

Resolution of the terms in the case (9,8)

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Abstract

The sequence of the elements of characteristic-free (Char.F.) Weyl resolution (We.Re.) for the case (9,8) is exact is the target of this treatise by prove that every definition maps are contracting homotopy.

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1. Introduction

Anywhere this treatise $\mathcal{R}$ commutative ring and has 1, $\mathcal{F}$ free $\mathcal{R}$ -module and $\mathcal{D}_i\mathcal{F}$ be the divided power algebra of degree i , $\Lambda\mathcal{F}$ is the exterior algebra of finitely generated free $\mathcal{R}$ -module $\mathcal{F}$ and for the partition $\lambda/\mu, \mathcal{K}_{\lambda/\mu}\mathcal{F} = \text{Im}(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu}$ is the Weyl map $d'_{\lambda/\mu}: \mathcal{D}\mathcal{F} \rightarrow \Lambda\mathcal{F}$, [1], [2]. The discussion of the contracting homotopy for the maps $\{\mathcal{S}_i\}$ in [3]. The partitions $(8,7)$, $(7,7)$, $(7,7)/(1,0)$, $(6,6,4;0,0)$ and $(7,7,4;0,0)$ studied respectively, in [4-7]. We stratify the same idea in [8] for the partition $(9,8)$. Also, we can employ that for $\mathcal{PSL}(2, \mathcal{U})$, and $\mathcal{SL}(2, \mathcal{U})$ where $\mathcal{U}=23, 29, 31, 37$ [9-12], Lie group, $\mathcal{SL}(2, \mathcal{U})$, $\mathcal{U}=81, 729, 121, 169, 125, 3125$ [13-16], dunkl weigted and S. U. T. $(2, p)$, for a prime $p \leq 9$, $\mathcal{SL}(2, 49)$ and univalent functions which are studied in [17, 18].

2. The score and their studied

The elements of Char.F. We.Re. for the partition $(9,8)$ are

$$\begin{aligned} \mathcal{M}_0 &= \mathcal{D}_9 \otimes \mathcal{D}_8, \mathcal{M}_1 = \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{8-b}; \text{ where } 1 \leq b \leq 8, \\ \mathcal{M}_2 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|}; \text{ where } 2 \leq |b| = b_1 + b_2 \leq 8 \\ \mathcal{M}_3 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|}; \text{ where } 3 \leq |b| = \sum_{i=1}^3 b_i \leq 8 \\ \mathcal{M}_4 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|}; \text{ where } 4 \leq |b| = \sum_{i=1}^4 b_i \leq 8 \\ \mathcal{M}_5 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|}; \text{ where } \\ &\quad 5 \leq |b| = \sum_{i=1}^5 b_i \leq 8, \\ \mathcal{M}_6 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|}; \\ &\quad \mathcal{D}_{8-|b|}, \text{ where } 6 \leq |b| = \sum_{i=1}^6 b_i \leq 8, \\ \mathcal{M}_7 &= \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{Z}_{21}^{(b_7)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|}; \text{ where } \\ &\quad 7 \leq |b| = \sum_{i=1}^7 b_i \leq 8, \\ \mathcal{M}_8 &= \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{D}_{18} \otimes \mathcal{D}_0 \end{aligned}$$

The homotopies $\{\mathcal{S}_i\}$; where $i=0,1,2,\dots,7$ are below with prove that are contracting

$$\begin{aligned} \mathcal{S}_0 \left(\begin{pmatrix} w \\ w' \end{pmatrix} \middle| \begin{matrix} 1^{(9)} & 2^{(\ell)} \\ 2^{(8-\ell)} \end{matrix} \right) &= \begin{cases} 0; & \text{if } \ell \leq 0 \\ \mathcal{Z}_{21}^{(\ell)} x \left(\begin{pmatrix} w \\ w' \end{pmatrix} \middle| \begin{matrix} 1^{(9+\ell)} \\ 2^{(8-\ell)} \end{matrix} \right); & \text{if } \ell = 1, 2, 3, 4, 5, 6, 7, 8 \end{cases} \\ \mathcal{S}_1 \left(\mathcal{Z}_{21}^{(\ell)} x \left(\begin{pmatrix} w \\ w' \end{pmatrix} \middle| \begin{matrix} 1^{(9+\ell)} & 2^{(m)} \\ 2^{(8-\ell-m)} \end{matrix} \right) \right) &= \begin{cases} 0; & \text{if } m = 0 \\ \mathcal{Z}_{21}^{(\ell)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{pmatrix} w \\ w' \end{pmatrix} \middle| \begin{matrix} 1^{(9+\ell+m)} \\ 2^{(8-\ell-m)} \end{matrix} \right); & \text{if } m = 1, 2, 3, 4, 5, 6, 7 \end{cases} \end{aligned}$$

$$\begin{aligned} \text{For } |\ell| = \ell_1 + \ell_2, \mathcal{S}_2 & \left(\mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|)} \\ 2^{(8-|\ell|-m)} \end{array} \right) \right) \\ &= \begin{cases} 0; & \text{if } m = 0 \\ \mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|+m)} \\ 2^{(8-|\ell|-m)} \end{array} \right); & \text{if } m = 1, 2, 3, 4, 5, 6 \end{cases} \end{aligned}$$

$$\begin{aligned} \text{For } |\ell| = \ell_1 + \ell_2 + \ell_3, \mathcal{S}_3 & \left(\mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \mathcal{Z}_{21}^{(\ell_3)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|)} \\ 2^{(8-|\ell|-m)} \end{array} \right) \right) \\ &= \begin{cases} 0; & \text{if } m = 0 \\ \mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \mathcal{Z}_{21}^{(\ell_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|+m)} \\ 2^{(8-|\ell|-m)} \end{array} \right); & \text{if } m = 1, 2, 3, 4, 5 \end{cases} \end{aligned}$$

and so as $\mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6, \mathcal{S}_7$. So we have the following diagram

$$\begin{array}{ccccccccccc} \mathcal{M}_8 & \xleftarrow[\mathcal{S}_7]{\partial_x} & \mathcal{M}_7 & \xleftarrow[\mathcal{S}_6]{\partial_x} & \mathcal{M}_6 & \xleftarrow[\mathcal{S}_5]{\partial_x} & \mathcal{M}_5 & \xleftarrow[\mathcal{S}_4]{\partial_x} & \mathcal{M}_4 & \xleftarrow[\mathcal{S}_3]{\partial_x} & \mathcal{M}_3 & \xleftarrow[\mathcal{S}_2]{\partial_x} & \mathcal{M}_2 & \xleftarrow[\mathcal{S}_1]{\partial_x} & \mathcal{M}_1 & \xleftarrow[\mathcal{S}_0]{\partial_x} & \mathcal{M}_0 \end{array}$$

$$\mathcal{S}_0 \partial_x \left(\mathcal{Z}_{21}^{(\ell)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+\ell)} \\ 2^{(8-\ell-m)} \end{array} \right) \right) = \binom{\ell+m}{m} \mathcal{Z}_{21}^{(\ell+m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+\ell+m)} \\ 2^{(8-\ell-m)} \end{array} \right), \text{ and}$$

$$\begin{aligned} \partial_x \mathcal{S}_1 \left(\mathcal{Z}_{21}^{(\ell)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+\ell)} \\ 2^{(8-\ell-m)} \end{array} \right) \right) &= - \binom{\ell+m}{m} \mathcal{Z}_{21}^{(\ell+m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+\ell+m)} \\ 2^{(8-\ell-m)} \end{array} \right) \\ &+ \mathcal{Z}_{21}^{(\ell)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+\ell)} \\ 2^{(8-\ell-m)} \end{array} \right). \text{ So } \mathcal{S}_0 \partial_x + \partial_x \mathcal{S}_1 = \text{id}_{\mathcal{M}_1} \end{aligned}$$

$$\begin{aligned} \mathcal{S}_1 \partial_x \left(\mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|)} \\ 2^{(8-|\ell|-m)} \end{array} \right) \right) &= \\ &- \binom{|\ell|}{\ell_2} \mathcal{Z}_{21}^{(|\ell|)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|+m)} \\ 2^{(8-|\ell|-m)} \end{array} \right) \\ &+ \binom{\ell_2+m}{m} \mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2+m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|+m)} \\ 2^{(8-|\ell|-m)} \end{array} \right), \text{ and for } |\ell| = \ell_1 + \ell_2 \end{aligned}$$

$$\begin{aligned} \partial_x \mathcal{S}_2 \left(\mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|)} \\ 2^{(8-|\ell|-m)} \end{array} \right) \right) &= \binom{|\ell|}{\ell_2} \mathcal{Z}_{21}^{(|\ell|)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|+m)} \\ 2^{(8-|\ell|-m)} \end{array} \right) - \\ &\binom{\ell_2+m}{m} \mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2+m)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|+m)} \\ 2^{(8-|\ell|-m)} \end{array} \right) + \\ &\mathcal{Z}_{21}^{(\ell_1)} x \mathcal{Z}_{21}^{(\ell_2)} x \left(\begin{array}{c} w \\ w' \end{array} \middle| \begin{array}{c} 1^{(9+|\ell|)} \\ 2^{(8-|\ell|-m)} \end{array} \right). \end{aligned}$$

Hence, $\mathcal{S}_1 \partial_x + \partial_x \mathcal{S}_2 = \text{id}_{\mathcal{M}_2}$.

By the same way, we can gain that, $\mathcal{S}_{i-1} \partial_x + \partial_x \mathcal{S}_i = \text{id}_{\mathcal{M}_i}$, $i=3, 4, \dots, 7$.

Thus $\{\mathcal{S}_i\}; i=0, 1, 2, \dots, 7$ is a contracting homotopy [3], which implies the exactness of the following complex

$$0 \rightarrow \mathcal{M}_8 \rightarrow \mathcal{M}_7 \rightarrow \mathcal{M}_6 \rightarrow \mathcal{M}_5 \rightarrow \mathcal{M}_4 \rightarrow \mathcal{M}_3 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_0$$

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