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The complement subgroup product graph of a cyclic group

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Abstract

In this paper, we introduced a new graph, it is called complement subgroup product graphs (CSPG) derived from cyclic groups. The set of vertices is the proper subgroup of group G and for each two proper subgroups H, K is connected if and only if HK=G. We study the formulas for vertex degrees, isolation properties of certain graphs, matrix degree representations, and edge calculations.

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1. Introduction and Preliminaries

In [1] and [2], Angsuman Das and Shaker, presented a new graph with name subgroup product graph of group, the new graph Γ_{sp} is structure of the set of vertices of group G and for any two subgroups of group G are join if $HK = G$, this mean any two subgroup is edge in this graph if satisfying the condition. In graph theory, inverse of a graph Γ is the complement graph $\overline{\Gamma}$ and the set of the vertices are same, but for any two vertices is join in graph is non-join in complement graph. It is well known, the subgroup product graph is define, Let

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$\Gamma(V, E)$ be a finite graph, where $V(\Gamma)$ is the set of all vertices and $E(\Gamma)$ is the set of all edges of the graph Γ. The product subgroup graph of a group, denoted by $\Gamma_{sp}(V, E)$, is a simple graph devoid of loops and edge multiplicities. Its set of vertices comprises all proper subgroups, with two vertices $< H >$ and $< K >$ joined if $< HK > \cong G$. A vertex without any edges is termed an isolated vertex. It is clear that, the product subgroup graph of cyclic group of order p, $\Gamma_{sp}(C_p)$ is null graph, and $\Gamma_{sp}(C_{pq}) \cong P_2$ is a path graph for each cyclic group of order pq are prime numbers. Let G be a finite group, the set of all subgroup we denoted by $Sub(G) = \{H \mid \forall H \leq G\}$, $TSub(G) = \{e, G\}$. and the set of all proper subgroup of group G is denoted by $PSub(G)$, it defined as the difference between $Sub(G)$ and $TSub(G)$, $PSub(G) = Sub(G) / TSub(G)$. In [1] computed the degree of vertex in product subgroup graph $deg_{sp}(H)$ for each vertex in graph $\Gamma_{sp}(G)$ by the following:

$$\deg(v) = \begin{cases} \deg(\langle a^{p} \rangle) = \left(2^{(r-1)} - 1\right) \\ \deg(\langle a^{p_i p_j} \rangle) = \left(2^{(r-2)} - 1\right) \\ \vdots \\ \deg\left(\langle a^{\prod_{1 \leq i \leq (r-2)} p_i} \rangle\right) = 3 \\ \deg\left(\langle a^{\prod_{1 \leq i \leq (r-1)} p_i} \rangle\right) = 1 \end{cases}$$

Where $V(\Gamma_{sp}(G)) = \{\langle a^{p} \rangle, \langle a^{p_i p_j} \rangle, \langle a^{p_i p_j p_k} \rangle, \langle a^{p_i p_j p_k p_h} \rangle, \ldots, \langle a^{\prod_{1 \leq i \leq (r-2)} p_i} \rangle, \langle a^{\prod_{1 \leq i \leq (r-1)} p_i} \rangle\}$ are proper subgroups. The matrix degree, it introduced by [3-5], Let Γ be a graph, the matrix degree is given by the following: $DM_{sp}(\overline{G}) = \begin{pmatrix} \mu(\lambda_1) & \mu(\lambda_1) & \cdots & \mu(\lambda_1) \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix}$ Where λ_i is a degree of vertices and $\mu(\lambda_i)$ ia s multy degree. For more information see refers [6-17].

2. Main Results

Definition 2.1: Suppose that H and K are a subgroups of group G, the complement product subgroup graph $\overline{\Gamma_{sp}}$ is the graph with vertex set $Psub(G)$, in which two distinct vertices H and K are joined if $HK \neq G$

Proposition 2.2: In the following are hold:

1- $\overline{\Gamma_{sp}}(C_p) \cong \Gamma_{sp}(C_p)$

2- $\overline{\Gamma_{sp}}(C_{pq})$ is empty graph.

Proof: It is clear that, the set of all vertices of $V(\Gamma_{sp}(C_p)) = \emptyset$ since there is no proper subgroups of the group C_p and $V\left(\overline{\Gamma_{sp}}(C_p)\right) = \emptyset$, since the number of vertices in graph $V(\Gamma_{sp}(C_p))$ equals to the number of vertices in its complement graph $V\left(\overline{\Gamma_{sp}}(C_p)\right)$, then there is no edge in this graph. For group C_{pq}, the set of all proper subgroups are $\{\langle a^p \rangle, \langle a^q \rangle\}$, this mean, the set of vertices is

$V(\overline{\Gamma_{sp}}(C_{pq})) = \{\langle a^p\rangle, \langle a^q\rangle\}$, since $\langle a^p\rangle.\langle a^q\rangle = C_{pq}$, then $\overline{\Gamma_{sp}}(C_{pq})$ is empty graph without edges.

Theorem 2.3: *Let $\overline{\Gamma_{sp}}(C_{pq})$ be a complement product subgroup graph of cyclic group of order $\prod_{i\geq 1} p_i$. The degree of vertices is given by the following*

$$\overline{\deg}(v) = \begin{cases} \deg(\langle a^p\rangle) = |V(G)| - \left(2^{(r-1)} - 1\right) \\ \deg(\langle a^{p_ip_j}\rangle) = |V(G)| - \left(2^{(r-2)} - 1\right) \\ \vdots \\ \deg\left(\langle a^{\prod_{1\leq i\leq (r-2)} p_i}\rangle\right) = |V(G)| - 3 \\ \deg\left(\langle a^{\prod_{1\leq i\leq (r-1)} p_i}\rangle\right) = |V(G)| - 1 \end{cases}$$

Proof: Suppose that, G is a finite group, by Proposition 2.1 in [2] if H, K is a proper subgroup of G such that $[G:H]$ and $[G:K]$ are coprime, this mean, $H \sim K \in E(\Gamma_{sp}(G))$. $\overline{\deg}(v) = \left|V\left(\Gamma_{sp}(G)\right)\right| - \deg(v)$ for each vertices in graph.

$$\overline{\deg}(v) = \left|V\left(\Gamma_{sp}(G)\right)\right| - \deg(v)$$

$$= \left|V\left(\Gamma_{sp}(G)\right)\right| - \begin{cases} \deg(\langle a^p\rangle) = \left(2^{(r-1)} - 1\right) \\ \deg(\langle a^{p_ip_j}\rangle) = \left(2^{(r-2)} - 1\right) \\ \vdots \\ \deg\left(\langle a^{\prod_{1\leq i\leq (r-2)} p_i}\rangle\right) = 3 \\ \deg\left(\langle a^{\prod_{1\leq i\leq (r-1)} p_i}\rangle\right) = 1 \end{cases}$$

$$\overline{\deg}(v) = \begin{cases} \deg(\langle a^p\rangle) = |V(G)| - \left(2^{(r-1)} - 1\right) \\ \deg(\langle a^{p_ip_j}\rangle) = |V(G)| - \left(2^{(r-2)} - 1\right) \\ \vdots \\ \deg\left(\langle a^{\prod_{1\leq i\leq (r-2)} p_i}\rangle\right) = |V(G)| - 3 \\ \deg\left(\langle a^{\prod_{1\leq i\leq (r-1)} p_i}\rangle\right) = |V(G)| - 1 \end{cases}$$

$$\tau\left(\prod_{1\leq i\leq r-1} p_i\right) - 1 = \tau(p_1)\dots\tau(p_{r-1}) - 1 = (2)\dots(2) - 1 = 2^{(r-1)} - 1$$

By similar for remains of vertices. To find the degree of all vertices in complement Graph G, we must be Note those Axioms: Firstly, the degree of vertex refer to the number of vertices that connect with vertex. Secondly, the number of vertices in $\Gamma_{sp}(G)$ compute as follow: $\sum_{j=1}^{r-1} C_j^r$ and in this graph $\Gamma_{sp}(\bar{G})$ has not loop, Hence We will exclude the vertex itself and the associated vertices to get the degree of vertex

$$\deg\left(\langle a^{\prod_{1\leq i\leq r} p_i}\rangle\right) = \left|V\left(\Gamma_{sp}(G)\right)\right| - \left(2^{(r-i)} - 1\right)$$

Theorem 2.4: *Let $G \cong C_n$ be a cyclic group and $n = \prod_{1\leq i\leq r} p_i$ is a positive integer number, the matrix degree is given by the following:*

$$M(\overline{\Gamma_{sp}}) = \begin{pmatrix} C_1^r & C_2^r & \cdots & C_{r-1}^r \\ |V(G)| - (2^{(r-1)} - 1) & |V(G)| - (2^{(r-2)} - 1) & \cdots & |V(G)| - 1 \end{pmatrix}$$

Corollary 2.5: *If G is a cyclic group of order $n = \prod_{1 \le i \le r} p_i$ is a positive integer number, then:*

$$E\left(\Gamma_{sp}(\bar{G})\right) = \frac{\sum_{i=1}^{r-1}\left(|V(G)| - \left(2^{(r-i)} - 1\right)\right)\ C_i^r}{2}$$

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