

Small subsemimodule and some related concepts

Alaa Hussein Ali *

Asaad Mohammed Ali Alhossaini †

Department of Mathematics

College of Education for Pure Sciences

University of Babylon

Babylon 51001

Iraq

Abstract

Previously, some authors have studied the concept of small submodules regarding semirings. This research aims to provide a comprehensive study of small modules and some related concepts as: cosemisimple semiring, semimodule with finite spanning dimension, small free semimodule and small projective semimodule. These concepts are implemented and studied in various modules. Then we convert it to semimodule and give some results.

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1. Introduction Preliminaries

Author belongs to an extensive list of scholars who have investigated small subsemimodules as part of their research According to [1] a subsemimodule N in a semimodule $\tilde{M}^a$, is defined as small when we denote this by $N \ll \tilde{M}^a$, and it requires that all subsemimodules L of M satisfy $N+L=\tilde{M}^a$, implies $L=\tilde{M}^a$.

Our research will examine conversion concepts from modules that include small projective semimodule, cosemisimple semiring, small free semimodule and semimodule with finite spanning dimension.

* ORCID-ID: 0000-0002-8084-5980

† ORCID-ID: 0009-0007-9709-3598

* E-mail: alaaasd476@gmail.com (Corresponding Author)

† E-mail: asaad_hosain@itnet.uobabylon.edu.iq

We will convert these module concepts together with their subsequent analysis from [10] and [7, 14-18], into semimodules to obtain various results.

The semiring Λ is a cosemisimple type of semiring.

The injective semimodule $\forall$ over Λ turns out to be small when it is projective.

Any Λ -semimodule turns out to be small projective when Λ is a semiring which produces only small free semimodules.

Remark 1.1: The following definitions: semimodule, subsemimodule, subtractive semimodule, homomorphism, projective semimodule and exact sequence are found in the sources [2, 3, 12] and [5].

Definition 1.2: [13] Let $f: \tilde{M}^a \rightarrow N$ be a hom-L of the Λ -semimodules. Then, f is said to be k -regular homomorphism assume that, for all $a_1, a_2 \in \tilde{M}^a$ with $f(a_1) = f(a_2)$, there are $k_1, k_2 \in \ker(f)$ and $a_1 + k_1 = a_2 + k_2$.

Remark 1.4: [11] A Modular Law for Semimodule exists where $L\Lambda$ -semimodule W contains subsemimodules $Q \leq W$ and Q' under the condition of W being subtractive and establishes the relationship $W \cap (Q + Q') = Q + (W \cap Q')$.

Exas 1.6:

1. Every finitely generated subsemimodules of $\mathbb{Q}$ as $\mathbb{Z}$ -semimodules is small in $\mathbb{Q}$.
2. The only subsemimodule, which is small in $\mathbb{Z}$ as $\mathbb{Z}$ -semimodule is $\{0\}$.

Definition 1.7: [1] Let W and H be semimodules. A surjective homomorphism $g: W \rightarrow H$ is said to be small, assume that $\ker(g) \ll W$.

Definition 1.8: [4] A semimodule $\tilde{M}^a$ qual assume that is as hollow when all of its p -subsemimodules possess small dimensions within $\tilde{M}^a$

Exa. 1.9

1. Each simple semimodule is hollow.
2. $\mathbb{Z}_6$ as $\mathbb{N}$ -semimodule is not hollow.

Exa 1.10: assume that $\tilde{M}^a$ is a h -semimodule and $g: \tilde{M}^a \rightarrow N$ is surjective homomorphism, then g is small.

2. Small Subsemimodule and Related concepts

In this section, we give some results of small subsemimodules, cosemisimple semiring, small projective semimodule, small free semimodule (S.F), semimodule with finite spanning dimension.

Lemma 2.1: Let $f: A \rightarrow B$ is k -quasiregular and $K \leq A$, Assume that $f(A) = f(K)$, then $A = K + \ker f$.

Lemma 2.2: Assume that $f: \tilde{M}^a \rightarrow N$ and $g: N \rightarrow L$ are k -quasiregular surjective homomorphism, then $g \circ f$ is k -quasiregular hom.

Lemma 2.3: Assume that f is surjective homomorphism, then $f(f^{-1}(H)) = H$.

Proposition 2.4: Let M, L and N be semimodules. Assume that $f: \tilde{M}^a \rightarrow N$ and $g: N \rightarrow L$ are k -quasiregular surjective homomorphism. Then $g \circ f$ is small assume that and only assume that both f, g is small.

Proof: ($\Rightarrow$) Let $g \circ f$ be a small surjective homomorphism and suppose that $\ker(f) + K = \tilde{M}^a$, where K is a subsemimodule of $\tilde{M}^a$, then $g(f(K)) = L = g(f(\tilde{M}^a))$.

Since $g(f(K)) = g(f(\tilde{M}^a))$, then by Lemma (2.1) $\Rightarrow \tilde{M}^a = K + \ker gf$, by assumption $\ker gf \ll M \Rightarrow M = K \Rightarrow \ker f \ll \tilde{M}^a \Rightarrow f$ is small.

To show that g is small surjective. Let $\ker(g) + K_1 = N$, where K_1 is a subsemimodule of N . Since f is surjective, assume that $K_2 = f^{-1}(K_1) \Rightarrow f(K_2) = K_1 \Rightarrow f(\tilde{M}^a) = f(K_2) + \ker g \Rightarrow g(f(\tilde{M}^a)) = g(f(K_2))$, since gf is k -quasiregular then by Lemma (2.1), $\tilde{M}^a = K_2 + \ker gf$ and $\ker gf \ll \tilde{M}^a \Rightarrow \tilde{M}^a = K_2 \Rightarrow K_1 = f(\tilde{M}^a) = N \Rightarrow N = K_1$. Therefore, both f and g are small surjective homomorphism.

($\Leftarrow$) Let f, g be small surjective homomorphism, and suppose that $\ker(g \circ f) + K = M$, where K is subsemimodule of M , then $g(f(K)) = g(f(\tilde{M}^a)) = g(N)$, by Lemma (2.1), $f(K) + \ker g = N$, since $\ker g \ll N \Rightarrow f(K) = N = f(M)$, by Lemma (2.1), $\tilde{M}^a = K + \ker f \Rightarrow M = K$.

Consequently, $g \circ f$ is a small surjective homomorphism.

Proposition 2.5: Assume that $Z \ll W \leq H$, then $Z \ll H$.

Proof: Let $Z + K = H$ where $K \leq H$. Then $(Z + K) \cap W = H \cap W = Z + K \cap W = W$ by Remark 1.4, thus $K \cap W = W$ since Z is small subsemimodule of W and hence $W \subseteq K \Rightarrow K = Z + K = H$.

Proposition 2.6: [1] Let $L \ll H$ and let $g: H \rightarrow W$ be a Λ -surjective. Assume that g is k -quasiregular, then $g(L) \ll W$.

Lemma 2.7: Assume that $N \leq K \leq \tilde{M}^a$, K is subtractive and $K/N = \tilde{M}^a/N$, then $K = M$.

Proof: Let $m \in \tilde{M}^a$, $m/N \in M/N = K/N$ then there exist $k \in K$ such that $m/N = k/N$ this implies that $m + n_1 = k + n_2$ for some $n_1, n_2 \in N$, since K subtractive, $k + n_2 \in K$ implies that $m + n_1 \in K$, $n_1 \in K$ then $m \in K$ thus $K = M$.

Proposition 2.8: Assume that $\tilde{M}^a$ is a subtractive semimodule, and $N \leq K \ll \tilde{M}^a$ then $K/N \ll \tilde{M}^a/N$.

Proof: Assume $L/N \leq M/N$ with $K/N + L/N = \tilde{M}^a/N$ then

$K + L/N = \tilde{M}^a/N$ since $K + L$ is subtractive, by Lemma 2.7. we have $K + L = \tilde{M}^a$ but $K \ll \tilde{M}^a$ so, $L = M$ hence $L/N = M/N$.

Corollary 2.9: Assume that $\tilde{M}^a$ is a subtractive hollow semimodule and $N \leq \tilde{M}^a$, then M/N is hollow.

Proof: Let $N \leq K \leq \tilde{M}^a$, since $\tilde{M}^a$ is hollow then $K \ll M$ and by Proposition (2.8), $K/\tilde{M}^a \ll \tilde{M}^a/N$.

Definition 2.10: [1] Let W be a semimodule then $\text{Rad}(W) = \sum\{H \leq W \mid H \ll W\}$.

A Jacobson radical is known to be defined by a maximum subsemimodule, but we have adopted the above definition for simplicity.

The following idea is introduced and then studied for modules in [10] We will modassume that y this idea for semimodule and give some results.

Definition 2.11: A semiring Λ is called cosemisimple assume that $\text{Rad}(W)=0$, for each Λ -semimodule W .

The following concept was introduced and then studied for modules in [10], we will convert this concept for semimodules and give some results.

Proposition 2.13: The following assertions are equivalent:

- (i) Λ is a cosemisimple semiring;
- (ii) Every subtractive semimodule over Λ is a small projective.

Proof: (i) $\Rightarrow$ (ii). Let $\tilde{M}^a$ be any subtractive semimodule over Λ , where Λ is a cosemisimple semiring. Consider the following diagram, where A, B are any semimodules over Λ (Fig 1).

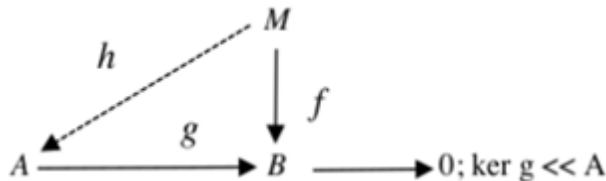


Figure 1

Diagram, A, B are any semimodules over Λ .

Since Λ is a cosemisimple, so $\text{Rad } A=0$, by Definition (2.11). Hence g is an isomorphism.

Let $h = g^{-1} \circ f$, then $goh = (gog^{-1}) \circ f = f$, thus $\tilde{M}^a$ is a small projective semimodule.

(ii) $\Rightarrow$ (i). Let $x \in \text{Rad}(\tilde{M}^a)$, then $\Lambda x \ll \tilde{M}^a$ (by definition of $\text{Rad}(M)$), by assumption $\pi: \tilde{M}^a \rightarrow M/\Lambda x$ splits (since $\tilde{M}^a/\Lambda x$ is small projective). So, there exists $\pi': \tilde{M}^a/\Lambda x \rightarrow \tilde{M}^a$ such that $\pi\pi' = 1_{M/\Lambda x}$.

Let $K = \text{Im } \pi' \leq \tilde{M}^a$, then for $m \in \tilde{M}^a$, $\pi(m) = m + \Lambda x$ and $\pi'(\pi(m)) = m' \in K$.

Now $\pi(\pi'(\pi(m))) = \pi(m')$, (since $\pi\pi' = 1_{M/\Lambda x}$), so $\pi(m) = \pi(m')$, since π is k -regular which implies $m + r_1x = m' + r_2x$ for some $r_1, r_2 \in \Lambda$ where $r_1x, r_2x \in \ker \pi = \Lambda x$, $m' + r_2x \in K + \Lambda x$, $r_2x \in K + \Lambda x$, this implies $r_1x \in K + \Lambda x$ and $m + r_1x \in K + \Lambda x$, since M subtractive then $m \in K + \Lambda x$, hence $M = K + \Lambda x$. But $\Lambda x \ll M$, which implies $K = M$, then $\Lambda x = 0$, that is, $x = 0$ (note that $K \cap \Lambda x = 0$, Indeed: take $rx \in K \cap \Lambda x$, since $K = \text{Im } \pi'$, then $rx = \pi'(x') \Rightarrow \pi(rx) = \pi\pi'(x') \Rightarrow x' = 0 \Rightarrow \pi'(x') = 0 \Rightarrow rx = 0$. Therefore, $\text{Rad}(M) = 0$).

The following concept was introduced and then studied for module in [10], we will convert this concept for semimodules and give some results.

Definition 2.14: A semimodule W is called small free semimodule (S.F), assume that zero is the only small subsemimodule in W .

Note that. Every simple semimodule is small free semimodule (S.F).

Proposition 2.15: Let Λ be a semiring such that all semimodules over Λ are small free semimodule (S.F), then any Λ -semimodule is small projective.

Proof: Since all semimodules are small free semimodule (S.F), thus Λ is a cosemisimple by Definition (2.11). Also by Proposition (2.13) every semimodule over Λ is small projective. $\square$

The proof of the following proposition as in the modules, See [6, p.78(1)].

Proposition 2.16: Consider the following commutative diagram of semimodules A_1, B_1, C_1 and A_2, B_2, C_2 (Fig 2).

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 \longrightarrow 0
 \end{array}$$

Figure 2
Diagram of semimodules A_1, B_1, C_1 and A_2, B_2, C_2 .

With both rows are exact and α is surjective, assume that g_1 is a small surjective, then so is g_2 . The following concept was introduced and then studied for modules in [7] we will introduce this concept for semimodule.

Definition 2.17: A-semimodule W is said to be with FSD, assume that $\forall$ strictly decreasing sequence, $W > W_0 > W_1 > \dots$ of subsemimodules of W , $\exists i$ s.t W_j is a small-subsemimodule in W , for every $j \geq i$.

Definition 2.18: [8] A semimodule that satisfies the descending chain condition on its post of subsemimodules is called Artinian semimodule.

Exa 2.19:

1. Every hollow semimodule is with finite spanning dimension.
2. Every Artinian semimodule is with finite spanning dimension.

We end this section, by the following proposition, the proof of it is as in the modules, See [9].

Proposition 2.20: Assume that W has finite spanning dimension, then any homomorphic image of W has finite spanning dimension.

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