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Investigating the metric and edge metric dimensions of H-naphthalenic nanotubes

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Abstract

If the distances between two vertices in a simple connected network are different, then a vertex x resolves the pair u and v. A set S of vertices in G is referred to as a resolving set if every pair of distinct vertices in G can be identified by at least one vertex in S. The metric dimension (MD) of G is the minimum number of vertices required for a resolving set. Moreover, an edge metric generator is any subset S of vertices that can distinguish between any two distinct edges, e_1 and e_2, according to their respective distances. An edge metric dimension (EMD), $\dim_e(G)$, is an edge metric generator of the least size. This study aims to explore the metric dimension (MD) and edge metric dimension (EMD) of the H-Naphthalenic Nanotube.

Subject Classification: *05C85, 05C90, 05C62, 05C12.*

Keywords: *Metric generator, MD, EMD, H-naphtalenic nanotube.*

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1. Introduction

A simple connected graph G consists of a set of vertices V and a set of edges E, indicated as (V, E). The distance $d(a_1, a_2)$ between any two vertices a_1 and a_2 in V is defined as the number of edges on the shortest path linking them. A vertex v in V can discriminate between a_1 and a_2 if its distances from these two vertices differ. If each pair of distinct vertices in G can be uniquely identified by at least one vertex in S, then S is a resolving set for G. The minimal size of such a resolving set, often called a metric basis, is the MD of G, represented as $\dim(G)$.

The smaller of the distances, $d(e, v)$, from v to a_1 and a_2, is the distance between a vertex v in V and an edge e in E, which connects vertices a_1 and a_2. A vertex x in V can distinguish between two edges e_1 and e_2 if the distances from x to these edges are different. A subset S of V is termed an edge resolving set for G if it distinguishes every pair of distinct edges in G. The EMD of G, denoted by $\dim_e(G)$, is the minimum size of such an edge resolving set.

The concept of the MD was first introduced by Slater in 1975, where he initially referred to it as "finding sets" aimed at uniquely determining the locations of intruders within networks. In the following year, Harary and Melter [1] independently proposed the idea of MD, describing these metric generators as resolving sets. They suggested that a subset of vertices could distinguish between not only different vertices but also distinct edges within a graph. To formalize this idea, Kelenc et al. introduced the EMD, a parameter that uses graph metrics to distinguish pairs of edges by assessing their separation based on a selected set of vertices.

MD has been applied across various scientific disciplines. For example, MD and drug discovery were associated by Chartrand et al. [3]. Khuller et al. [2] proposed utilizing MD for robot navigation within network structures rather than in Euclidean space. Chvatal's exploration of the MD of Hamming graphs has also provided insights into its application in complex digital games, such as mastermind games. Erdős and Lindström highlighted MD's utility in coin-weighing problems, and research into fault-tolerant MD has proven relevant for sensor networks [4-9].

The idea of resolving sets is expanded upon by EMD, which also promotes deeper investigation of MD across different graph types. Ali et al., for instance, investigated MD in Möbius networks and cycle-related graphs. While Liu et al. concentrated on MD in cocktail party, Toeplitz, and jellyfish graphs, Kuziak et al. examined the strong MD of graphs. Ahmad et al. investigated MD in a number of graphs connected to cycles. After studies on barycentric subdivisions of Cayley graphs by Koam and Ahmad, EMD became more and more popular in resolvability discussions. The convex polytope graph was investigated by Ahsan et al. and Zhang and Gao. Many graph families have been examined for Mixed MD, which satisfies both metric and EMD criteria. In these references [10, 20-26], authors computed mixed MDs in graphs with rotational symmetry, yielding exact values and reported results on mixed MDs for particular path-related graphs, and calculated mixed MDs for the generalized Petersen graph $P(n, 2)$.

Finding a graph's EMD is typically NP-hard. There is no uniform association between metric and EMDs.

2. Importance of Metric Dimension

The concept of the metric dimension (MD) plays a pivotal role in graph theory, offering profound insights across multiple scientific domains. As a measure that captures the uniqueness of vertex identification within a graph, MD has found extensive applications in various fields. Its utility extends beyond theoretical investigations in mathematics, influencing practical applications in computer science, engineering, and beyond. For instance, in network design and analysis, MD provides a critical tool for optimizing routing algorithms, enhancing security protocols, and improving fault tolerance. The significance of MD is further underscored by its applications in chemistry, where it aids in understanding molecular structures, and in physics, where it contributes to the study of complex systems. The interdisciplinary impact of MD is evident from the publication trends observed in recent years, with a significant proportion of research emanating from mathematics, followed by computer science and engineering. According to the Scopus database, the distribution of papers across different fields is as follows:

- **Mathematics:** 42.5%
- **Computer Science:** 25.3%
- **Engineering:** 8.4%
- **Physics:** 6.4%
- **Decision Sciences:** 3.1%
- **Material Science:** 2.3%

2.1 Applications in Real Life

2.1.1 Network Design and Optimization

Minimizing the MD is crucial in designing optimized routing algorithms for communication networks. By reducing the MD, networks can achieve faster data transmission, lower latency, and improved reliability. This understanding is particularly important for enhancing the performance of large-scale data centers, where efficient routing is essential [11].

2.1.2 Construction and Structural Engineering

In the construction industry, the EMD plays a significant role in designing resilient structures such as bridges and high-rise buildings. By identifying and reinforcing critical edges within the structural graph of a building, engineers can optimize material usage, reduce costs, and increase the structure's durability [12].

2.1.3 Computer Science and Robot Navigation

In computer science, especially in the realm of robot navigation, the MD is key to efficient pathfinding. A lower MD implies that fewer waypoints or

landmarks are necessary, which simplifies the navigation process and reduces computational demands. This is particularly beneficial in the development of autonomous systems [13].

2.1.4 Chemistry and Molecular Structures

The study of graph properties like MD and EMD is essential in the field of chemistry, particularly for understanding molecular structures. In the case of H-Naphthalenic Nanotubes, these dimensions offer insights into the chemical stability and reactivity of the material. Such knowledge is instrumental in designing new materials with specific properties, such as enhanced strength or conductivity [14-19].

In short, The investigation into the MD and EMD of H-Naphthalenic Nanotubes sheds light on both theoretical and practical aspects of graph theory. The findings from this research have broad implications across various domains, including network design, construction, computer science, and chemistry. By bridging abstract mathematical concepts with practical applications, this study contributes to potential advancements in technology, science, and industry.

3. H-Naphthalenic Nanotubes

The term “nano” denotes the metric prefix for one-billionth (10-9). Carbon nanotubes, or CNTs, are nanoscale carbon allotropes with a length-to-diameter ratio greater than one million. Arc discharge, laser ablation, and chemical vapor deposition (CVD) are a few of the techniques that have been developed for the large-scale manufacture of CNTs. Among these, CVD stands out as the most effective technique for producing substantial quantities of CNTs. Carbon, in its natural state, exists in two primary crystalline forms: diamond and graphite. Advances in production techniques have facilitated the generation of commercially available, large-scale CNTs, with CVD emerging as the most promising method for their synthesis [27-29].

Nanoscience has significantly impacted our daily lives through various innovations. For example, advancements include stain-resistant fabrics inspired by the nanoscale features of lotus leaves and computer hard drives that utilize magnetic strips with a thickness of only 20 nanometers. The remarkable electrical and thermal conductivity, high aspect ratio, and mechanical strength of carbon nanotubes provide a variety of unique features and uses. One notable application is the use of carbon nanotubes as probes in Atomic Force Microscopy (AFM), where they address numerous experimental and modeling challenges associated with localized measurement techniques.

Historically, considerable effort has been directed toward utilizing carbon nanotubes as nanoprobes. Despite extensive research, many questions remain about their optimal application. Carbon nanotubes are employed in various nanoelectromechanical systems, including nanoscale mechanical memory and electrical motors. Since their discovery by Iijima in 1991, interest in carbon nanotubes has continued to grow, highlighting their potential and ongoing relevance in advanced technological applications.

H-naphthalenic nanotubes are a novel class of nanostructures that have garnered significant interest due to their unique geometric and electronic properties. These nanotubes are derived from the naphthalene molecule, which consists of two fused benzene rings.The 'H' in H-naphthalenic nanotubes is named to indicate difference in disposition of the naphthalene units in the tubular structure and gives H-nanotubes different properties from other types of nanotubes.

The arrangement of H-naphthalenic nanotubes also known as 'H-naphthalenic nanotubes structure' that has effects on some physical properties. These nanotubes have diverse interesting characteristics, for example, electronic band structure and mechanical stability, therefore, could be considered as prospective nanotubes for using in nanotechnology and material science (see Figure 1).

Theorem 1: *Let $G_{q,l}$ represent the graph of an $q \times l$ two-dimensional H-naphthalenic nanotube. The MD of this graph is 2.*

Proof: In this context, q and l denote the number of rows and columns, respectively. Consider the set of vertices:

$$p_{\xi(1,1)}, p_{\xi(1,2)}, p_{\xi(1,3)}, \cdots, p_{\xi(1,l)}, p_{\xi(2,1)}, p_{\xi(2,2)}, p_{\xi(2,3)}, \cdots, p_{\xi(2,l)}, \cdots, p_{\xi(q,1)},$$
$$p_{\xi(q,2)}, p_{\xi(q,3)}, \cdots, p_{\xi(q,l)}\}$$

Let $T = \{p_{\xi(2,2)}, p_{\xi(2,5n-1)}\}$ be a proposed resolving set. We demonstrate that T indeed serves as a resolving set. Here, let $b = q + l$, $d_\eta = 2q$ and $k_\xi = 5n + q - l$.

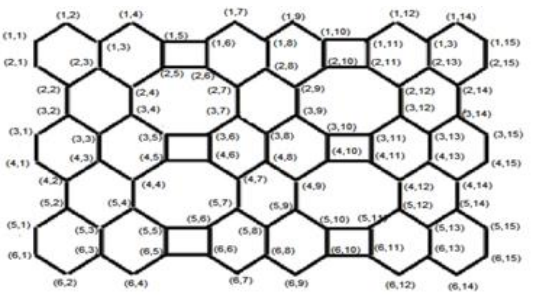

Figure 1
H-Naphthalenic nanosheet for n=3.

Case 1: n is odd

First Row, If q and l are both odd:

$$r(p_{\xi(q,l)}|T) = \begin{cases} \left(b, k_\xi - 1\right) \text{for } l = 1 \\ \left(\mathrm{b} - 2, \mathrm{k}_\xi - 1\right) \text{for all} \\ \left(\mathrm{b} - 2, \mathrm{d}_\eta\right) \text{for the last term} \end{cases}$$

- If q is odd and l is even $r(p_{\xi(q,l)}|T) = \begin{cases} (b, k_\xi - 1) \text{ l} = 1 \\ (\text{b} - 2, \text{k}_\xi - 1) \text{ for all} \\ (\text{b} - 2, 3\text{q}) \text{ for last term} \end{cases}$

Second Row, If q is even and l is odd:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 3, k_\xi - 3) & \text{for} l = 1 \\ (b - 4, k_\xi - 3) & \text{for all } l \\ (b - 4, d_\eta - 3) & \text{for the last term} \end{cases}$$

- If q is even and l is even $r(p_{\xi(q,l)}|T) = \{(b - 4, k_\xi - 3)$ forall

Other Rows, - If q is odd and l is odd

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 3) & \text{for } q \geq l \\ (\text{b} - 4, \text{k}_\xi - 3) & \text{for } q < l \\ (b - 4, d_\eta - 3) & \text{for } b \geq 5n + 3 \end{cases}$$

If q is odd and l is even:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 5,\ k_\xi - 3) \text{ first two values} \\ (d_\eta - 3,\ k_\xi - 3) \text{ for } q \geq l \\ (\text{b} - 4,\ k_\xi - 3) \text{ for } q < l \\ (\text{b} - 4,\ d_\eta - 3) \text{ for } b \geq 5n + 2 \\ (\text{b} - 4,\ d_\eta - 5) \text{ for last two values when } b \geq 5n + 3 \end{cases}$$

Note: For $b \geq 5n + 2$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

If q is even and l is odd:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 5, k_\xi - 3), \text{first two values} \\ (d_\eta - 3, k_\xi - 3) \text{ for } q \geq l \\ (\text{b} - 4, k_\xi - 3) \text{ for } q < l \\ (\text{b} - 4,\ d_\eta - 3) \text{ for } b \geq 5n + 2 \\ (\text{b} - 4,\ d_\eta - 5) \text{ for last two values when } b \geq 5n + 2 \end{cases}$$

Note: For $b \geq 5n + 2$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

If q is even and l is even:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 2) \text{ for } q \geq l \\ (\text{b} - 4, k_\xi - 3) \text{ for } q < l \\ (\text{b} - 4, d_\eta - 4) \text{ for } b \geq 5n + 3 \end{cases}$$

Case 2: n is even

First Row, If q and l are both odd:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (b, k_\xi - 1) & \text{for } l = 1 \\ (b-2, k_\xi - 1) & \text{forall} \\ (b-2, d_\eta) & \text{for the last term} \end{cases} \quad (1)$$

If q is odd and l is even:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (b, k_\xi - 1) & \text{for } l = 1 \\ (b-2, k_\xi - 1) & \text{for all} \\ (b-2, d_\eta) & \text{for the last term} \end{cases} \quad (2)$$

Second Row, If q is even and l is odd:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (b-2, k_\xi - 3) & \text{for } l = 1 \\ (b-4, k_\xi - 3) & \text{for all} \end{cases} \quad (3)$$

If q is even and l is even:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (b-4, k_\xi - 3) & \text{for all} \\ (b-4, d_\eta - 3) & \text{for the last term} \end{cases} \quad (4)$$

Other Rows, If q is odd and l is odd

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 3) & \text{for } q \geq l \\ (b-4, k_\xi - 3) & \text{for } q < l \\ (b-4, d_\eta - 3) & \text{for } b \geq 5n+2 \\ (b-4, d_\eta - 5) & \text{for the last two terms } b \geq 5n+2 \end{cases} \quad (5)$$

Note: For $b \geq 5n + 2$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

If i is odd and m is even:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 5, k_\xi - 3) & \text{for } q \geq l, \text{first two values} \\ (d_\eta - 3, k_\xi - 3) & \text{for } q \geq l \\ (b-4, k_\xi - 3) & \text{for } q < l \\ (b-4, d_\eta - 4) & \text{for } b \geq 5n+2 \end{cases} \quad (6)$$

If q is even and l is odd:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 5, k_\xi - 3) & \text{for } q \geq l, \text{first three values} \\ (d_\eta - 3, k_\xi - 3) & \text{for } q \geq l \\ (b-4, k_\xi - 3) & \text{for } q < l \\ (b-4, d_\eta - 4) & \text{for } b \geq 5n+3 \end{cases} \quad (7)$$

If q is even and l is even:

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 3) & \text{for } q \geq l \\ (b - 4, k_\xi - 3) & \text{for } q < l \\ (b - 4, d_\eta - 3) & \text{for } b \geq 5n + 1 \\ (b - 4, d_\eta - 5) & \text{for the last three terms } b \geq 5n + 1 \end{cases} \quad (8)$$

Note: For $b \geq 5n + 1$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

Hence, the MD of this graph is 2, which means that only two vertices are needed to uniquely determine the positions of all other vertices in the graph. This result is significant because it simplifies the complexity of navigation and identification within the graph structure, which is important in various applications such as network design, molecular chemistry, and data structures. The minimal resolving set, as demonstrated, effectively provides the necessary information to distinguish all vertices, confirming that the MD of the H-naphthalenic nanotube is indeed 2.

Theorem 2: *Let $G_{q,l}$ represent the graph of an $q \times l$ two-dimensional H-naphthalenic nanotube. The EMD of this graph is 2.*

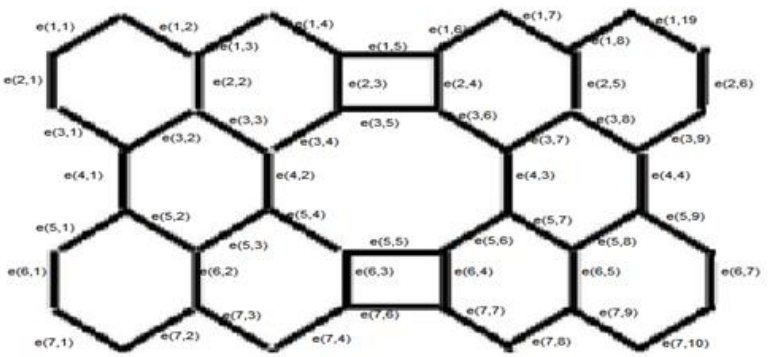

Figure 2
H-Naphthalenic nanosheet for n=3.

Proof: In this context, q and l denote the number of rows and columns, respectively. Consider the set of edges (see Figure 2):

$$\{e(1,1), e(1,2), e(1,3), \dots, e(1,l), e(2,1), e(2,2), e(2,3), \dots, e(2,l), \dots, e(q,1), e(q,2), e(q,3), \dots, e(q,l)\}$$

Let $T = \{e_{(3,2)}, e_{(2,5n-1)}\}$ be a proposed resolving set. We demonstrate that T indeed serves as a resolving set. Here, let $b = i + m$, $d_\eta = 2q$ and $k_\xi = 5n + q - l$.

Case 1: n is odd

First Row, If q and l are both odd

$$r(p_{\xi(q,l)}|T) = \begin{cases} (b, k_\xi - 1) \text{ for } l = 1 \\ (\mathrm{b} - 2, \mathrm{k}_\xi - 1) \text{ for all} \\ (\mathrm{b} - 2, \mathrm{d}_\eta) \text{ for the last term} \end{cases}$$

- If q is odd and l is even $r(p_{\xi(q,l)}|T) = \begin{cases} (b, k_\xi - 1) \, \mathrm{l} = 1 \\ (\mathrm{b} - 2, \mathrm{k}_\xi - 1) \text{ for all} \\ (\mathrm{b} - 2, 3\mathrm{q}) \text{ for last term} \end{cases}$

Second Row, If q is even and l is odd

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 3, k_\xi - 3) & \text{for} l = 1 \\ (b - 4, k_\xi - 3) & \text{for all } l \\ (b - 4, d_\eta - 3) & \text{for the last term} \end{cases}$$

- If q is even and l is even $r(p_{\xi(q,l)}|T) = \{(b - 4, k_\xi - 3) \quad \text{for all}$

Other Rows, If q is odd and l is odd

$$r(e(q,l)|T) = \begin{cases} (d_\eta - 4, k_\xi - 3) \text{ for } q \geq l \\ (\mathrm{b} - 4, k_\xi - 3) \text{ for } q < l \\ (\mathrm{b} - 4, d_\eta - 4) \text{ for } b \geq 5n + 3 \end{cases}$$

If q is odd and l is even

$$r(e(q,l)|T) = \begin{cases} (d_\eta - 5, k_\xi - 3) \text{ for } q \geq l \text{ first two values} \\ (d_\eta - 3, k_\xi - 3) \text{ for } q \geq l \\ (\mathrm{b} - 4, k_\xi - 3) \text{ for } q < l \\ (\mathrm{b} - 4, d_\eta - 3) \text{ for } b \geq 5n + 2 \\ (\mathrm{b} - 4, d_\eta - 5) \text{ for last two values when } b \geq 5n + 3 \end{cases}$$

Note: For $b \geq 5n + 2$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

If q is even and l is odd

$$r(e(q,l)|T) = \begin{cases} (d_\eta - 5, k_\xi - 3) \text{ first two values} \\ (2i - 3, k_\xi - 3) \, for \, q \geq l \\ (b - 4, k_\xi - 3) \, for \, q < l \\ (b - 4, d_\eta - 3) \, for \, b \geq 5n + 2 \\ (b - 4, d_\eta - 5) \, for \, last \, two \, values \, when \, b \geq 5n + 2 \end{cases}$$

Note: For $b \geq 5n + 2$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

If q is even and l is even,

$$r(p_{\xi(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 2) \text{ for } q \geq l \\ (\mathrm{b} - 4, k_\xi - 3) \text{ for } q < l \\ (\mathrm{b} - 4, d_\eta - 4) \text{ for } b \geq 5n + 3 \end{cases}$$

Case 2: n is even

First Row, If q and l are both odd

$$r(e_{(q,l)}|T) = \begin{cases} (b-2, k_\xi - 1) & \text{for all} \\ (b-2, 2i) & \text{for the last term} \end{cases} \quad (9)$$

If q is odd and l is even

$$r(e_{(q,l)}|T) = \begin{cases} (b-1, k_\xi - 1) & \text{for } l = 1 \\ (b-2, k_\xi - 1) & \text{for all} \\ (b-2, q) & \text{for the last term} \end{cases} \quad (10)$$

Second Row, If q is even and l is odd

$$r(e_{(q,l)}|T) = \begin{cases} (b-3, k_\xi - 3) & \text{for } l = 1 \\ (b-4, k_\xi - 3) & \text{for all} \end{cases} \quad (11)$$

If q is even and l is even

$$r(e_{(q,l)}|T) = \begin{cases} (b-4, k_\xi - 3) & \text{for all} \\ (b-4, d_\eta - 4) & \text{for the last term} \end{cases} \quad (12)$$

Other Rows, If q is odd and l is odd

$$r(e_{(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 3) & \text{for } q \geq l \\ (b-4, k_\xi - 3) & \text{for } q < l \\ (b-4, d_\eta - 3) & \text{for } b \geq 5n + 2 \\ (b-4, d_\eta - 5) & \text{for the last two terms when } b \geq 5n + 2 \end{cases} \quad (13)$$

Note: For $b \geq 5n + 2$, apply the conditions of the final step first to those values that are also subject to the penultimate step. This ensures that the criteria for the last step are satisfied before addressing the second-to-last condition.

If q is odd and l is even

$$r(e_{(q,l)}|T) = \begin{cases} (d_\eta - 5, k_\xi - 3) & \text{for } q \geq m, \text{first two values} \\ (2i - 3, k_\xi - 3) & \text{for } i \geq m \\ (b-4, k_\xi - 3) & \text{for } q < l \\ (b-4, d_\eta - 4) & \text{for } b \geq 5n + 2 \end{cases} \quad (14)$$

If q is even and l is odd

$$r(e_{(q,l)}|T) = \begin{cases} (d_\eta - 5, k_\xi - 3) & \text{for the first three values when } q \geq l \\ (d_\eta - 3, k_\xi - 3) & \text{for } q \geq l \\ (b - 4, k_\xi - 3) & \text{for } q < l \\ (b - 4, d_\eta - 4) & \text{for } b \geq 5n + 3 \end{cases} \quad (15)$$

If q is even and l is even

$$r(e_{(q,l)}|T) = \begin{cases} (d_\eta - 4, k_\xi - 3) & \text{for } q \geq l \\ (b - 4, k_\xi - 3) & \text{for } q < l \\ (b - 4, d_\eta - 3) & \text{for } b \geq 5n + 1 \\ (b - 4, d_\eta - 5) & \text{for the last three terms when } b \geq 5n + 1 \end{cases} \quad (16)$$

The EMD of this graph is 2, a fact that means that two edges are enough to place all the other edges in this graph. This finding is important since it goes a long way in simplifying on the challenges of identification and or navigation within the confines of the graph framework that is very useful in fields such as networks, chemical structures of molecules, and data structures among others. The minimal resolving set actually provides just the information necessary to distinguish all edges and confirms that indeed, the EMD of the H-naphthalenic nanotube must be 2.

4. Importance of Results

Hence the results of this study on the MD and edge EMD of H-Naphthalenic Nanotubes are significant for theoretical and experimental astronomy and physics. Through description of the MD and EMD of these nanotubes, this work offers profound information about the structural features of these nanotubes that is extremely important for the envisaged roles they play as building blocks in nanotechnology and materials science. Specific features of the vertices and edges of graph of these nanotubes can also be identified more effectively, allowing for better physical parameters modelling of nanotubes such as the electrical conductivity and chemical activity. These models can result into improvements in the efficiency of the electronic equipment, useful catalytic activities and better drug delivery systems anchored on these nanostructures.

Moreover, this study of MD and EMD within the H-Naphthalenic Nanotubes is potential enrichment into the discipline of graph theory as practical applications of those mentioned notions for various nanomaterials are presented. The results of this study may lead to more studies on the metric and edge MDs other nanostructures, paving knowledge to geometric and topological nature. Therefore, the given work gives new development of an important branch of knowledge, namely MD and EMD while stressing the importance of their application for enhancing the efficiency of nanotubes as well as their production and utilization across the spectrum of technologies and industries.

5. Conclusion

We have carefully investigated the H-Naphthalenic Nanotube's metric dimension (MD) and equivalent mass (EMD) in this work. According to our investigation, the H-Naphthalenic Nanotube's MD always stays at 2, regardless of the particular structural changes. This MD consistency is especially interesting since it highlights the nanotube's intrinsic resolution with a small number of vertices, which may have important consequences for its implementation in a variety of material sciences and nanotechnology applications. Future studies into the metric and edge metric features of other complex nanostructures will benefit greatly from the methods and strategies used in this work. Subsequent research endeavors may expand upon these discoveries by examining diverse nanotube topologies or by examining their pragmatic uses in domains like sensor networks and nanoelectronics.

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