

Result for the terms of the characteristic free resolution for the partition (9,9)

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Abstract

By applying and proving that $S_{i-1} \partial_x + \partial_x S_i = \text{id}_{M_i}$ for $i = 0, 1, 2, \dots, 8$ are contracting homotopy by considered two-rowed Weyl module, also prove the exactness of the sequence of the terms of Weyl resolution of characteristic-free resolution for (9,9).

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1. Introduction

$\mathcal{R}$ has 1 and commutative ring, $\mathcal{F}$ is a free $\mathcal{R}$ -module and $\mathcal{D}_i\mathcal{F}$ be the divided power algebra of degree i , $\Lambda\mathcal{F}$ is the exterior algebra of finitely generated free $\mathcal{R}$ -module $\mathcal{F}$ and for the partition λ/μ , $\mathcal{K}_{\lambda/\mu}\mathcal{F} = \text{Im}(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu}$ is the Weyl map $d'_{\lambda/\mu}: \mathcal{D}\mathcal{F} \rightarrow \Lambda\mathcal{F}$, [1], [2]. The author in [3] defined contracting homotopy $\{\mathcal{S}_i\}$. The partitions $(3,3,2)$, $(8,7,3)$, $(6,6,4;0,0)$ and $(7,7,4;0,0)$ studied respectively, in [4-7]. We applied the same idea in [8] for the partition $(9,9)$. Also, we can employ that for 2-visible submodules and modules, $\mathcal{PSL}(2, \mathfrak{U})$, where $\mathfrak{U} = 23, 29, 31, 37, \mathcal{SL}(2, \mathfrak{U})$, $\mathfrak{U} = 23, 29, 31, 37, 121, 169, \mathcal{S.U.T.}(2, \mathfrak{p})$, for a prime $\mathfrak{p} \leq 9$ and rad-quasi-prime submodules, which are studied by authors in [9-20].

2. The Outcomes

The terms of Weyl resolution for the partition $(9,9)$ are

$$\begin{aligned} \mathcal{M}_0 &= \mathcal{D}_9 \otimes \mathcal{D}_9, \mathcal{M}_1 = \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b}; \text{ where } 1 \leq b \leq 9, \\ \mathcal{M}_2 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|}; \text{ where } 2 \leq |b| = b_1 + b_2 \leq 9 \\ \mathcal{M}_3 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|}; \text{ where } 3 \leq |b| = \sum_{i=1}^3 b_i \leq 9 \\ \mathcal{M}_4 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|}; \text{ where } 4 \leq |b| = \sum_{i=1}^4 b_i \leq 9 \\ \mathcal{M}_5 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|}; \text{ where } \\ 5 \leq |b| &= \sum_{i=1}^5 b_i \leq 9, \mathcal{M}_6 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{D}_{9+|b|} \\ &\otimes \mathcal{D}_{9-|b|}; \mathcal{D}_{9-|b|}, \text{ where } 6 \leq |b| = \sum_{i=1}^6 b_i \leq 9, \mathcal{M}_7 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} \\ &x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{Z}_{21}^{(b_7)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|}; \text{ where } 7 \leq |b| = \sum_{i=1}^7 b_i \leq 9, \\ \mathcal{M}_8 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{Z}_{21}^{(b_7)} x \mathcal{Z}_{21}^{(b_8)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|}; \\ \text{where } 8 \leq |b| &= \sum_{i=1}^8 b_i \leq 9, \end{aligned}$$

$$\mathcal{M}_9 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{D}_{18} \otimes \mathcal{D}_0$$

The homotopies $\{\mathcal{S}_i\}$; where $i = 0, 1, 2, \dots, 8$ are below with prove that are contracting

$$\begin{aligned} \mathcal{S}_0 \left(\begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9)} & 2^{(\ell)} \\ 2^{(9-\ell)} & \end{vmatrix} \right) &= \begin{cases} 0; & \text{if } \ell \leq 0 \\ \mathcal{Z}_{21}^{(\ell)} x \left(\begin{vmatrix} w \\ w' \end{vmatrix} \begin{vmatrix} 1^{(9+\ell)} \\ 2^{(9-\ell)} \end{vmatrix} \right); & \text{if } \ell = 1, 2, 3, 4, 5, 6, 7, 8, 9 \end{cases} \\ \mathcal{S}_1 \left(\mathcal{Z}_{21}^{(\ell)} x \left(\begin{vmatrix} w \\ w' \end{vmatrix} \begin{vmatrix} 1^{(9+|\ell|)} & 2^{(m)} \\ 2^{(9-|\ell|-m)} & \end{vmatrix} \right) \right) \\ &= \begin{cases} 0; & \text{if } m = 0 \\ \mathcal{Z}_{21}^{(\ell)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{vmatrix} w \\ w' \end{vmatrix} \begin{vmatrix} 1^{(9+\ell+m)} \\ 2^{(9-\ell-m)} \end{vmatrix} \right); & \text{if } m = 1, 2, 3, 4, 5, 6, 7, 8 \end{cases} \\ \text{For } |\ell| = \ell_1 + \ell_2, \mathcal{S}_2 \left(\mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \left(\begin{vmatrix} w \\ w' \end{vmatrix} \begin{vmatrix} 1^{(9+|\ell|)} & 2^{(m)} \\ 2^{(9-|\ell|-m)} & \end{vmatrix} \right) \right) \end{aligned}$$

$$= \begin{cases} 0 ; \text{ if } m = 0 \\ Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|+m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} ; \text{ if } m = 1, 2, 3, 4, 5, 6, 7; \end{cases}$$

For $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3$,

$$\begin{aligned} & \mathcal{S}_3 \left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \right) \\ &= \begin{cases} 0 ; m = 0 \\ Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|+m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} ; m = 1, 2, 3, 4, 5, 6; \end{cases} \end{aligned}$$

and so as $\mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6, \mathcal{S}_7, \mathcal{S}_8$. So we have the following diagram

$$\begin{array}{ccccccccccccccc} \mathcal{M}_9 & \xleftarrow[\mathcal{S}_8]{\partial_x} & \mathcal{M}_8 & \xleftarrow[\mathcal{S}_7]{\partial_x} & \mathcal{M}_7 & \xleftarrow[\mathcal{S}_6]{\partial_x} & \mathcal{M}_6 & \xleftarrow[\mathcal{S}_5]{\partial_x} & \mathcal{M}_5 & \xleftarrow[\mathcal{S}_4]{\partial_x} & \mathcal{M}_4 & \xleftarrow[\mathcal{S}_3]{\partial_x} & \mathcal{M}_3 & \xleftarrow[\mathcal{S}_2]{\partial_x} & \mathcal{M}_2 & \xleftarrow[\mathcal{S}_1]{\partial_x} \\ & & & & & & & & & & & & & & & & \mathcal{M}_1 & \xleftarrow[\mathcal{S}_0]{\partial_x} & \mathcal{M}_0 \end{array}$$

Now we have

$$\mathcal{S}_0 \partial_x \left(Z_{21}^{(\kappa)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \right) = \begin{pmatrix} \kappa + m \\ m \end{pmatrix} Z_{21}^{(\kappa+m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+\kappa+m)} \\ 2^{(9-\kappa-m)} \end{vmatrix},$$

and

$$\begin{aligned} \partial_x \mathcal{S}_1 \left(Z_{21}^{(\kappa)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \right) &= - \begin{pmatrix} \kappa + m \\ m \end{pmatrix} Z_{21}^{(\kappa+m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+\kappa+m)} \\ 2^{(9-\kappa-m)} \end{vmatrix} + \\ Z_{21}^{(\kappa)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix}. \end{aligned}$$

$$\begin{aligned} & \mathcal{S}_1 \partial_x \left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \right) \\ &= - \begin{pmatrix} |\kappa| \\ \kappa_2 \end{pmatrix} Z_{21}^{(|\kappa|)} x Z_{21}^{(m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|+m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \\ &+ \begin{pmatrix} \kappa_2 + m \\ m \end{pmatrix} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|+m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix}, \text{ and for } |\kappa| = \kappa_1 + \kappa_2 \\ & \partial_x \mathcal{S}_2 \left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \right) = \begin{pmatrix} |\kappa| \\ \kappa_2 \end{pmatrix} Z_{21}^{(|\kappa|)} x Z_{21}^{(m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|+m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} \\ & - \begin{pmatrix} \kappa_2 + m \\ m \end{pmatrix} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+m)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|+m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix} + \\ & Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x \begin{pmatrix} w \\ w' \end{pmatrix} \begin{vmatrix} 1^{(9+|\kappa|)} & 2^{(m)} \\ 2^{(9-|\kappa|-m)} \end{vmatrix}. \end{aligned}$$

It is clear that $\mathcal{S}_1 \partial_x + \partial_x \mathcal{S}_2 = \text{id}_{\mathcal{M}_2}$

By that same way, we can get that, $\mathcal{S}_{i-1} \partial_x + \partial_x \mathcal{S}_i = \text{id}_{\mathcal{M}_i}$, $i = 3, 4, \dots, 8$.

Thus $\{\mathcal{S}_i\}$; $i = 0, 1, 2, \dots, 8$ is a contracting homotopy [3], which implies the exactness of the following complex

$$0 \longrightarrow \mathcal{M}_9 \longrightarrow \mathcal{M}_8 \longrightarrow \mathcal{M}_7 \longrightarrow \mathcal{M}_6 \longrightarrow \mathcal{M}_5 \longrightarrow \mathcal{M}_4 \longrightarrow \mathcal{M}_3 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0$$

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