry in linear space [2, 9, 14]. Representation theory studies abstract algebraic structures and their modules by presenting their elements as morphisms of vector spaces [4, 12, 15].

The group structure of a hyperoctahedral group is isomorphic to that of the symmetry group of a hypercube. The hyperoctahedral group is still seen as a coxeter group of type B_n, and then as the weyl group associated with sympletic groups and orthogonal groups of $(2n+1)-$dimensions,

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$n \in \mathbb{N}$. Moreover, $B_n = \mathfrak{S}_2 \wr \mathfrak{S}_n$ where $\mathfrak{S}_n$ represents the group of permutations of n elements [13].

The Galois group of field extension is a proper group associated to this field extension [11]. In this paper, we find one of the key theorems in Galois theory of important use in the linear portrayal of the Galois group $\mathcal{G}$.

The purpose of this paper is to give a representation of Artin by the ring of irreducible characters of B_n. Our work will be structured as follows:

Section 2 gives a clear definition of the hyperoctahedral group B_n as (a wreath product and a semi-direct product): $B_n = \mathbb{Z}_2 \wr \mathfrak{S}_n = \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n$ where φ is a homomorphism.

Section 3 recalls a representation of the hyperoctahedral group B_n on the tensor product of the vector space V.

Section 4 is devoted to the character of a linear portrayal of the Galois group $\mathcal{G}$ by B_n.

2. Preliminaries on the hyperoctahedral group B_n

The symmetric group $\mathfrak{S}_n$ can be considered as a matrix group where $n \times n$-matrices are permutation matrices (only one 1 per row and column, 0 elsewhere)[13, 14].

Let $\sigma \in \mathfrak{S}_n, \ \forall i \in [n]; \ \sigma(i) = j, \ j \in [n]$.

$\sigma(i) = j \Rightarrow a_{ji} = a_{\sigma(i)i} = 1$ at the i^{th} column. This practice generalize for the hyperoctahedral group, B_n, with the difference that the 1 of each row and column can be replaced by -1 (matrices of signed permutations).

Take the following matrix as an example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

that can be noted $(1,1,-1;(1)(23)) = (f,\pi)$ where $\pi \in \mathfrak{S}_3, \ f:[3] \rightarrow \{1,-1\}$ and $f(i)$ denotes the non-zero element of the i^{th} row of the matrix. We can generalize this notation by the following definition.

2.1 *Hyperoctahedral group as a wreath product*

Definition 1: Assume G is a finite group, with H being a subgroup of $\mathfrak{S}_n$. Let's pose:

$$G \wr H = \{(f,\pi) | f:[n] \to G; \pi \in H\}.$$

This group is characterized by the following structure

$$(f,\pi)(f',\pi') = (ff'_\pi, \pi\pi')$$

where f'_π denotes $f' \circ \pi^{-1}$ and (ff'') (i) = $f(i)$ $f''(i)$, $i \in [n]$. The identity element can be described as (e, id_H) and $e(i) = id_G$, $i \in [n]$. The inverse $(f,\pi)^{-1} = (f^{-1}_{\pi^{-1}}; \pi^{-1})$.

B_n is isomorphic to $\mathfrak{S}_2 \wr \mathfrak{S}_n$, when the notation is abused by replacing in the permutation matrix $\{id_{\mathfrak{S}_2} \text{ and } (12)\}$ by 1 and –1 respectively. In other words, we have: $B_n = \mathbb{Z}_2 \wr \mathfrak{S}_n$.

The hyperoctahedral group B_n has subgroups:

- $G^* = \{(f, id_H) | f \in G^n\}$;
- $H' = \{(e_{G^n}, \pi) | \pi \in H\}$;
- $Diag(G^*) = \{(f, id_H) | f(i) = g, i \in [n], g \in G\}$.

In general, if we have a finite group G and H a subgroup of $\mathfrak{S}_n$, the wreath product $G \wr H$ is the right semi-direct product of G^n by H with the following operation:

$$(g'_1, \ldots, g'_n, s')(g_1, \ldots, g_n, s) = (g'_{s(1)}g_1, \ldots, g'_{s(n)}g_n, s's).$$

It is easy to verify that $(1_G, \ldots, 1_{G'}, 1_H)$ is the identity element in $G \wr H$; the inverse of $(g_1, \ldots, g_n, s)$ is $(g^{-1}_{s^{-1}(1)}, \ldots, g^{-1}_{s^{-1}(n)}, s^{-1})$.

Given the coxeter group of type B_n; $B_n = \mathbb{Z}_2 \wr \mathfrak{S}_n$ is the wreath product associated with of $\mathbb{Z}_2$ with the symmetric group $\mathfrak{S}_n$. So the following sequence is exact and short

$$0 \longrightarrow \mathbb{Z}_2^n \xrightarrow{i} B_n \xrightarrow{\pi} \mathfrak{S}_n \longrightarrow 0$$

Moreover, the section s verifies $\pi \circ s = id_{\mathfrak{S}_n}$.

2.2 *Hyperoctahedral group as a semi-direct product*

Proposition 1: Let $0 \longrightarrow \mathbb{Z}_2^n \xrightarrow{i} B_n \xrightarrow{\pi} \mathfrak{S}_n \longrightarrow 0$ be a short, exact and split sequence of groups. There exists a homomorphism $\varphi : \mathfrak{S}_n \to Aut(\mathbb{Z}_2^n)$ and an isomorphism $\psi : \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n \to B_n$ such that we have the following chain complex diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2^n \arrow[r, "\bar{i}"] \arrow[d, "id"'] & \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n \arrow[r, "\bar{\pi}"] \arrow[d, "\psi"'] & \mathfrak{S}_n \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & \mathbb{Z}_2^n \arrow[r, "i"] & B_n \arrow[r, "\pi"] & \mathfrak{S}_n \arrow[r] & 0
\end{tikzcd}$$

Proof 1: The proof will be carried out on three steps.

i) Construction of a homomorphism φ

Let $\sigma \in \mathfrak{S}_n$, if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n$ where $x_i \in \mathbb{Z}_2$, $i \in [n]$, $s(\sigma)$ $i(x)$ $s(\sigma)^{-1} \in Ker(\pi) = Im(i)$ because $\pi(s(\sigma)\ i(x)\ s(\sigma)^{-1}) = \sigma(\pi \circ i\ (x))\sigma^{-1} = e_{\mathfrak{S}_n}$.

So, there is a distinct component $x' \in \mathbb{Z}_2^n$ verifying

$$s(\sigma)i(x)s(\sigma)^{-1} = i(x').$$

Let $f_\sigma : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ be the application that sends x to x'. It is a homomorphism because if x_1, $x_2 \in \mathbb{Z}_2^n$ and $f_\sigma(x_1) = x_1'$, $f_\sigma(x_2) = x_2'$, we have

$$s(\sigma)i(x_1x_2)s(\sigma)^{-1} = (s(\sigma)i(x_1)s(\sigma)^{-1})(s(\sigma)i(x_2)s(\sigma)^{-1}).$$

On the other hand, $\forall \sigma_1$ and $\sigma_2 \in \mathfrak{S}_n$, we have $f_{\sigma_1\sigma_2} = f_{\sigma_1} \circ f_{\sigma_2}$.

Indeed $\forall x \in \mathbb{Z}_2^n$,

$$s(\sigma_1\sigma_2)i(x)s(\sigma_1\sigma_2)^{-1} = s(\sigma_1)(s(\sigma_2)i(x)s(\sigma_2)^{-1})(s(\sigma_1^{-1})).$$

Like $f_{e_{\mathfrak{S}_n}} = id_{\mathbb{Z}_2^n}$, we see that $\forall \sigma \in \mathfrak{S}_n$, f_σ is an automorphism of $\mathbb{Z}_2^n$ with inverse $f_{\sigma^{-1}}$.

We define $\varphi : \mathfrak{S}_n \to Aut(\mathbb{Z}_2^n)$ by setting $\varphi(\sigma) = f_\sigma$, the above equality shows that φ is a homomorphism.

ii) Construction of the isomorphism ψ

We define $\psi : \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n \to B_n$ by setting: $\psi(x,\sigma) = i(x)\ s(\sigma)\quad \forall x \in \mathbb{Z}_2^n$ and $\sigma \in \mathfrak{S}_n$, it is a homomorphism because if (x,σ_1) and $(x',\sigma_2) \in \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n$,

$$\begin{aligned}\psi((x,\sigma_1)(x',\sigma_2)) &= \psi(x\varphi(\sigma_1)(x'),\sigma_1\sigma_2)\\ &= i(x\varphi(\sigma_1)(x'))s(\sigma_1\sigma_2)\\ &= i(x)s(\sigma_1)i(x')s(\sigma_1)^{-1}s(\sigma_1\sigma_2)\\ &= \psi(x,\sigma_1)\psi(x',\sigma_2)\end{aligned}$$

It is one-to-one function because if $(x,\sigma) \in \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n$ is such that $\psi(x,\sigma) = i(x)\ s(\sigma) = e_{B_n}$, thus $\pi(i(x)\ s(\sigma)) = \pi \circ s(\sigma) = \sigma = e_{\mathfrak{S}_n}$. We deduce that $i(x) = e_{B_n}$ and therefore $x = e_{\mathbb{Z}_2^n}$. It is surjective because for all $\omega \in B_n$, $s\circ\pi(\omega) \in B_n$.

In general, $s\circ\pi(\omega)$ is not equal to ω but $\omega(s\circ\pi(\omega))^{-1} \in Ker(\pi)$. So, there exists an element $x \in \mathbb{Z}_2^n$ for which $\omega(s\circ\pi(\omega))^{-1} = i(x)$. So,

$$\omega = i(x)(s\circ\pi(\omega)) = \psi(x,\pi(\omega)).$$

iii) Diagram commutes

If $x \in \mathbb{Z}_2^n$, $\psi\circ\bar{i}(x) = \psi(x,e) = i(x)$, which shows that the left square commutes. Let us show that the one on the right commutes for projections and sections.

If $(x,\sigma) \in \in \mathbb{Z}_2^n \rtimes_\varphi \mathfrak{S}_n$, then $\bar{\pi}(x,\sigma) = \sigma$, $\pi\circ\psi(x,\sigma) = \pi(i(x)\ s(\sigma)) = \sigma$. Conversely, if $\sigma \in \mathfrak{S}_n$ $\psi\circ\bar{s}(\sigma) = \psi(e,\sigma) = i(e)s(\sigma) = s(\sigma)$.

Corollary 1: *Given the group* B_n; $H,K \subset B_n$, *two subgroups verifying:*

- $B_n = HK$;
- $H \lhd B_n$, normal subgroup;
- $H \cap K = \{e\}$.

Then $\varphi : K \to Aut(H)$ is linear such that B_n is structurally identical to $H \rtimes_\varphi K$.

Proof 2: Let $k \in K$, the same arguments used in the previous proof show that the map $f_k : H \to H$ given by $f_k(h) = khk^{-1}\quad \forall h \in H$ is indeed with values in H by normality of H and that it is an automorphism of

H. Moreover, the map $\varphi : K \to Aut(H)$ which to $k \in K$ associates f_k is linear. Let $\psi : H \rtimes_{\varphi} K \to B_n$ such that $\psi(h,k) = hk \quad \forall(h,k) \in H \rtimes_{\varphi} K$. We confirm that it is a homomorphism that is surjective by $B_n = HK$ and also that $H \lhd B_n$.

3. Representation of B_n within the vector space V

Consider V as a K-vector space, K a field. Consider the permutations in their cyclic decomposition (into disjoint cycles):

$$\pi = \prod_{\nu=1}^{c(\pi)} (j_\nu \pi(j_\nu) \ldots \pi^{l_\nu - 1}(j_\nu))$$

where $c(\pi)$ is the number of disjoint cycles, j_ν minimal in its cycle and l_ν is the length of the cycle whose index is ν [1, 17]. We consider the following function which, with each cycle of π associates an element of $\mathbb{Z}_2$:

$$g_\nu(f,\pi) = f(j_\nu) f(\pi^{-1}(j_\nu)) \ldots f(\pi^{-l_\nu+1}(j_\nu)) \in \mathbb{Z}_2.$$

Let D denote a representation of $\mathbb{Z}_2$ acting on V, it is possible to take the tensor product of V with itself n times:

$$\otimes^n V = \underbrace{V \otimes \ldots \otimes V}_{n \; times}.$$

If $\{b_1, b_2, \ldots, m\}$ is a basis of V then $\otimes^n V = < b_\varphi | \varphi \in m^n >$ where $b_\varphi = b_{\varphi(1)} \otimes b_{\varphi(2)} \otimes \ldots \otimes b_{\varphi(m)}$. We can consider V as being a $\mathbb{Z}_2$-module:

$$\forall a \in \mathbb{Z}_2, v \in V \quad a \cdot v = D(a)v.$$

And hence $\otimes^n V$ as a B_n-module by the action of $(f,\pi) \in B_n$ on an element b_φ of the basis:

$$(f,\pi) b_\varphi = f(1) b_{\varphi(\pi^{-1}(1))} \otimes f(2) b_{\varphi(\pi^{-1}(2))} \otimes \ldots \otimes f(n) b_{\varphi(\pi^{-1}(n))}$$

and by multilinearity

$$(f,\pi)(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = f(1) v_{\pi^{-1}(1)} \otimes f(2) v_{\pi^{-1}(2)} \otimes \ldots \otimes f(n) v_{\pi^{-1}(n)}.$$

We denote the representation defined above by $(\#^n D)^{\sim}$. The character of this representation is worth:

$$\chi^{(\#^n D)^\sim}(f,\pi)=\prod_{\nu=1}^{c(\pi)}\chi^D(g_\nu(f,\pi));$$

$$\chi^{(\#^n D)^\sim}(f,\pi)=\prod_{k=1}^{n}\chi^D(g^k)^{a_k(\pi)}.$$

And we generally also have the following permutation representation for $G\leq \mathfrak{S}_m$, $H\leq \mathfrak{S}_n$

$$\theta : G \wr H \to \mathfrak{S}_{m^n}$$

$$(f,\pi)\mapsto(\varphi\mapsto\varphi')$$

where $\varphi'\in m^n$ is defined by $\varphi'(i)=f(i)(\varphi(\pi^{-1}(i)))$, $i\in[n]$.

Thus according to Brauer's Theorem [1] any character of B_n is formed by combining χ_i^* linearly on $\mathbb{Z}$ induces by $\chi_i\in H_i$ of B_n.

4. linear portrayal of the Galois group $\mathcal{G}$

Consider A as a ring of irreducible characters of B_n, A_K its corresponding valuation ring where v_K is the discrete valuation and K the associated field and (K,v_K) is complete [5, 10]. I denotes the maximal ideal of A_K, $\overline{K}=A_K/I$ its residual field and $U_K=A_K-I=A_K^\times$.

Consider L as a finite separable field over K. We designates by A_L a integral extension of A_K in L [3]. We suppose that the residual extension $\overline{L}/\overline{K}$ is separable. Let us designate by $e_{L/K}$ a branching index of P_L and $f_{L/K}$ its residual degree.

Thus

$$e_{L/K}\cdot f_{L/K}=[L:K].$$

Consider L/K as the finite Galois field. Let $\mathcal{G}=gal(L/K)$, Galois group, $\mathcal{G}$ operates on A_L. We know that there exists $x\in A_L$ such that A_L considered as A_K-algebraic structure. Let $i_{\mathcal{G}}$ be a function $\mathcal{G}$ defined by the formula

$$i_{\mathcal{G}}(s)=v_L(s(x)-x).$$

We will put $f=[\overline{L}:\overline{K}]$; if $s\in\mathcal{G}$ distinct from 1,

$$a_{\mathcal{G}}(s) = \begin{cases} -f \cdot i_{\mathcal{G}}(s), & \text{if } s \neq 1; \\ f\sum_{s\neq 1} i_{\mathcal{G}}(s), & \text{otherwise.} \end{cases}$$

So, we have:

$$\sum_{s\in\mathcal{G}} a_{\mathcal{G}}(s) = 0, \quad \Rightarrow \quad (a_{\mathcal{G}}, i_{\mathcal{G}}) = 0.$$

Theorem 1: *Let* $a_{\mathcal{G}}$ *represent the character of a linear portrayal of* $\mathcal{G}$.

$$a_{\mathcal{G}} = \sum c_{\chi}\chi$$

where χ *is an element of the set of irreducible characters of* $\mathcal{G}$.

This linear portrayal of the group $\mathcal{G}$ is the Artin representation attached to L/K and K is the field corresponding to the corresponding valuation ring A_K with A the ring generated by the irreducible characters of the hyperoctahedral group. It will be noted on the other hand that it is only defined via its character up to isomorphism.

The $a_{\mathcal{G}}$ function is a central function. We can therefore write it:

$$a_{\mathcal{G}} = \sum c_{\chi}\chi$$

where $\chi \in \mathcal{G}$. We have:

$$c_{\chi} = (a_{\mathcal{G}}, \chi) = \frac{1}{g}\sum_{s\in\mathcal{G}} a_{\mathcal{G}}(s)\chi(s^{-1}) = \frac{1}{g}\sum_{s\in\mathcal{G}} a_{\mathcal{G}}(s^{-1})\chi(s).$$

In general, if φ is a $\mathcal{G}$-central map, we will set:

$$f(\varphi) = (\varphi, a_{\mathcal{G}}).$$

Remark 1: Theorem 1 is restated as: If $\chi \in \mathcal{G}$, then $f(\chi) \in \mathbb{N}$

Proposition 2: Consider $\mathcal{G}_i$ as the i^{th} branching subgroup of $\mathcal{G}$. Consider $u_i \in \mathcal{G}_i$, $u_i^* \in \mathcal{G}$ induced by u_i. We have:

$$a_{\mathcal{G}} = \sum_{i=0}^{\infty} \frac{1}{(\mathcal{G}_0 : \mathcal{G}_i)} u_i^*.$$

Proof: Letting $g_i = Card(\mathcal{G}_i)$. The reasoning above shows that $u_i^*(s) = 0$ if s is not an element of $\mathcal{G}_i$,

$$u_i^*(s) = -\frac{g}{g_i} = -f \cdot \frac{g_0}{g_i} \text{ if } s \in \mathcal{G}_i, s \neq 1,$$

and $\sum_{s\in\mathcal{G}} = u_i^*(s) = 0.$ If $s \in \mathcal{G}_k - \mathcal{G}_{k+1}$, the sum of the line is $-f(k+1)$, and so is $a_{\mathcal{G}}(s)$.

For $s = 1$, the equality comes from the fact that two members are orthogonal.

As soon as φ is a $\mathcal{G}$-central map:

$$\varphi(\mathcal{G}_i) = \frac{1}{g_i}\sum_{s\in\mathcal{G}_i}\varphi(s), \qquad g_i = Card(\mathcal{G}_i).$$

Corollary 2: *Consider φ as a $\mathcal{G}$-central map, we have:*

$$f(\varphi) = \sum_{i=0}^{\infty}\frac{g_i}{g_0}(\varphi(\{1_{\mathcal{G}}\}) - \varphi(\mathcal{G}_i)).$$

This follows from proposition 2, considering that:

$$(\varphi, u_i^*) = (\varphi|_{\mathcal{G}_i}, u_i) = \varphi(\{1_{\mathcal{G}}\}) - \varphi(\mathcal{G}_i)$$

Corollary 3: *If $\chi \in \mathcal{G}$ from the $\mathbb{Z}$-module generated by the irreducible characters of B_n, then $f(\chi)$ is a positive integer.*

Proof 4: Proposition 2 shows that $g_0 a_{\mathcal{G}}'$ represents a straightfoward depiction of $\mathcal{G}$. So $g_0 \cdot f(\chi) \in \mathbb{N}$.

Proposition 3: Consider H as a subgroup of $\mathcal{G}$ isomorphic to sub-extension K'/K ; $\delta_{K'/K}$ its discriminant in K'/K. We have:

$$a_{\mathcal{G}}|_H = \lambda r_H + f_{K'/K} \cdot a_H, \qquad \text{with } \lambda = v_K(\delta_{K'/K}).$$

Where $r_H \in H$.

Proof 5: Consider $s \neq 1 \in H$, We have:

$$a_{\mathcal{G}}(s) = -f_{L/K} i_{\mathcal{G}}(s), \quad a_H(s) = -f_{L/K'} i_H(s), \quad r_H(s) = 0,$$

and as $i_{\mathcal{G}}(s) = i_H(s)$, it gives

$$a_{\mathcal{G}}(s) = \lambda r_H(s) + f_{K'/K} \cdot a_H(s).$$

Take now $s=1$. We first have:

$$a_G(1)=f_{L/K}v_L(D_{L/K})=v_K(\delta_{L/K}).$$

Similarly, $a_H(1)=v_{K'}(\delta_{L/K'})$. Therefore:

$$v_K(\delta_{L/K})=[L:K']v_K(\delta_{K'/K})+f_{K'/K}v_{K'}(\delta_{L/K'}).$$

It results by $\delta_{L/K}$ -transitivity [3]:

$$\delta_{L/K}=(\delta_{K'/K})^{[L:K']}\cdot N_{K'/K}(\delta_{L/K'}).$$

Corollary 4: *Consider* ψ *as the character of* H *and* $\psi^*\in\mathcal{G}$ *induced by* ψ.

$$f(\psi^*)=v_K(\delta_{K'/K})\psi(1)+f_{K'/K}f(\psi).$$

Proof: Indeed:

$$\begin{aligned} f(\psi^*) &= (\psi^*,a_{\mathcal{G}})=(\psi,a_{\mathcal{G}}|_H) \\ &= (\psi,\lambda r_H+f_{K'/K}\cdot a_H) \\ &= \lambda(\psi,r_H)+f_{K'/K}(\psi,a_H) \\ &= \lambda\psi(1)+f_{K'/K}f(\psi),\ \text{with } \lambda=v_K(\delta_{K'/K}) \end{aligned}$$

Proposition 4: Consider $\chi\in\mathcal{G}$, $c_\chi\in\mathbb{N}$ such that $\chi|_{\mathcal{G}_{c_\chi}}$ isn't trivial ($c_\chi=-1$ when $\chi=1$). Thus, we have:

$$f(\chi)=\varphi_{L/K}(c_\chi)+1$$

where $\forall u$ integer

$$\varphi_{L/K}(u)=\frac{1}{g_0}\sum_{s\in\mathcal{G}}Inf(i_{\mathcal{G}},u+1)-1.$$

Proof 7: If $i\le c_\chi$, then $\chi(\mathcal{G}_i)=0$, hence $\chi(1_{\mathcal{G}})-\chi(\mathcal{G}_i)=1$; if $i>c_\chi$, then $\chi(\mathcal{G}_i)=1$, so $\chi(1_{\mathcal{G}})-\chi(\mathcal{G}_i)=0$. Applying corollary 2 to proposition 2, we get:

$$f(\chi)=\sum_{i=0}^{i=c_\chi}\frac{g_i}{g_0}=\varphi_{L/K}(c_\chi)+1.$$

Corollary 5: *Consider* $H = ker(\chi)$ *and* K' *such that* $L/K \cong H$, *consider* $c'_{\chi} \in \mathbb{N}$ *such that* $(G/H)_{c'_{\chi}} \neq \{1_{\mathcal{G}}\}$. *So,* $f(\chi) = \varphi_{K'/K}(c'_{\chi}) + 1$, *and it is an integer* ≥ 0.

Proof 8: Herbrand's theorem [3] establishes that $c'_{\chi} = \varphi_{L/K'}(c_{\chi})$. $f(\chi) = \varphi_{K'/K}(c'_{\chi}) + 1$ therefore results simply from the φ-transitivity. Since $\varphi_{K'/K}(c'_{\chi}) \in \mathbb{N}$, it follows from the Hasse-Arf theorem [3], since G/H is abelian.

Here is finally a proof of the theorem 1.

Proof 9: It is a question of proving that: $f(\chi) \in \mathbb{N}$ when $\chi \in \mathcal{G}$. We already know from corollary 3 to proposition 2, that it is a positive rational number; it remains to show that it is an integer. In line with to Brauer's theorem [1],

$$\chi = \sum n_i \chi_i^*,\ n_i \in \mathbb{Z}$$

$\chi_i \in H_i$ of $\mathcal{G}$, characters of degree 1. Thus, we are left to demonstrate that $f(\chi^*) \in \mathbb{N}$ when $\chi \in \mathcal{G}$ of degree 1. Under these hypotheses $f(\chi)$ is an integer according to Corollary 5 to Proposition 4, and it is therefore the same for $f(\chi^*)$, according to the Corollary of Proposition 3.

5. Conclusion

In this paper, it was a question of giving a linear portrayal of the Galois group $\mathcal{G}$ by the ring of irreducible character of the hyperoctahedral group B_n.

I first appears that the group $B_n = \mathbb{Z}_2 \wr \mathfrak{S}_n = \mathbb{Z}_2^n \rtimes_{\varphi} \mathfrak{S}_n$ where φ is a homomorphism defined by $\varphi : \mathfrak{S}_n \to Aut(\mathbb{Z}_2^n)$.

Finally, the function $a_{\mathcal{G}}$ defined by

$$a_{\mathcal{G}}(s) = \begin{cases} -f \cdot i_{\mathcal{G}}(s), & \text{if } s \neq 1; \\ f \sum_{s\neq 1} i_{\mathcal{G}}(s), & \text{otherwise.} \end{cases}$$

$a_{\mathcal{G}}(s)$ is an element of a linear portrayal of the Galois group $\mathcal{G}$ from the hyperoctahedral group B_n; where f is the index of $\overline{K}$ in $\overline{L}$, $s \in \mathcal{G}$.

In a forthcoming article, we will see that the structutures of Θ-valent modal logic [6, 7, 16] can be modeled using combinatorial structures that have symmetrical properties, such as those described by hyperoctahedral group.

References

[1] R. Brauer and J. Tate, “On the characters of finite groups,” *Ann. of Math.,* vol. 62, pp. 1–7 (1955).

[2] J. Baker, Matrix Groups: An Introduction to Lie Group Theory, Springer, London (2002).

[3] J. P. Serre, Corps locaux, Actualités scientifiques et Industrielles, Chap IV, pp. 69–87 (1968).

[4] J. L. Basdevant and J. Dalibard, Mécanique Quantique, Les Editions de l’Ecole Polytechnique (2004).

[5] E. Artin, “Die gruppentheoretische Struktur der diskriminanten algebraischer Zahlkörper,” *J. reine ang. Math.*, vol. 164, pp. 1–11 (1931).

[6] P. B. Gabriel Cédric, “Quasi-Cyclic Codes Over Finite Chain mΘ Pseudo Field F(pkZ, 1),” *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences*, vol. 23, no. 3, pp. 85–101 (2023).

[7] P. B. Gabriel Cédric, “The mΘ protocol F5 and Hamming mΘ codes,” *London Journal of Research in Science,* vol. 23, no. 13, pp. 21–36 (2023).

[8] F. G. Frobenius, Gesammelte Abhandlungen, vol. 3, Springer, Berlin (1968).

[9] B. C. Hall, Lie Groups, Lie Algebras and Representations: An Elementary Introduction, Springer, New York (2003).

[10] E. Artin and J. Tate, Class Field Theory, Benjamin, New York (1967).

[11] S. Lang, Algebra, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York (2002).

[12] A. Knapp, Lie Groups, Beyond an Introduction, Birkhäuser, New York, 1996; 2nd ed. (2002).

[13] S. Porier, Fonctions symétriques, ensembles de descentes et classes de conjugaison dans les produits en couronne, Ph.D. dissertation, Université du Québec à Montréal, Canada (1995).

[14] G. Rauch, Les groupes finis et leurs représentations, Ellipses, Paris (2000).

[15] W. Rossmann, Lie Groups: An Introduction Through Linear Groups, Oxford University Press, Oxford (2002).

[16] G. C. Pemha Binyam, L. Um Emilie, and Y. J. Ndje, “The mΘ Quadratic Character in the mΘ Set ZnZ,” *Mathematics and Computer Science,* vol. 8, no. 1, pp. 11–18 (2023).

[17] A. Rougé, Introduction à la physique subatomique, Les Editions de l’Ecole Polytechnique, Palaiseau (2005).

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