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Outcome for the sequence of the terms in the state (9,6)

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Abstract

$\mathcal{R}$ is a commutative ring has 1, $\mathcal{F}$ is a free R-module and $\mathcal{D}_i\ \mathcal{F}$ be the divided power algebra of degree i. For the two-rowed Weyl module (W.M.) the contracting homotopy and the exactness of the series of the elements of Weyl resolution (W.R.) of characteristic-free resolution for the state (9,6) are discuss in this work.

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1. Introduction

The Weyl map is $d'_{\lambda/\mu}: \mathcal{DF} \rightarrow \wedge\mathcal{F}$ (exterior algebra of finitely generated free $\mathcal{R}$-module $\mathcal{F}$) and for the partition λ/μ, $\mathcal{K}_{\lambda/\mu}\mathcal{F} = \mathrm{Im}(d'_{\lambda/\mu})$, [1, 2]

The contracting homotopy $\{\mathcal{S}_\mathrm{i}\}$ illustrate in [3]. The partitions (3,3,2), (8,7,3), (6,6,4;0,0) and (7,7,4;0,0) studied respectively in [4-7]. We applied the same idea in [8] for the partition (9,6). Also we can apply that for 2-visible submodules and modules, $\mathcal{PSL}(2,\mathfrak{U})$, where $\mathfrak{U}$=23, 29, 31, 37, [9-11], $\mathcal{SL}(2,\mathfrak{U})$, $\mathfrak{U}$=23, 29, 31, 37, 121, 169,[12-14], $\mathcal{S}.\mathcal{U}.\mathcal{T}.(2,\wp)$, for a prime $\wp \leq 9$ and rad-quasi-prime submodules which are study by authors in [15-20].

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2. The exactness of the series of the elements of W.R.

The elements of W.R. for the state (9,6) are

$$\mathcal{M}_0 = \mathcal{D}_9 \otimes \mathcal{D}_6 \text{ , } \mathcal{M}_1 = \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b}; \text{ where } 1 \leq b \leq 6,$$

$$\mathcal{M}_2 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|}; \text{ where } 2 \leq |b| = b_1 + b_2 \leq 6$$

$$\mathcal{M}_3 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|}; \text{ where } 3 \leq |b| = \textstyle\sum_{i=1}^{3} b_i \leq 6$$

$$\mathcal{M}_4 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|}; \text{ where } 4 \leq |b| = \textstyle\sum_{i=1}^{4} b_i \leq 6$$

$$\mathcal{M}_5 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|}; \text{ where}$$

$$5 \leq |b| = \textstyle\sum_{i=1}^{5} b_i \leq 6, \mathcal{M}_6 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0$$

The homotopies $\{\mathcal{S}_i\}$; where i=0,1,2,…,5 are below with prove that are contracting

$$\mathcal{S}_0\left(\left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+h+m)} \; 2^{(k)} \\ 2^{(6-k-m)} \end{matrix}\right)\right) = \begin{cases} 0 \, ; \text{ if } k \leq 0 \\ \mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k)} \\ 2^{(6-k)} \end{matrix}\right) \, ; \text{ if } k = 1,2,3,4,5,6 \end{cases}$$

$$\mathcal{S}_1\left(\mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+h+m)} \; 2^{(m)} \\ 2^{(6-k-m)} \end{matrix}\right)\right)$$

$$= \begin{cases} 0 \, ; \text{ if } m = 0 \\ \mathcal{Z}_{21}^{(k)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k+m)} \\ 2^{(6-k-m)} \end{matrix}\right) \, ; \text{ if } m = 1,2,3,4,5 \end{cases}$$

For $|k| = k_1 + k_2$, $\mathcal{S}_2\left(\mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \; 2^{(m)} \\ 2^{(6-|k|-m)} \end{matrix}\right)\right)$

$$= \begin{cases} 0 \, ; \text{ if } m = 0 \\ \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \\ 2^{(6-|k|-m)} \end{matrix}\right) \, ; \text{ if } m = 1,2,3,4 \end{cases};$$

For $|k| = k_1 + k_2 + k_3$, $\mathcal{S}_3\left(\mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \; 2^{(m)} \\ 2^{(6-|k|-m)} \end{matrix}\right)\right)$

$$= \begin{cases} 0 \, ; \text{ if } m = 0 \\ \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \\ 2^{(6-|k|-m)} \end{matrix}\right) \, ; \text{ if } m = 1,2,3 \end{cases};$$

and so as $\mathcal{S}_4$, $\mathcal{S}_5$. So we have the following diagram

$$\mathcal{M}_6 \underset{\mathcal{S}_5}{\overset{\partial_x}{\rightleftarrows}} \mathcal{M}_5 \underset{\mathcal{S}_4}{\overset{\partial_x}{\rightleftarrows}} \mathcal{M}_4 \underset{\mathcal{S}_3}{\overset{\partial_x}{\rightleftarrows}} \mathcal{M}_3 \underset{\mathcal{S}_2}{\overset{\partial_x}{\rightleftarrows}} \mathcal{M}_2 \underset{\mathcal{S}_1}{\overset{\partial_x}{\rightleftarrows}} \mathcal{M}_1 \underset{\mathcal{S}_0}{\overset{\partial_x}{\rightleftarrows}} \mathcal{M}_0$$

Now we have

$$\mathcal{S}_0 \partial_x \left(\mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k)} & 2^{(m)} \\ 2^{(6-k-m)} & \end{matrix}\right)\right) = \binom{k+m}{m} \mathcal{Z}_{21}^{(k+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k+m)} \\ 2^{(6-k-m)} \end{matrix}\right),$$

and

$$\partial_x \mathcal{S}_1\left(\mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k)} \; 2^{(m)} \\ 2^{(6-k-m)} \end{matrix}\right)\right) = -\binom{k+m}{m} \mathcal{Z}_{21}^{(k+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k+m)} \\ 2^{(6-k-m)} \end{matrix}\right) +$$

$$\mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+k)} \; 2^{(m)} \\ 2^{(6-k-m)} \end{matrix}\right). \text{ So } \mathcal{S}_0 \partial_x + \partial_x \mathcal{S}_1 = \mathrm{id}_{\mathcal{M}_1}.$$

$$\mathcal{S}_1 \partial_x \left(\mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|)} \; 2^{(m)} \\ 2^{(6-|k|-m)} \end{matrix}\right)\right)$$

$$= -\binom{|k|}{k_2} \mathcal{Z}_{21}^{(|k|)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \\ 2^{(6-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+m}{m} \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \\ 2^{(6-|k|-m)} \end{matrix}\right), \text{ and}$$

$$\text{For } |k| = k_1 + k_2,\ \partial_x \mathcal{S}_2 \left(\mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|)} \; 2^{(m)} \\ 2^{(6-|k|-m)} \end{matrix}\right)\right) =$$

$$= \binom{|k|}{k_2} \mathcal{Z}_{21}^{(|k|)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \\ 2^{(6-|k|-m)} \end{matrix}\right)$$

$$- \binom{k_2+m}{m} \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|+m)} \\ 2^{(6-|k|-m)} \end{matrix}\right) +$$

$$\mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(9+|k|)} \; 2^{(m)} \\ 2^{(6-|k|-m)} \end{matrix}\right). \text{ So } \mathcal{S}_1 \partial_x + \partial_x \mathcal{S}_2 = \mathrm{id}_{\mathcal{M}_2}.$$

By that same way, we can get that, $\mathcal{S}_{i-1}\partial_x + \partial_x \mathcal{S}_i = \mathrm{id}_{\mathcal{M}_i}$, i=3, 4, 5.

Thus $\{\mathcal{S}_i\}$; i=0,1,2, …, 5 is a contracting homotopy [3], which implies the exactness of the following complex

$$0 \to \mathcal{M}_6 \to \mathcal{M}_5 \to \mathcal{M}_4 \to \mathcal{M}_3 \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0$$

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