

On some properties of separation axioms in topological gama-rings

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Abstract

In this article we present separation axioms T_2 -space, $T_{2\frac{1}{2}}$ -space, T_3 -space, completely regular, $T_{3\frac{1}{2}}$ -space, T_4 -space, completely normal and T_5 -space in topological Γ ring and study the characteristic of them also the relation among them.

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1. Introduction

The topological Γ ring is the latest topics and is considered one of the complex topics, as it links the topic of topology and the topic of rings in addition to the topic of gamma. Through these three topics, algebraic topics are studied and topological topics are sometimes studied, as is the case in this research. One of the most crucial ideas in commutative topological algebra is the idea of separation axioms in subject topological Γ ring. This article presents and examines the ideas of T_2 -space, $T_{2\frac{1}{2}}$ -space, T_3 -space, $T_{3\frac{1}{2}}$ -space, T_4 -space and T_5 -

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space in topological Γ ring where $(\mathcal{G}, \tau_{\mathcal{G}})$ is topological space see[7] and $(\mathcal{G}, +, \cdot)$ is Γ ring see [1-6, 8-11].

2. T_0 - T_1 - T_2 -Spaces in Topological Gama Ring

Definition 2.1: A topological Γ ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is referred to be T_0 -space if and only if there is an open set $\mathcal{K}^*$ such that $s \in \mathcal{K}^*$ and $m^* \notin \mathcal{K}^*$ for any $s, m^* \in \mathcal{G}$ such that $s \neq m^*$.

Example 2.2: A co-finite $\mathbb{Z}$ -ring $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}})$ is T_0 -space, $\tau_{\mathbb{R}} = \{U \subset \mathbb{R} : \mathbb{R} - U \text{ is finite}\} \cup \{\mathbb{R}, \emptyset\}$

Example 2.3: The $\mathbb{Z}_7$ -ring $(\mathbb{Z}_{21}, +_{21}, \cdot_{21}, \tau_{\mathbb{Z}_{21}})$ on discrete topological space is T_0 -space.

Lemma 2.4: In topological Γ ring the T_0 is hereditary property

Proof: Suppose $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_0 -topological Γ ring and H^* is Γ subring of $\mathcal{G}$. If H^* is Γ subring of $\mathcal{G}$ and $g, g^* \in H^*$, then $g, g^* \in \mathcal{G}$. Since $\mathcal{G}$ is T_0 space, so that there is an open set $\mathcal{K}^*$ in $\mathcal{G}$ $\ni g \in \mathcal{K}^*$ and $g^* \notin \mathcal{K}^*$. Hence, $U^* = H^* \cap \mathcal{K}^* \in \mathcal{T}_A$, $x \in H^* \cap \mathcal{K}^*$ and $x \notin H^* \cap \mathcal{K}^*$. Therefore topological Γ -subring is T_0 -space.

Lemma 2.5: In the topological Γ ring the T_0 -space is topological property.

Definition 2.6: A topological Γ ring $\mathcal{G}$ is referred to as T_1 -space if and only if for any $x^*, y^* \in \mathcal{G}$ where as $x^* \neq y^*$, following that, there are two open sets, D and W , $x^* \in D$, $y^* \notin D$ and $x^* \notin W$, $y^* \in W$.

Example 2.7: A discrete topological $\mathbb{Z}_{10}$ -ring $(\mathbb{Z}_{30}, +_{30}, \cdot_{30}, \tau_{\mathbb{Z}_{30}})$ is T_1 -space.

Lemma 2.8: In topological Γ ring the T_1 -space is hereditary property.

Lemma 2.9: In topological Γ ring the T_1 -space is topological property.

Remark 2.10: In topological Γ ring every T_1 -space is T_0 -space

Lemma 2.11: In topological Γ ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_1 iff every singleton is closed.

Definition 2.12: A topological Γ ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is referred to as T_2 -space iff for any $x, y \in \mathcal{G}$ which as $x \neq y$, so that, there are two open sets, D and W , $D \cap W = \emptyset$, such that $x \in D$, $y \notin D$ and $x \notin W$, $y \in W$.

Example 2.13: A usual topological $\mathbb{Z}$ -ring $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}})$ is T_2 space.

Example 2.14: A topological $\mathbb{Z}$ -ring $(\mathbb{Z}_2, +_2, \cdot_2, \tau_{\mathbb{Z}_2})$, $\tau_{\mathbb{Z}_2} = \{\emptyset, \mathbb{Z}_2, \{0\}, \{1\}\}$ is T_2 -space

Lemma 2.15: In topological Γ -ring the T_2 -space is hereditary property.

Proof: Suppose that $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ be T_2 -topological Γ ring and A is Γ subring of $\mathcal{G}$. If $x, y \in A$, and A is Γ subring of $\mathcal{G}$ then $x, y \in \mathcal{G}$. Since $\mathcal{G}$ is T_2 -space, so that, there are two open sets T and W , $T \cap W = \emptyset$ in $\mathcal{G}$ such that $x \in T$, $y \notin T$ and $x \notin W$, $y \in W$. Following that $x \in T \cap A$, $y \notin T \cap A$ also $y \in W \cap A$, $x \notin W \cap A$, where $T \cap A = D^*$, $W \cap A = W^* \in \tau_A$, since $T \cap W = \emptyset$, therefore $D^* \cap W^* = T \cap A \cap W \cap A = \emptyset$. Thus the topological Γ -subring A is T_2 -space.

Lemma 2.16: In topological Γ ring the T_2 -space is topological property.

Remark 2.17: In topological Γ ring every T_2 -space is T_1 -space.

3. T_3 -Space in Topological Gama Ring

Definition 3.1: A topological Γ ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is referred to as regular iff for any closed set P , $x \in \mathcal{G}$ which as $x \notin P$, so that, there are two open sets K and L such that $P \subseteq K$, $x \in L$, $K \cap L = \emptyset$

Example 3.2: A $\mathbb{Z}$ -ring $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}})$ on discrete topological space is regular.

Example 3.3: $\mathbb{Z}_{10}$ -ring $(\mathbb{Z}_{30}, +_{30}, \cdot_{30}, \tau_{\mathbb{Z}_{30}})$ on co-finite topological space is regular topological $\mathbb{Z}_{10}$ -ring.

Definition 3.4: A topological Γ ring $((\mathcal{G}, +, \cdot, \tau_{\mathcal{G}}))$ is referred to as T_3 -space if and only if $\mathcal{G}$ is T_1 -space and regular space

Example 3.5: A discrete $\mathbb{Z}_{10}$ -ring $(\mathbb{Z}_{50}, +_{50}, \cdot_{50}, \tau_{\mathbb{Z}_{50}})$ is T_3 -topological $\mathbb{Z}_{10}$ ring.

Theorem 3.6: In topological Γ ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ every T_3 -space is T_2 -space.

Proof : Let $x \neq y \in \mathcal{G}$, since $\mathcal{G}$ is T_1 -space therefore $\{y\}$ is closed by Lemma 2.11 and $x \notin \{y\}$. As $\mathcal{G}$ is a regular space, then $P, L \in \tau_{\mathcal{G}}$, such that $x \in P$, $\{y\} \subseteq L$, $K \cap L = \emptyset$. That is $\mathcal{G}$ is T_2 -space.

Corollary 3.7: In topological Γ ring $\mathcal{G}$ every T_3 -space is T_1 -space.

Corollary 3.8: In topological Γ ring $\mathcal{G}$ every T_3 -space is T_0 -space.

Theorem 3.9: *In topological Fring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ the T_3 -space is hereditary property*

Proof: Immediate.

Theorem 3.10: *In the topological Fring the T_3 -space is topological property.*

Lemma 3.11: *Every T_2 -space is T_3 -space.*

4. T_4 -Topological Gama Ring

Definition 4.1: Let $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ be topological Fring then $\mathcal{G}$ is called normal iff, for any pair closed sets F_1 and F_2 , $F_1 \cap F_2 = \emptyset$, there are corresponding open sets K, L of $\mathcal{G}$, $K \cap L = \emptyset$, which as $F_1 \subseteq K$ and $F_2 \subseteq L$.

Example 4.2: The indiscrete topological Z-ring R is normal.

Example 4.3: A discrete topological Z-ring $(Z_7, +_7, \cdot_7, \tau_{Z_7})$, is normal space.

Definition 4.4: A topological Fring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is called T_4 -space if and only if it's normal and T_1 -space.

Example 4.5: A topological Z-ring $(Z_5, +_5, \cdot_5, \tau_{Z_5})$ where $\tau_{Z_5} = \{Z_5, \emptyset, \{0\}, \{2\}, \{3\}, \{0,2\}, \{0,3\}, \{2,3\}, \{0,2,3\}\}$ is T_4 -topological Z-ring.

Lemma 4.6: *In topological Fring every T_4 -space is T_3 -space.*

Proof: Assume that a topological Fring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_4 -space. To prove $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_3 -space. Let F be closed in $\mathcal{G}$, $p \notin F$, $p \in \mathcal{G}$. Since $\{p\}$ is closed by Lemma 2.11 such that $F \cap \{p\} = \emptyset$ and $\mathcal{G}$ is T_4 -space, $\mathcal{G}$ has two disjoint open sets, K, L , and these are $F \subseteq K$, $\{p\} \subseteq L$, that is $F \subseteq K$, $p \in L$. Thus $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_3 -space.

Corollary 4.7: *In topological Fring every T_4 -space is T_2 -space.*

Corollary 4.8: *In topological Fring every T_4 -space is T_1 -space.*

Corollary 4.9: *In topological Fring every T_4 -space is T_0 -space.*

Lemma 4.10: *In topological Fring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ every T_3 -space is T_4 -space.*

Lemma 4.11: *In topological Fring the T_4 -space is hereditary property.*

Lemma 4.12: *In the topological Γ -ring the T_4 -space is topological property.*

5. Completely Normal and T_5 -topological Gama Ring

Definition 5.1: Let $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ be a topological Γ ring and $A^*, B^* \subseteq \mathcal{G}$ be two non-empty sets. so, we say that A^*, B^* are separated sets if $A^* \cap \overline{B^*} = \emptyset, \overline{A^*} \cap B^* = \emptyset$.

Example 5.2: The Q-ring R on discrete topology satisfy the separated.

Definition 5.3: A Γ topological ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is called completely normal space if and only if for each two separated $A^*, B^* \subseteq \mathcal{G}$, there exist $K, L \in \tau_{\mathcal{G}}$ s.t. $A^* \subseteq K, B^* \subseteq L$ and $K \cap L = \emptyset$.

Example 5.4: The discrete topological Z-ring R is completely normal.

Theorem 5.5: Every completely normal topological Γ ring is normal topological Γ -ring.

Theorem 5.6: The property of being completely normal topological Γ -ring is a hereditary property.

Definition 5.7: A topological Γ -ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is said to be a T_5 -space if:

1. $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is completely normal space.
2. $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_1 -space.

Example 5.8: A topological Q-ring R on discrete topology is T_5 -space.

Theorem 5.9: In topological Γ -ring every T_5 -space is a T_4 -space.

Proof: Straightforward.

Corollary 5.10: In topological Γ -ring every T_5 -space is T_3 -space.

Proof: By Theorem (5.9) and lemma (4.6)

Corollary 5.11: In topological Γ -ring every T_5 -space is T_2 -space.

Proof: By Theorem (5.9) and lemma (4.7)

Corollary 5.12: In topological Γ -ring every T_5 -space is T_1 -space.

Proof: By Theorem (5.9) and lemma (4.8)

Corollary 5.13: In topological Γ -ring every T_5 -space is T_0 -space.

Proof: By Theorem (5.9) and lemma (4.9)

Theorem 5.14: *Hereditary property in topological Γ -ring is being a T_5 -space.*

Proof: By Theorem (5.6) and Lemma (2.8).

6. $T_{\frac{1}{2}}$ Topological Gama Ring

Definition 6.1: Let be a topological Γ -ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$. There are two open sets K, L such that $\lambda \in K, \nu \in L$ and $\overline{K} \cap \overline{L} = \emptyset$. for any pair of singletons $\lambda, \nu \in \mathcal{G}$ such that $\lambda \neq \nu$, therefore, $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is $T_{\frac{1}{2}}$.

Example 6.2: A discrete topological $\mathcal{Q}$ -ring R is $T_{\frac{1}{2}}$.

Theorem 6.3: The property of being $T_{\frac{1}{2}}$ -space is a hereditary property

Theorem 6.4: *In topological Γ ring every $T_{\frac{1}{2}}$ -space is T_2 -space.*

Proof: Let $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is $T_{\frac{1}{2}}$ -space. Assume that $\alpha \neq \beta$ and $\alpha, \beta \in \mathcal{G}$, since $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is $T_{\frac{1}{2}}$ -space, then there are two open sets K^*, L^* exist such that $\alpha \in K^*, \beta \in L^*, \overline{K^*} \cap \overline{L^*} = \emptyset$. As $K^* \subseteq \overline{K^*}, L^* \subseteq \overline{L^*}$. Therefore $\alpha \in K^*, \beta \in L^*$ and $\overline{K^*} \cap \overline{L^*} = \emptyset$. Thus $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_2 space.

Corollary 6.5: *In topological Γ ring every $T_{\frac{1}{2}}$ -space is T_1 -space.*

Proof: By theorem 5.5 and by Theorem 6.4.

Corollary 6.6: *In topological Γ ring every $T_{\frac{1}{2}}$ -space is T_0 -space.*

Proof: By theorem 5.5 and by Corollary 6.5.

7. Completely Regular and $T_{\frac{3}{2}}$ Topological Gama Ring

Definition 7.1: For any element x in the topological Γ ring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$, if there are closed sets P such that $x \notin P$ and two open sets K and L , $x \in K$ and $P \subseteq L$, then the topological Γ ring is completely regular and $\overline{K} \cap \overline{L} = \emptyset$.

Example 7.2: A discrete topological $\mathcal{Z}$ -ring $(Q, +, \cdot, \tau_Q)$ is completely regular space.

Theorem 7.3: *In topological Fring the completely regular is hereditary property.*

Definition 7.4: A topological Fring $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is said to be a $T_{\frac{3}{2}}^1$ -space if:

1. $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is completely regular space.
2. $(\mathcal{G}, +, \cdot, \tau_{\mathcal{G}})$ is T_1 -space.

Example 7.5: A discret topological $\mathbb{Z}_3$ -ring $\mathbb{Z}_6$ is $T_{\frac{3}{2}}^1$ -space.

Theorem 7.6: *In topological Fring every $T_{\frac{3}{2}}^1$ -space is T_3 .*

Corollary 7.7: *In topological Fring every $T_{\frac{3}{2}}^1$ -space is T_2 .*

Corollary 7.8: *In topological Fring every $T_{\frac{3}{2}}^1$ -space is T_1 .*

Proof: Clear by using the Corollary 7.7.

Corollary 7.9: *In topological Fring every $T_{\frac{3}{2}}^1$ -space is T_0 .*

Proof: By Corollary 7.8.

Theorem 7.10: *In topological Fring the property of being $T_{\frac{3}{2}}^1$ -space is hereditary property*

Proof: By Theorem 7.3 and Lemma 2.8.

Conclusion

The present work delineates the characteristics of separation axioms in topological Γ -rings. It establishes novel characterizations. Moreover, there is a correlation between this separation axioms on topological Γ -rings.

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