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S^*K-nonsingular modules

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Abstract

This note proposes and examines the notion of S^*k-nonsingular module which is a new type of module theory. The characterization of this property over rings and modules is investigated. Moreover, a few ties between these modules and some further classes are discussed.

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Keywords: *e-small submodule, s-essential submodule; S^*k-nonsingular module.*

1. Introduction

In this article, consider R is an associative ring has 1 unless otherwise stated, also, W is a left R-module. R-Mod denotes the class of all left modules over a ring R and ''$\leq$'' denotes to the submodule. An R-submodule U of $W \in R$-Mod is called essential (small) in W, for short $U \trianglelefteq W$ (resp., $U \ll W$) if, $U \neq 0$ (resp., $U \neq W$) and for all $L \leq W$ with $U \cap L = 0$ (resp., $U + L = W$) imply $L = 0$ (resp., $L = W$), see [2]. In [6] a $U \leq W \in R$-Mod is known an s-essential (e-small) and we can write $U \trianglelefteq_{sm} W$ (resp., $U \ll_e W$) if, for any $L \ll W$ (resp., $L \trianglelefteq W$) with $U \cap L = 0$ (resp., $U + L = W$) implies $L = 0$ (resp., $L = W$). And $\sum_{L \ll_e W} L \leq W$ is named the generalized radical of $W \in R$-Mod which denoted as $Rad_e(W)$. If for any $W \in R$-Mod, $Rad_e(W) = 0$ then R is named a g-V-ring, see [3]. The are some concepts in modules that include in their definition the subclass of e-small submodules as a main case, see [1] and

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[4]. Recall [5-14] $W \in R$-Mod is named k-nonsingular if, for each $f \in End_R(W)$ such that $kerf \trianglelefteq W$ implies $f = 0$. This concept led us to introduce the concept of S^*k-nonsingular modules. $W \in R$-Mod is called to be $S^*\kappa$-nonsingular if for any $f \in End_R(W)$, $kerf \trianglelefteq_{sm} W$ implies $Imf \ll_e W$. Some results, examples and relations about this concept are discussed in this work.

2. S*K-nonsingular modules

Definition 2.1: $W \in R$-Mod is named S^*k-nonsingular if for all $f \in End_R(W)$ with $kerf$ is s-essential in W implies Imf is e-small in W. So, we say a ring R is S^*k-nonsingular if $R \in R$-Mod is $S^*\kappa$-nonsingular.

Examples 2.2:

(1) Evidently, the semisimple or, simple module implies S^*k-nonsingular.

(2) For every prime number p, $\mathbb{Z}_{p^2} \in \mathbb{Z}$-Mod is S^*k-nonsingular.

(3) $\mathbb{Z} \in \mathbb{Z}$-Mod is not S^*k-nonsingular, since we know that $\mathbb{Z} \in \mathbb{Z}$-Mod is uniform, then every nonzero submodule in $\mathbb{Z} \in \mathbb{Z}$-Mod is s-essential. Furthermore, because that (0) is the only small submodule in $\mathbb{Z}$, therefore (0) is also s-essential in $\mathbb{Z}$. So, if we consider a $\mathbb{Z}$-homomorphism $i: \mathbb{Z} \longrightarrow \mathbb{Z}$ where i is an identity mapping, hence i has zero kernel which is s-essential in $\mathbb{Z}$, while $Imi = \mathbb{Z}$ does not e-small in $\mathbb{Z}$, since the only e-small in $\mathbb{Z}$ is the zero submodule.

(4) There is no relation between the two concepts S^*k-nonsingular and k-nonsingular modules, where we proved by (3) that $\mathbb{Z} \in \mathbb{Z}$-Mod is not S^*k-nonsingular but is clearly k-nonsingular. Furthermore, by (2) $\mathbb{Z}_{p^2} \in \mathbb{Z}$-Mod is $S^*\kappa$-nonsingular but is not κ-nonsingular, because that $f \in End_{\mathbb{Z}}(\mathbb{Z}_{p^2})$ defined by $f(\bar{x}) = p\bar{x}$ for all $\bar{x} \in \mathbb{Z}_{p^2}$, then $kerf = \langle p \rangle \trianglelefteq \mathbb{Z}_{p^2}$ but $f \neq 0$ for any prime number p. The next is to identify S^*k-nonsingular modules.

Theorem 2.3: *A module $W \in R$-Mod is S^*k-nonsingular if and only if for every $f \in Hom_R(W/V, W)$ with $V \trianglelefteq_{sm} W$ implies $Imf \ll_e W$.*

Proof: $\Longrightarrow$) Let $V \trianglelefteq_{sm} W$ and $h \in Hom_R(W/V, W)$. Then $g = h \circ \pi \in End_R(W)$, where π is the natural map from W onto W/V. As $g(V) = h(\pi(V)) = h(V) = 0$, so $V \subseteq kerg$, and hence $kerg \trianglelefteq_{sm} W$, see [6, Proposition 2.7(1-(a))]. However, W is S^*k-nonsingular implies that $g(W) \ll_e W$, so $h(\pi(W)) = h(W/V) \ll_e V$, that means $Imh \ll_e W$.

$\Longleftarrow$) If $h \in End_R(W)$ with $kerh \trianglelefteq_{sm} W$. We can consider a homomorphism $\psi: W/ker\, h \longrightarrow W$ by $\psi(m + kerh) = h(m)$ for each $m \in W$. By hypothesis, we deduce that $\psi(W/ker(h)) \ll_e W$, thus $h(W) \ll_e W$ so W is S^*k-nonsingular.

Corollary 2.4: *If $W \in R$-Mod is S^*k-nonsingular and let $L \trianglelefteq_{sm} W$, then $rW \ll_e W$ for all $r \in l_R(L)$.*

Proof: Let $r \in l_R(L)$, and define $g: W/L \longrightarrow W$ by $g(x+L) = rx$ for each $x \in W$. If $x_1 + L = x_2 + L \in W/L$ then $x_1 - x_2 \in L$, and so $rx_1 - rx_2 \in rL = 0$ implies $g(x+L) = g(x_2 + L)$. Evidently, g is an R-homomorphism. From Theorem 2.3, $Img \ll_e W$, i.e., $rW \ll_e W$ for each $r \in l_R(L)$.

Proposition 2.5: If $W \in R$-Mod and $Rad_e(W) = 0$. Then W is S^*k-nonsingular if and only if for each $f \in End_R(W)$, $kerf$ is s-essential in W implies $f = 0$.

Proof: Let $W \in R$ -Mod be S^*k -nonsingular with $Rad_e(W) = 0$. If $f \in End_R(W)$ has s-essential kernel in W, then $Imf \ll_e W$, and $Imf \subseteq Rad_e(W)$, $Imf = 0$, $f = 0$. Conversely, let $f \in End_R(W)$, $kerf$ is s-essential in W. By assumption, $f = 0$ that implies $Imf = 0 \ll_e W$, hence W is S^*k-nonsingular. □

The next corollary is from Proposition 2.5.

Corollary 2.6: *If R is a g-V-ring with $W \in R$-Mod. Then W is S^*k-nonsingular if and only if for each $f \in End_R(W)$, $kerf$ is s-essential in W implies $f = 0$.*

Proposition 2.7: For any ring R, consider the next assertions:

(1) For all s-essential ideal V in R, $l_R(V) \ll_e R$.

(2) $R \in R$-Mod is S^*k-nonsingular. Then $1 \Longrightarrow 2$. If R is uniform, then $2 \Longrightarrow 1$.

Proof: $1 \Longrightarrow 2$ Let $h \in End_R(R)$ with $kerh \trianglelefteq_{sm} R$, to illustrate that $Imh \ll_e R$. Since $1 \in R$ then there is an $x \in R$ with $h(1) = x$, therefore, $h(r) = rx$ for all $r \in R$. Thus, $h(R) = Rx$, as well as so $ker(h) = l_R(x) \trianglelefteq_{sm} R$. By (1), $l_R(ker(h)) = l_R(l_R(x)) = l_R(l_R(Rx)) \ll_e R$, but $Rx \subseteq l_R(l_R(Rx))$ then by [6, Proposition 2.5(1-(i))] $Imh \ll_e R$ and that R is S^*k-nonsingular.

$2 \Longrightarrow 1$ Let $A \trianglelefteq_{sm} R$ with $l_R(A) + B = R$ where $B \trianglelefteq R$. So, $1 = b + c$ for some $b \in l_R(A)$, $c \in B$. Define $\varphi: R \longrightarrow R$ as $\varphi(a) = ba$ for each $a \in R$. Clearly, φ is an R-homomorphism. Thus, $\varphi(A) = rA = 0$, and then $A \subseteq ker\varphi$, hence, $ker\varphi \trianglelefteq_{sm} R$ by [6, Proposition 2.7(1-(a))], as $A \trianglelefteq_{sm} R$. By S^*k-nonsingular of R, $Im\varphi \ll_e R$; that is $bR \ll_e R$ implies $bR \ll R$ because R is uniform. Moreover, $R = bR + cR$, thus $R = cR$. So $1 = cd$ for some $d \in R$, therefore, $1 \in B$ and then $B = R$, hence $l_R(A) \ll_e R$.

Corollary 2.8: *Let $S \in S$-Mod is uniform. Then S is S^*k-nonsingular if and only if, for every $\alpha \in End_S(S)$, there is an $a \in S$, $l_S(a) \trianglelefteq_{sm} S$ implies $Sa \ll_e S$.*

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