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SAS-*N*-flat modules

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Abstract

New kind of modules called SAS-***N***-flat module is considered and investigated in this article as a proper generalization of ***N***-flat module. Many properties and characterizations of this kind of modules are given and some known results are extended.

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1. Introduction

In what follows, we let R will be an associative ring with identity element and all the modules considered are will be unitary R-modules. The category of left (resp. right) R-modules is denoted by R-Mod (resp. Mod-R). If the soc(N / K) is not zero, for all submodules K of a module N, then we say that N is semiartinian [5]. For a module M, we use $Sa(M) = \sum_{N \leq^{sa} M} N$. A submodule H of a module M is said to be small if adding it with any submodule K of M, then $K = M$ [1]. The notation $B \leq^{fg} L$ (resp. $B \leq^{sa} L$, $B \leq^{sas} L$, $B \leq^{fgsas} L$, $B \leq^{csas} L$, $B \leq^{fg} L$) signifies that B is a finitely generated (resp. semiartinian, semiartinian small, finitely generated semiartinian small, cyclic semiartinian small) submodule of a module L. The symbol M^* refer to $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

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Module theory is a branch of algebra and has various applications in mathematics. References [1-3,5,8-14] and [16] provide fundamentals and related concepts in this field. Injectivity and flatness are aspects in module theory with many connections, between them. Many generalizations of injectivity were introduced by authors (see, for instance, [3,4,6,9-11,13] and [15-21]).

Also, some general concepts of flat modules are introduced by many authors as opposite concepts of some generalizations of injectivity (see, for instance, [7,10,15]). The notation iff (resp. f.g.f., T.F.S.E., C.T.F.S.) means if and only if (resp. finitely generated free, the following statements are equivalent, consider the following statements).

In [3], the proposed a generalized version of N-injectivity known as SAS-N-injectivity. "A right R-module M is called SAS-N-injective (where $N \in$ Mod-R) if every right R-homomorphism from any $K \leq^{sas} N$ into M extends to N. M is called SAS-injective, when M is SAS-R-injective".

In the present paper, we present and consider SAS-N-flat modules as a general concept of N-flatness. The terminology used to refer to the class of SAS-N-injective left (resp. SAS-injective left, SAS-N-flat right, SAS-flat right) modules is represented by the symbol ${}_R$(SAS-N-$\mathbb{I}$) (resp. ${}_R$(SAS-$\mathbb{I}$), (SAS-N-$\mathbb{F})_R$, (SAS-$\mathbb{F})_R$). We provide some characterizations of SAS-N-flat modules, in particular, we prove that $D \in$ (SAS-N-$\mathbb{F})_R$ iff the exactness of the sequence $0 \longrightarrow D\otimes_R K \xrightarrow{I_D \otimes i_K} D\otimes N$ holds for any $K \leq^{fgsas} N$. Additionally, we discuss various properties concerning SAS-N-flat and SAS-flat modules. For instance, we prove that if F is a free module with basis $\{x_j | j \in \wedge\}$ and A is a submodule of F, then F/A is SAS-flat iff for any $a \in A$ such that $a = \sum_{i=1}^{n} x_{ji} r_i$, $r_i \in R$ and $Rr_i \leq^{sas} {}_R R$, $(i = 1, \dots\dots, n)$, $n \in \mathbb{Z}^+$, then $a \in AI_a$, where $I_a = \sum_{i=1}^{n} Rr_i$.

2. SAS-N-Flat Modules

The notion of SAS-N-flat modules is considered and investigated in follows.

Definition 2.1: A right R-module M is called SAS-N-flat ($N \in R$-Mod), if for any $K \leq^{sas} N$, the canonical mapping $M\otimes_R K \longrightarrow M\otimes_R N$ is injective. Also, if M is SAS-R-flat, then we say that M is SAS-flat.

Examples 2.2:

1. For any $N \in R$-Mod with $Sa(N) = 0$ or $\mathrm{J}(N) = 0$, we have that all modules are SAS-N-flat.
2. It is clear that every flat module is SAS-flat. For any $n \geq 2$, we have $\mathbb{Z}_n$ is SAS-flat $\mathbb{Z}$-module (by (1)), but it is not flat.

Theorem 2.3: *A module $L \in$ (SAS-N-$\mathbb{F})_R$ iff $L^* \in$ ${}_R$(SAS-N-$\mathbb{I}$), where $N \in R$-Mod.*

Proof: Straightforward.

Proposition 2.4: Then the next conditions hold for $M, L \in$ Mod-R and $K, N \in$ R-Mod:

(1) If K is a subset of N and $M \in$ (SAS-N-$\mathbb{F})_R$, then $M \in$ (SAS-K-$\mathbb{F})_R$.

(2) If $M \cong L$ and $M \in$ (SAS-N-$\mathbb{F})_R$, then $L \in$ (SAS-N-$\mathbb{F})_R$.

(3) If $K \cong N$ and $M \in$ (SAS-K-$\mathbb{F})_R$, then $M \in$ (SAS-N-$\mathbb{F})_R$.

Proof:

(1) Let $M \in$ (SAS-N-$\mathbb{F})_R$ and $K \subseteq N$. By Theorem 2.3, $M^* \in {}_R$(SAS-N-$\mathbb{I}$). Since K is a subset of N, it follows from [2, Theorem 2.3] that $M^* \in {}_R$(SAS-K-$\mathbb{I}$). Thus, from Theorem 2.3 we get $M \in$ (SAS-K-$\mathbb{F})_R$.

(2) Let $\alpha: M \longrightarrow L$ be an isomorphism. Suppose that $M \in$ (SAS-N-$\mathbb{F})_R$. Thus, we conclude from Theorem 2.3 that $M^* \in {}_R$(SAS-N-$\mathbb{I}$). Since $M \cong L$, we get $M^* \cong L^*$ and hence $L^* \in {}_R$(SAS-N-$\mathbb{I}$). According to Theorem 2.3, $L \in$ (SAS-N-$\mathbb{F})_R$.

(3) Let K be an isomorphic to N and $M \in$ (SAS-K-$\mathbb{F})_R$. By Theorem 2.3, M^* ${}_R$(SAS-K-$\mathbb{I}$). Consequently, from Theorem 2.3 in [2, p. 32], we get M* ${}_R$(SAS-N-$\mathbb{I}$) and so $M \in$ (SAS-N-$\mathbb{F})_R$. □

Corollary 2.5: *The SAS-flatness is an algebraic property.*

Proof: By applying Proposition 2.4(2), when we take $N = {}_RR$. □

Proposition 2.6: A module $M \in$ (SAS-N-$\mathbb{F})_R$ iff for every $K \leq^{fgsas} N$, where $N \in R$-Mod, the canonical mapping $M \otimes_R K \longrightarrow M \otimes N$ is monic.

Proof: Straightforward.

Corollary 2.7: *A module $\boldsymbol{M} \in (SAS\text{-}\mathbb{F})_{\boldsymbol{R}}$ iff the canonical mapping $\boldsymbol{M \otimes_R L} \longrightarrow \boldsymbol{M \otimes_R R}$ is monic, for all $\boldsymbol{L} \leq^{\boldsymbol{fgsas}} {}_RR$.*

Proposition 2.8: C.T.F.S. for $M \in$ Mod-R and $N \leq M$:

(1) $M/N \in$ (SAS-$\mathbb{F})_R$.

(2) For every $I \leq^{fgsas}$ ${}_R$R, $\mathrm{Tor}_1(M/N, R/I) = 0$.

(3) If $L \leq^{fgsas}$ ${}_R$R, then $N \cap ML = NL$.

Then **(1)⟺(2)⟹(3)** and **(3)** implies **(1)**, if $M \in$ (SAS-$\mathbb{F})_R$.

Proof: Straightforward.

Proposition 2.9: C.T.F.S. for $N \in R$-Mod:

(1) $J(N) \cap Sa(N) = 0$.

(2) R-Mod$= {}_R$(SAS-N-$\mathbb{I}$).

(3) All pure injective modules in R-Mod are in ${}_R$(SAS-N-$\mathbb{I}$).

(4) Mod-$R =$ (SAS-N-$\mathbb{F})_R$.

Then **(1)⟺(2), (2)⟹(3)** and **(3)⟺(4)**. All statements are equivalent, if all pure submodules of any SAS-N-injective left module are summand.

Proof: **(1)⟺(2)**. This is according to [2, Proposition 2.8].

(2)⟹(3). This is obvious.

(3)⟹(4). Let $M \in$ Mod-R. According to [14, 34.6, p. 289], we conclude that M^* is pure injective. Consequently, we get from the hypothesis that $M^* \in {}_R$(SAS-N-$\mathbb{I}$). According to Theorem 2.3 we can assert that $M \in$ (SAS-N-$\mathbb{F})_R$.

(4)⟹(3). Suppose that K is pure injective. Consequently, we get from the hypothesis that $K^* \in$ (SAS-N-$\mathbb{F})_R$. According to Theorem 2.3, we conclude that $K^{**} \in {}_R$(SAS-N-$\mathbb{I}$). We get that there is a pure monomorphism $\alpha: K \longrightarrow K^{**}$ (from [14, 34.6, p. 289]). Consequently from pure injectivity of K, we get that α is a split monomorphism (by [14, 34.6, p. 289]) and so $\alpha(K)$ is a summand of K^{**}. Thus, we conclude from $K^{**} \in {}_R$(SAS-N-$\mathbb{I}$) that $\alpha(K) \in {}_R$(SAS-N-$\mathbb{I}$) (according to [2, Theorem 2.3, p. 32]). Since K is an isomorphic to $\alpha(K)$, we get that $K \in {}_R$(SAS-N-$\mathbb{I}$) (from applying [2, Theorem 2.3, p. 32].

(3)⟹(2). Suppose all pure submodules of any module belong to ${}_R$(SAS-N-$\mathbb{I}$) are summand. Consequently, from [14, 34.6, p. 289], we get that M^{**} is pure injective, for any $M \in R$-Mod. We conclude from hypothesis that $M^{**} \in {}_R$(SAS-N-$\mathbb{I}$). Thus, we get that there is a pure monomorphism $\beta: M \longrightarrow M^{**}$ (according to [14, 34.6, p. 289]). Thus, $\beta(M) \leq^p M^{**}$ and hence from hypothesis, we conclude that $\beta(M)$ is a summand of M^{**} and so $\beta(M) \in {}_R$(SAS-N-$\mathbb{I}$) (by [5, Corollary 2.4, p. 33]). Since M is an isomorphic to $\beta(M)$, we get that M is SAS-N-injective (by using [2, Theorem 2.3, p. 32]). □

Remark 2.10: If K is a submodule of M, where $M \in$ (SAS-N-$\mathbb{F})_R$ and $N \in R$-Mod, then it is not necessary that the modules M/K and K are SAS-N-flat, for example: let $M = \mathbb{Z}_4$ as $\mathbb{Z}_4$-module, $K =<\overline{2}>$ is a submodule of M and $N = \mathbb{Z}_4$ as $\mathbb{Z}_4$-module. Clearly, $K =<\overline{2}> \leq^{sas} N = \mathbb{Z}_4$ as $\mathbb{Z}_4$-module. Since M is projective $\mathbb{Z}_4$-module, we have M is SAS-flat $\mathbb{Z}_4$-module. By [5, Proposition(1), p.256], $\mathrm{I}_{M/K} \otimes \mathrm{i}_K = 0$. Since $\mathrm{K}^2 =<\overline{0}> \neq$ K, it follows from [5, Preposition(2), p.256] that $\mathbb{Z}_4/<\overline{2}> \otimes <\overline{2}> \neq 0$, hence there is $0 \neq x \in \mathbb{Z}_4/<\overline{2}> \otimes <\overline{2}>$. Since $I_{\mathbb{Z}_4/<\overline{2}>} \otimes i_{<\overline{2}>} : \mathbb{Z}_4/<\overline{2}> \otimes <\overline{2}> \longrightarrow \mathbb{Z}_4/<\overline{2}> \otimes \mathbb{Z}_4$ is a zero $\mathbb{Z}$-homomorphism, we have $(I_{\mathbb{Z}_4/<\overline{2}>} \otimes i_{<\overline{2}>})(x) = 0$ with $x \neq 0$. Thus $I_{\mathbb{Z}_4/<\overline{2}>} \otimes i_{<\overline{2}>}$ is not monomorphism and hence the $\mathbb{Z}_4$-module $\mathbb{Z}_4/<\overline{2}>$ is not an SAS-$\mathbb{Z}_4$-flat (=SAS-flat). Since $K =<\overline{2}> \cong \mathbb{Z}_4/<\overline{2}>$, we have that K is not SAS-$\mathbb{Z}_4$-flat (=SAS-flat).

Proposition 2.11: If every f.g. submodule of M is SAS-N-flat, ($N \in R$-Mod, $M \in$ Mod-R), then:

(1) $M \in (\text{SAS-}N\text{-}\mathbb{F})_R$.

(2) All submodules of M are SAS-N-flat.

Proof:

(1) By [14, 24.7, p.200], M is a direct limit of all its f.g. submodules. Thus M is a direct limit of SAS-N-flat modules and consequently we get that $M \in (\text{SAS-}N\text{-}\mathbb{F})_R$.

(2) Let K be a submodule of M. If L $\leq^{\text{fg}}$ K, thus L $\leq^{\text{fg}}$ M. Consequently from hypothesis, we get L is SAS-N-flat. This, all f.g. submodule of K are SAS-N-flat. By **(1)** above, K is SAS-N-flat. □

"A ring R is called a right semihereditary if every finitely generated right ideal of R is projective" [6, 2.28, p.43].

Corollary 2.12: *All right ideals of any right semihereditary ring R are SAS-N-flat right R-modules, where $N \in R\text{-}Mod$.*

Proof: Let R be a right semihereditary ring and $N \in R$-Mod. Thus all finitely generated right ideals of R_R are projective right R-modules. Hence all finitely generated right ideals of R_R are SAS-N-flat right R-modules. By Proposition 2.11(2), all right ideals of R are SAS-N-flat. □

Proposition 2.13: $\ker(\lambda)$ is SAS-flat, for each SAS-flat left R-module M and $\lambda \in \text{End}(M)$ iff the class $_R(\text{SAS-}\mathbb{F})$ is closed under kernels of homomorphisms.

Proof: (⟹). Let $g \in \text{Hom}_R(N, M)$ with $N, M \in {}_R(\text{SAS-}\mathbb{F})$. Define $\lambda: N \oplus M \longrightarrow N \oplus M$ by $\lambda((a, b)) = (0, g(a))$. By hypothesis, $\ker(\lambda) = \ker(g) \oplus M$ is SAS-flat and so $\ker(g)$ is SAS-flat.

(⟸). Clear.

Proposition 2.14: Let F be a free right R-module with basis $\{x_j | j \in \wedge\}$ and let A be a submodule of F. Then T.F.S.E.:

(1) F/A is SAS-flat.

(2) For any $a \in A$ such that $a = \sum_{i=1}^{n} x_{ji} r_i$, $r_i \in R$ and $Rr_i \leq^{sas} {}_R R$, $(i = 1, \dots \dots, n)$, $n \in \mathbb{Z}^+$, then $a \in AI_a$, where $I_a = \sum_{i=1}^{n} Rr_i$.

Proof: **(1)⟹(2)**. Suppose that F/A is SAS-flat. Let $a \in A$ such that $a = \sum_{i=1}^{n} x_{ji} r_i, r_i \in R$ and $Rr_i \leq^{sas} {}_R R$, for all $i = 1,2, \dots, n, n \in \mathbb{Z}^+$. Let $I_a = \sum_{i=1}^{n} Rr_i$. Thus $I_a \leq^{sas} {}_R R$. Since $x_{ji} \in F$ and $r_i \in I_a$, for all $i = 1, \dots \dots, n$, we have $a \in FI_a$ and hence $a \in A \cap FI_a$. Since $I_a \leq^{fgsas} {}_R R$ and F/A is SAS-flat (by hypothesis), we have from Proposition 2.8 that $A \cap FI_a = AI_a$. Thus $a \in AI_a$.

(2)⟹(1). Let $L \leq^{fgsas} {}_R R$. Let $a \in A \cap FL$, thus $a \in A$ and $a \in FL$. Hence $a = \sum_{i=1}^{n} x_{ji} t_i$, where $x_{ji} \in F$ and $t_i \in L \subseteq R$. Put $I_a = \sum_{i=1}^{m} R_i t_i$. Since $t_i \in L$ and $L \leq^{sas} {}_R R$, we have $Rt_i \leq^{sas} {}_R R$. By hypothesis, $a \in AI_a$. Since $AI_a \subseteq AL$, it follows that $a \in AL$ and hence $A \cap FL \subseteq AL$. Since $AL \subseteq A \cap FL$ is true always, we have $A \cap FL = AL$. Since $F \in {}_R$(SAS-$\mathbb{F}$), consequently, from Proposition 2.8, we get F/A is SAS-flat. □

Corollary 2.15: *Let A be a right ideal of a ring R. Then T.F.S.E.:*

(1) *R/A is an SAS-flat right R-module.*

(2) *For any $a \in A$ with $I_a = Ra \leq^{sas} {}_R R$, then $a \in AI_a$.*

Proof: By using Proposition 2.14, when we take $F = R_R$. □

Proposition 2.16: Let F be a free right R-module with basis $\{x_j | j \in \wedge\}$ and let A be a submodule of F. Then T.F.S.E.:

(1) F/A is an SAS-flat R-module.

(2) If $a \in A$ such that $a = \sum_{i=1}^{n} x_{ji} r_i, r_i \in R$ and $Rr_i \leq^{sas} R, (i = 1,2, \ldots, n), n \in \mathbb{Z}^+$, then there is $\theta \in \mathrm{Hom}_R(F, A)$ with $\theta(a) = a$.

Proof: **(1)⟹(2)**. Let F/A be SAS-flat. Let $a \in A$ such that $a = \sum_{i=1}^{n} x_{ji} r_i, r_i \in R$ and $Rr_i \leq^{sas} R, (i = 1,2, \ldots, n), n \in \mathbb{Z}^+$. Since F/A is SAS-flat, it follows from Proposition 2.14 that $a \in AI_a$, where $I_a = \sum_{i=1}^{n} Rr_i$. Thus $a = \sum_{i=1}^{n} a_i r_i$, where $a_i \in A$. Define $\theta: F \longrightarrow A$ by $\theta(x_{ji}) = a_i$ and $\theta(x_k) = 0$ for all $x_k \neq x_{ji}$. By [6, Theorem 4.23, p. 129], θ is a right R-homomorphism. Thus $\theta(a) = \theta(\sum_{i=1}^{n} x_{ji} r_i) = \sum_{i=1}^{n} \theta(x_{ji}) r_i = \sum_{i=1}^{n} a_i r_i = a$.

(2)⟹(1). Let $a \in A$ such that $a = \sum_{i=1}^{n} x_{ji} r_i, r_i \in R$ and $Rr_i \leq^{sas} R, (i = 1,2, \ldots, n), n \in \mathbb{Z}^+$. Thus, there is $\theta \in \mathrm{Hom}_R(F, A)$ with $\theta(a) = a$ (by hypothesis). Thus $\theta(\sum_{i=1}^{n} x_{ji} r_i) = a$ and so $a = \sum_{i=1}^{n} \theta(x_{ji}) r_i$. Since $\theta(x_{ji}) \in A$ and $r_i \in I_a$, where $I_a = \sum_{i=1}^{n} Rr_i$, we have $a \in AI_a$. By Proposition 2.14, F/A is an SAS-flat module.

We can easily prove the next proposition, which gives equivalent statements of SAS-flat module.

Proposition 2.17: Let $M \in$ Mod-R. Then T.F.S.E.:

(1) $M \in$ (SAS-$\mathbb{F}$)$_R$.

(2) For any f.g.f. right R-module F and $K \leq^{csas} F$, any $\alpha \in \mathrm{Hom}_R(F/K, M)$ factors through a f.g.f. module.

(3) For any f.g.f. right R-module F and $H \leq^{csas} F$ and any diagram with an epimorphism $\lambda: L \to D$, there is $\tau \in \mathrm{Hom}_R(F/H, L)$ with $\varphi\vartheta = \lambda\tau$.

$$\begin{tikzcd} F/H \arrow[r, "\vartheta"] \arrow[d, dashed, "\tau"'] & M \arrow[d, "\varphi"] & \\ L \arrow[r, "\lambda"] & D \arrow[r] & 0 \end{tikzcd}$$

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