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Score for the complex terms in the partition (9,9,3)

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Abstract

$\mathcal{R}$ has 1 and commutative ring, $\mathcal{F}$ is a free $\mathcal{R}$-module and $\mathcal{D}_i\ \mathcal{F}$ be the divided power algebra of degree **i**. By applying capelli identities, diagrams and place polarization we prove that the sequence of the complex terms is complex for the partition (9,9,3).

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1. Introduction

$\wedge\mathcal{F}$ is the exterior algebra of finitely generated free $\mathcal{R}$-module $\mathcal{F}$ and for the partition λ/μ, $\mathcal{K}_{\lambda/\mu}\mathcal{F} = \mathrm{Im}(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu}$ is the Weyl map $d'_{\lambda/\mu}: \mathcal{DF} \to \wedge\mathcal{F}$, [1,2]. The author in [3] defined divided power algebra of the place polarization $(\partial_{21}^{(\hbar)})$ and $(\partial_{32}^{(e)})$. The partitions (8,7), (8,6,3)/(u,1) where u=1 and 2, (6,6,4;0,0) and (7,7,4;0,0) studied respectively, in [4-7]. We applied the same idea in [8] for the partition (9,9,3). Also, we can employ that for $\mathcal{PSL}(2,\mathfrak{U})$, and $\mathcal{SL}(2,\mathfrak{U})$ where $\mathfrak{U}$=23,29,31,37 [9-12], Lie group, $\mathcal{SL}(2,\mathfrak{U})$, $\mathfrak{U} = 81, 729, 121, 169, 125, 3125$ [13-16], dunkl weigted, $\mathcal{S}.\mathcal{U}.\mathcal{T}.(2,\wp)$, for a prime $\wp \leq 9$ and $\mathcal{SL}(2,49)$ and PG(2, 8) which are studied by authors in [17-20], also for PG(2,8) [21-25]. We want these Capelli identities

$$\partial_{32}^{(2)}\,\partial_{21} = \partial_{21}\,\partial_{32}^{(2)} + \partial_{32}\,\partial_{31} \qquad \ldots(1)$$

$$\partial_{32}\,\partial_{21}^{(2)} = \partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\,\partial_{31} \qquad \ldots(2)$$

$$\partial_{21}^{(2)}\,\partial_{32} = \partial_{32}\,\partial_{21}^{(2)} - \partial_{21}\,\partial_{31} \qquad \ldots(3)$$

$$\partial_{32}\,\partial_{31} = \partial_{31}\,\partial_{32} \qquad \ldots(4)$$

2. The Results

The formulation of the partition (9,9,3) of the characteristic zero resolution is

$$\begin{array}{ccccccc} & & \mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F} & & \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} & & \\ \mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} & \to & \oplus & \to & \oplus & \to & \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} \\ & & \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} & & \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_2\mathcal{F} & & \end{array}$$

Take the pursue diagram:

$$\begin{tikzcd}
\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} \arrow[r, "d_1"] \arrow[d, "\hbar_1"'] \arrow[dr, phantom, "\mathcal{Q}"] & \mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F} \arrow[r, "d_2"] \arrow[d, "\hbar_2"] \arrow[dr, phantom, "\mathcal{T}"] & \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} \arrow[d, "\hbar_3"] \\
\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} \arrow[r, "g_1"'] & \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_2\mathcal{F} \arrow[r, "g_2"'] & \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}
\end{tikzcd}$$

If we assign $d_1(v) = \partial_{32}(v)$, $\hbar_1(v) = \partial_{21}(v)$ and $\hbar_2(v) = \partial_{21}^{(2)}(v)$

We want to assign the map g_1 and by (1) activate the diagram $\mathcal{Q}$ commute, thus

$$g_1 \circ \partial_{21} = \partial_{21}^{(2)} \circ \partial_{32} = \partial_{32} \circ \partial_{21}^{(2)} - \partial_{31} \circ \partial_{21} = (\tfrac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}) \circ \partial_{21}$$

Hence, $g_1 = \frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}$.

If we assign the map $g_2(v) = \partial_{32}(v)$ and $h_3(v) = \partial_{21}(v)$, we want assign d_2 and by (2) to activate the diagram $\mathcal{T}$ commute. Thus $h_3 \circ d_2 = g_2 \circ h_2$

$$\partial_{21} \circ d_2 = \partial_{32} \circ \partial_{21}^{(2)} = \partial_{21}^{(2)} \circ \partial_{32} + \partial_{21} \circ \partial_{31} = \partial_{21} \circ (\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31})$$

Hence, $d_2 = \frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}$.

Assign $w(v) = \partial_{32}^{(2)}(v)$ and take the following

$$\begin{tikzcd}
\mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \arrow[r, "d_1"] \arrow[d, "h_1"'] & \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \arrow[r, "d_2"] \arrow[d, "h_2"] & \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \arrow[d, "h_3"] \\
\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \arrow[r, "g_1"'] \arrow[urr, "w"] \arrow[ur, phantom, "\mathcal{H}"] & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \arrow[r, "g_2"'] \arrow[ur, phantom, "\mathcal{N}"] & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}
\end{tikzcd}$$

Proposition 2.1: $\mathcal{H}$ is commutative.

Proof: $d_2 \circ d_1 = (\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}) \circ \partial_{32} = \partial_{32}^{(2)} \circ \partial_{21} - \partial_{32} \circ \partial_{31} + \partial_{31} \circ \partial_{32}$
$= \partial_{32}^{(2)} \circ \partial_{21} = w \circ h_1$

Proposition 2.2: $\mathcal{N}$ is commutative.

Proof: $g_2 \circ g_1 = \partial_{32} \circ (\frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}) = \partial_{21} \circ \partial_{32}^{(2)} + \partial_{32} \circ \partial_{31} - \partial_{32} \circ \partial_{31}$

$$= \partial_{21} \circ \partial_{32}^{(2)} = h_3 \circ w$$

Proposition 2.3: The complex of the sequence

$$\begin{array}{ccccccc}
 & & \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} & & \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} & & \\
0 \to \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} & \xrightarrow{\sigma_3} & \oplus & \xrightarrow{\sigma_2} & \oplus & \xrightarrow{\sigma_1} & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\
 & & \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} & & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} & &
\end{array}$$

Where $\sigma_3(x) = \big(d_1(x), h_1(x)\big)$, $\sigma_2((x_1, x_2)) = (d_2(x_1) - w(x_2), g_1(x_2) - h_2(x_1))$ and $\sigma_1((x_1, x_2)) = \big(h_3(x_1) + g_2(x_2)\big)$

Proof: By the injective of ∂_{21} and ∂_{32} [3] and from (1) - (4) we gain σ_3 is injective

$$(\sigma_2 \circ \sigma_3)(x) = \sigma_2\big(d_1(x), h_1(x)\big)$$

$$=\Big(d_2\big(\partial_{32}(x)\big) - w\big(\partial_{21}(x)\big), g_1\big(\partial_{21}(x)\big) - h_2\big(\partial_{32}(x)\big)\Big)$$

$$d_2\big(\partial_{32}(x)\big) - w\big(\partial_{21}(x)\big) = (\partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31} + \partial_{31}\partial_{32} - \partial_{32}^{(2)}\partial_{21})(x) = 0$$

$$g_1\big(\partial_{21}(x)\big) - h_2\big(\partial_{32}(x)\big) = (\partial_{21}^{(2)}\partial_{32} + \partial_{21}\partial_{31} - \partial_{31}\partial_{21} - \partial_{21}^{(2)}\partial_{32})(x) = 0$$

We gain $(\sigma_2 \circ \sigma_3)(x) = 0$, and $(\sigma_1 \circ \sigma_2)(x_1, x_2) = (\partial_{32} \circ \partial_{21}^{(2)} - \partial_{21} \circ \partial_{31}$ $+\partial_{21} \circ \partial_{31} - \partial_{32} \circ \partial_{21}^{(2)})(x_1) + (\partial_{32}^{(2)}\partial_{21} + \partial_{32}\partial_{31} - \partial_{32}\partial_{31} - \partial_{21}\partial_{32}^{(2)})(x_2) =$ 0. So the sequence is complex.

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