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Exploring metric dimensions in chemical structures : Insights and applications

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Abstract

In this article, we dive into the metric dimension of various lattice networks, focusing on Bakelite, Backbone DNA, and Polythiophene networks. The metric dimension is a crucial graph invariant that helps us understand how uniquely we can identify the vertices in a network. Our detailed analysis and calculations reveal that the metric dimension for Bakelite, Polythiophene, and Backbone DNA networks is consistently two. This means that, within these lattice structures, a simple pair of vertices is enough to pinpoint the location of all other vertices. These insights shed light on the structural properties of these molecular networks and could have practical implications for areas like biological systems and organic electronics. Plus, this study sets the stage for future research in graph theory and the understanding of molecular structures.

Subject Classification: *05C12, 05C76, 05C90, 68R10.*

Keywords: *Metric dimension, Vertex identification, Polythiophene network, Backbone DNA network, Bakelite network, Graph theory, Lattice networks, Organic electronics.*

1. Introduction

One of the main problems in various disciplines is finding an agent's position within the network exactly. One way of going around this is by deploying sensors in various nodes of the network. The agents then use relative distances to some of these sensors to estimate their locations accurately. In a formal sense, if an agent has information regarding the topology of the network, can they establish their location from the sensors. On the other hand, what placement of the sensors guarantees that any agent, whether in a random starting region, can always predict its position. These questions have been researched much in combinatorics under different names such as MD, dominating sets, and identifying codes. In the recent decades, this area has attracted much attention.The study of resolving sets has seen substantial contributions from many researchers. For instance, M. Ali [1] calculated the MDs of two graph families with constant MDs. Maryum explored the MD of graph connections

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[2], while Shahida computed the MD of graphs associated with paths [3]. Further research has been conducted in this area by [4-9].

The idea of MD has emerged in the focus in graph theory, combinatorics, and computer science due relevance, interesting findings, and numerous open problems. However, recently there has been a trend towards a new form of MD, in which the key focus is made not on vertices of a graph, but on its edges. The approach is referred to as the edge metric dimension (EMD) because vertices are utilized to differentiate edges. It was first introduced in [10], and has attracted a lot of researchers' attention, encouraging them to enter this field and thus producing more papers.

To put it simply, the MD of a graph G, written as $dim(G)$, represents the smallest group of vertices S such that for every vertex ζ in G, there is a unique way to identify it using distances from the vertices in S. In practical terms, for any two vertices η and ξ in G, at least one vertex σ in S will have a different distance to η than it does to ξ. This is the essence of how MD works. It allows us to tell vertices apart based on how far away they are from a special set of points in the graph.

Over the years, many researchers have explored the MD of different types of graphs. For example, the MD of prism graphs was thoroughly studied in, revealing how the structure of these graphs influences their MD. Similarly, looked at the MD of the Möbius ladder, another graph with unique features. The MD of a more complex family of graphs, the generalized Petersen graph $P(m, 4)$, was calculated in [11].

Other well-known graph families have also been examined. For instance, worked on the MD of fan graphs, represented as $fn = K_1 + P$ for $n \geq 1$. Additionally, studied the MD of gear graphs, also known as Jahangir graphs J_{2n}, and discussed how these graphs are connected and partitioned. A key finding in [12] showed that path graphs always have an MD of 1, meaning that no matter how many vertices a path graph has, its MD remains constant. Similarly, cycle graphs with $n \geq 3n$ vertices always have an MD of 2, highlighting how their circular structure affects this measurement.

Further research has expanded this understanding to more types of graphs. In [12], for example, antiprism graphs An for n≥5n were identified as a family of regular graphs with an MD of 3. Meanwhile, [13] focused on the MD of trees, and explored unicyclic graphs. Lastly, [14] looked at the MD of middle and total graphs of paths, providing new insights into how MD behaves in these different graph structures.

Overall, this growing area of research reflects the depth and variety of the MD concept. Each study has contributed valuable insights, helping us better understand both the theoretical and practical aspects of MD in different kinds of graphs. As researchers continue to explore this field, it holds great promise for uncovering new findings and applications [15].

2. Applications and Importance of The metric dimension (MD)

It is an invariant of graphs, which proved to be essential and applicable in numerous theoretical and practical situations.

2.1 *Engineering and Industrial Applications*

MD is used in engineering in the sense of designing and planning various network structures. For instance in communication networks where routing or sensors are placed based on MD it results to efficient routing and reliability against faults. This decreases the expenditure of the network hence cutting down the cost of deploying the network and on the other improves on the reliability on the communication network. In industrial applications MD is used where a minimal number of sensors is placed so as to give unique information regarding a whole system, for instance a power grid or transport infrastructure [16-24].

2.2 *Mapping and Geographical Information Systems (GIS)*

MD is used throughout mapping and geographical information systems for geolocation and navigation. It is also applicable in placement of monitoring stations or emergency units in urban planning and in disaster management. The ability of assigning unique identification to the locations in terms of the distances is crucial in map making as well as the distribution of resources.

2.3 *Artificial Intelligence and Robotics*

In the field of AI and robotics specifically mobile robotics, MAIN can be widely used in environment modelling and localization. It is used in robots that are fitted with distance sensors; the robots can then calculate the map location accurately within the real world hence allow the carrying out of a task by itself. This is especially the case in all the circumstances whereby GPS signals cannot be accessed including indoors and mars' rovers.

2.4 *Medical Imaging and Diagnostics*

In medical imaging especially MRI and CT, image reconstruction is an area that is underpinned by the MD concepts. Since number to measurements in question is kept to the minimal while at the same time allowing for the identification of the specific anatomical features, MD is beneficial in the sense that it also lowers the scan time and patient discomfort.

2.5 *Data Analysis and Clustering*

It is also used in the analysis of data especially in the clustering and classification of data. It helps in finding out important characteristics in large databases and helps in the right classification of similar data. This application is most useful in areas like bioinformatics where sequence or structural analysis of genes or proteins involves the identification of the unique features in the data set.

2.6 *Cost Management in Networking*

In cost management especially in networking and telecommunication MD is a guideline for proper positioning of the network components so that the total cost of the network is kept as low as possible while the network connectivity is kept strong. This includes defining the number of nodes that is sufficient to cover the network adequately, which has a straight forward relationship with the cost of infrastructure.

2.7 *Artificial Intelligence and Machine Learning*

The MD is also used in machine learning algorithms in feature extraction and feature space reduction. It is used in defining the smallest number of features that can be used to differentiate between classes within a given data set to enhance the efficiency of classifiers.

In other words, these applications of MD in these different fields go a long way in establishing its significance as one of the basic concepts of graph theory. Its applicability to the identification of vertices within a network has potential applications across the design and management of networks, as well as artificial intelligence systems and medical diagnosis. As this area of research progresses, there is likely to be further development of new and creative uses of MD and its role in both theoretical and applied sciences will remain secure.

3. Polythiophene network ($\boldsymbol{PT(m)}$)

Polythiophene, revolutionized this field by offering materials that can shift from non-conductive to conductive states (For example see Figure 1.). This happens when polythiophene is oxidized, a process that makes it conductive, increasing its usefulness in various electronic applications.What makes polythiophene special is its ability to transition between conducting and semiconducting states, which makes it ideal for use in electrical and optical devices. Out of all π-conjugated polymers, polythiophene has been one of the most thoroughly studied, both in theory and through experimentation. Its wide range of practical applications includes biosensors, hydrogen storage, water purification systems, and light-emitting diodes (LEDs). The groundbreaking work of Alan J. Heeger, Alan MacDiarmid, and Hideki Shirakawa on polythiophenes and other similar conductive polymers earned them the Nobel Prize in Chemistry.

In addition to its electronic applications, polythiophenes (PTs) have been used in medical research, particularly in the treatment of prion disorders, and they have shown the ability to detect metal ions in various environments.

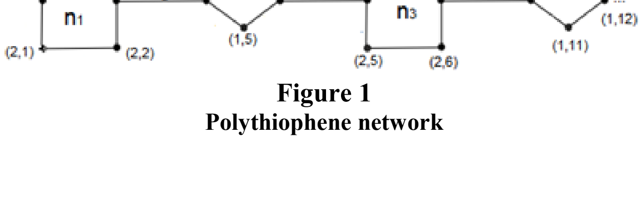

Figure 1
Polythiophene network

Theorem 1: *Let $G_{p,t(m)}$ represent the graph corresponding to a 2D lattice structure of a Polythiophene network. The MD (MD) of a $p \times l$ two-dimensional lattice Polythiophene network is 2.*

Proof: In this context, l and p represent the number of rows and vertices, respectively. Consider the set of vertices:

$$\{y_{\xi(1,1)}, y_{\xi(1,2)}, y_{\xi(1,3)}, \cdots, y_{\xi(1,t)}, y_{\xi(2,1)}, y_{\xi(2,2)}, y_{\xi(2,3)}, \cdots, y_{\xi(2,t)},$$
$$y_{\xi(3,1)}, y_{\xi(3,2)}, y_{\xi(3,3)}, \cdots, y_{\xi(3,t)}, \cdots, y_{\xi(p,1)}, y_{\xi(p,2)}, y_{\xi(p,3)}, \cdots, y_{\xi(p,t)}\}$$

Here, p represents the number of rows, while t indicates the number of edges." Define the set $T = \{y_{\xi(1,1)}, y_{\xi(1,3n+1)}\}$. We will demonstrate that T acts as a resolving set for the graph.

Let $a_{\xi(i)} = 2i$, $k_{\eta(n)} = 3n$, and $b_{\zeta} = p + t$. After generalizing, we consider two cases based on whether n is even or odd.

Case 1: $\boldsymbol{n}$ is even: For the first row, the representation of $r(y_{\xi(p,t)}|Z)$ is given by:

$$r(y\_(p,t) \mid Z) = (b\_ - 2, k\_(n) + 1 - m) \text{ for all } m\, Z$$

For the second row, the representation of $r(y_{\xi(p,t)}|Z)$ is:

$$r(y\_(p,t) \mid Z) = (m, k\_(n) + 1 - m) \text{ for all } m\, Z$$

Case 2: $\boldsymbol{n}$ is odd: For the first row, the representation of $r(y_{\xi(p,t)}|Z)$ is:

$$r(y\_(p,t) \mid Z) = (b\_ - 2, k\_(n) + i + 1 - m) \text{ for all } m\, Z$$

For the second row, the representation of $r(y_{\xi(p,t)}|Z)$ is:

$$r(y\_(p,t) \mid Z) = (m, k\_(n) + 1 - m) \text{ for all } m \text{ Z}$$

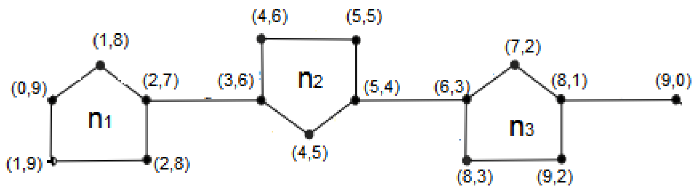

Figure 2
MD of Polythiophene network

Thus, we conclude that the MD of the Polythiophene network *PT(m)* is 2 (For example see Figure 2.).

The MD of a graph is a quantitative characteristic of the distinguishability of the vertices of the graph using distances from a selected subset of vertices, called the distance set. In this proof, the author has demonstrated that for the given $2D$ lattice structure of a Polythiophene network described by the graph $G(p, t(m))$, the position of any vertex in the graph is uniquely determined from the position of just two vertices. This implies that the best possible size of such a resolving set is the minimum number of vertices needed to define it as an MD of the graph and is equal to 2. This result shows that the metric properties of the Polythiophene lattice structure are inherently regular and simple.

4. Backbone DNA network *BBDNA(m)*

This section provides an overview of the chemical structure of the Backbone DNA Network ***BBDNA(m)***and explains its topology, which is visually represented in the Figure 3. The ***BBDNA(m)*** network is shown to have a consistent Metric Dimension (MD), which plays an important role in understanding its geometric properties.

In Figure 5, the labeling pattern of the backbone DNA network is illustrated. DNA is made up of two primary components: the backbone and the nitrogenous bases. Structurally, DNA (deoxyribonucleic acid) consists of alternating phosphate groups, nitrogenous basesadenine, cytosine, guanine, and thymine and a deoxyribose sugar, which is a five-carbon (pentose) molecule. These components are linked together by phosphodiester bonds, forming long chains of nucleotides. It's important to note that DNA does not contain six-carbon sugars (hexose).

In terms of network structure, the Backbone DNA Network has a size defined as 8m-3 and an order of 7m. Additionally, DNA is negatively charged, and in eukaryotic cells, it interacts with positively charged histone proteins. This attraction between DNA and histones helps compact and organize DNA within the cell.

Functionally, DNA is responsible for encoding genetic information, supporting replication, allowing for mutation and recombination, and controlling gene expression. These essential biological roles highlight the significance of understanding the topological features, such as the constant *MD*, of the ***BBDNA(m)*** network (For example see Figure 3.).

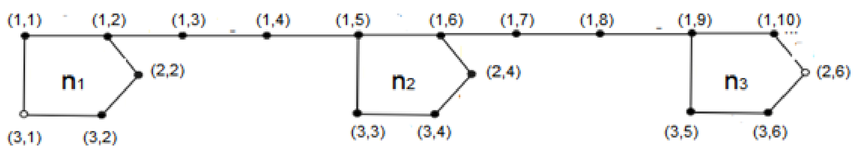

Figure 3
Backbone DNA network *BBDNA*.

Theorem 2: *Let* $G_{p,t(m)}$ *denote the graph representing a* $2D$ *lattice structure of a Backbone DNA network BBDNA(m). The MD of a* $p \times t$ *two-dimensional lattice Backbone DNA network BBDNA(m) is 2.*

Proof: In this context, p represents the number of rows and t denotes the number of columns. Consider the set of vertices:

$$\{y_{\xi(1,1)}, y_{\xi(1,2)}, y_{\xi(1,3)}, \cdots, y_{\xi(1,m)}, y_{\xi(2,1)}, y_{\xi(2,2)}, y_{\xi(2,3)}, \cdots, y_{\xi(2,m)},$$

$y_{\xi(3,1)}, y_{\xi(3,2)}, y_{\xi(3,3)}, \cdots, y_{\xi(3,m)}, \cdots, y_{\xi(i,1)}, y_{\xi(i,2)}, y_{\xi(i,3)}, \cdots, y_{\xi(i,m)}\}$

Here, i indicates the number of rows, while m specifies the number of columns.

Define the set $Z = \{y_{\xi(1,1)}, y_{\xi(1,4n+1)}\}$. We will show that Z acts as a resolving set for the graph.

Let $a_{\xi(i)} = 2i$, $k_{\eta(n)} = 3n$, and $b_{\zeta} = p + t$. We examine two cases based on whether n is even or odd.

For the first row, the distance representation $r(y_{\xi(i,m)}|Z)$ is:

$$r(y\_(i, m) \mid Z) = (m - 1, 4n + i - m) \text{ for all}$$

For the second row, the distance representation $r(y_{\xi(i,m)}|Z)$ is:

$$r(y\_(p, t) \mid Z) = (2m - 2, 4n + 2 - 2(m - 1)) \text{ for all}$$

For the third row, the distance representation $r(y_{\xi(i,m)}|Z)$ is:

$$r(y\_(p, t) \mid Z) = \begin{cases} (\mathrm{m}, 4\mathrm{n} + 1) \text{ for m} = 1,2 \\ (\mathrm{m} + 2, 4\mathrm{n} - 3) \text{ for m} = 3,4 \\ (\mathrm{m} + 4, 4\mathrm{n} - 7) \text{ for m} = 5,6 \\ (\mathrm{m} + 6, 4\mathrm{n} - 11) \text{ for m} = 7,8 \\ (\mathrm{m} + 8, 4\mathrm{n} - 13) \text{ for m} = 9,10 \end{cases}$$

and so on.

Therefore, the MD of the Backbone DNA network *BBDNA(m)* is 2.

The MD of a graph is a measure of how well a set of vertices can uniquely identify all other vertices based on their distances to this set. For the Backbone DNA network $BBDNA(m)$, represented by the graph $G_{p,t(m)}$, it has been shown that a set of just two vertices is sufficient to uniquely determine the location of any other vertex in the graph. This means that the minimal number of vertices required to form a resolving set for this graph is 2, indicating that the MD of the Backbone DNA network *BBDNA(m)* is 2 (For example see Figure 4.).

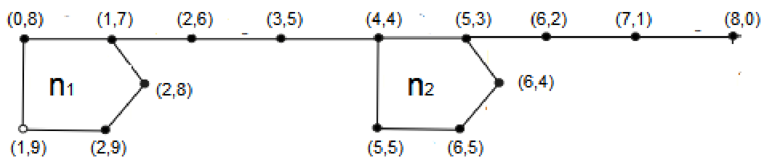

Figure 4
MD of Backbone DNA network $BBDNA$ for n =2.

5. Bakelite network *B(i×m)*

In 1907, phenol-formaldehyde resin, commonly known as Bakelite, was developed. This marked the beginning of the modern plastics industry. Leo Hendrik Baekeland, an American scientist of Belgian origin, was recognized for inventing Bakelite when he patented this phenol-formaldehyde thermoset. The commercialization of the first entirely synthetic polymers was achieved with phenol-formaldehyde polymers, widely known as plastics. Today, about half of the production of thermosetting polymers is dedicated to their use as adhesives.

Bakelite, a versatile chemical substance, finds extensive applications in various areas of human life. Some of its significant uses include: In the electrical industry, Bakelite is used to manufacture non-conductive components for various electrical appliances, such as light bulbs, supports, radios, phones, sockets, bases for electron tubes, and other silicon products. It is also employed in the production of kitchenware items like frying pans and specialized spoons. In the sports industry, Bakelite is used to make billiard balls, chess sets, poker chips, and other game pieces. It is commonly used in making toys, clocks, and jewelry. Its high resistance to heat and electricity makes Bakelite ideal for industrial and automotive components.

In this study, I will determine the MD of the Bakelite network.

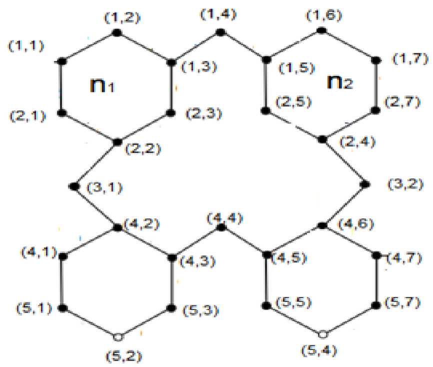

Figure 5
Bakelite network *B(i×m)*.

Theorem 3: *Let* $G_{i,m}$ *represent the graph corresponding to a 2D lattice structure of a Bakelite network **B(i×m)**. The MD of a **i×m** two-dimensional lattice Bakelite network **B(i×m)** is 2.*

Proof: Consider the set of vertices:

$$\{y_{\xi(1,1)}, y_{\xi(1,2)}, y_{\xi(1,3)}, \dots, y_{\xi(1,m)}, y_{\xi(2,1)}, y_{\xi(2,2)}, y_{\xi(2,3)}, \dots, y_{\xi(2,m)},$$
$$y_{\xi(3,1)}, y_{\xi(3,2)}, y_{\xi(3,3)}, \dots, y_{\xi(3,m)}, \dots, y_{\xi(i,1)}, y_{\xi(i,2)}, y_{\xi(i,3)}, \dots, y_{\xi(i,m)}\}$$

Here, i indicates the number of rows, while m specifies the number of columns. Define the set $Z = \{y_{\xi(1,1)}, y_{\xi(1,4n-1)}\}$. We will show that Z acts as a resolving set for the graph.

Let $a_{\xi(i)} = 2i$, $k_{\eta(n)} = 4n + i - m - 2$, and $b_{\zeta} = i + m$. according the Figure 5.

Now we will discuss the rows 1,4,7,10,13,16,....

For m is odd

First row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (1)$$

4th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (2)$$

7th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (3)$$

10th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 5, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ (a_{\xi(i)} - 1, k_{\eta(n)}) & \text{for} i \geq m, m = 9 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (4)$$

13th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 6, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 9,11 \\ (a_{\xi(i)}, k_{\eta(n)}) & \text{for} i \geq m, m = 13 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (5)$$

16th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 7, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 5, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m, m = 9,11 \\ (a_{\xi(i)} - 1, k_{\eta(n)}) & \text{for} i \geq m, m = 13,15 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (6)$$

and so on.. According the Figure 5

For m is even

First row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (7)$$

4th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 2 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 4 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (8)$$

7th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 5, k_{\eta(n)}) & \text{for} i \geq m, m = 2 \\ (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m, m = 4,6 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (9)$$

10th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 6, k_{\eta(n)}) & \text{for} i \geq m, m = 2 \\ (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 4,6 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 8,10 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (10)$$

13th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 7, k_{\eta(n)}) & \text{for} i \geq m, m = 2 \\ (a_{\xi(i)} - 5, k_{\eta(n)}) & \text{for} i \geq m, m = 4,6 \\ (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m, m = 8,10 \\ (a_{\xi(i)} - 1, k_{\eta(n)}) & \text{for} i \geq m, m = 12 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (11)$$

16th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 8, k_{\eta(n)}) & \text{for} i \geq m, m = 2 \\ (a_{\xi(i)} - 6, k_{\eta(n)}) & \text{for} i \geq m, m = 4,6 \\ (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 8,10 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 12,14 \\ (a_{\xi(i)}, k_{\eta(n)}) & \text{for} i \geq m, m = 16 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (12)$$

The equations presented depict a structured series that outlines the relationships between the parameters $a_{\xi(i)}$ and b_ζ based on whether the variable m is odd or even. For odd values of m, the adjustments to $a_{\xi(i)}$ decrease by increments as $i \geq m$, following a pattern that changes across rows, indicating a non-linear progression. On the other hand, if m is even the in $a_{(\xi(i))}$ alters, new values being affected by certain values of m. In both cases, the substraction of b_ζ when $i < m$ demonstrates a normalisation that tends to show that the overall molecular structure is constant. This demonstrates that the network has an inherent balance or cycle built in to it somewhere. As such, this series of equations gives not only a sight into the non-uniformity of the metrics within the network, but also into the discrete patterns of how the odd and even cases are distinct from one another.

Now we will discuss the rows 2,5,8,11,14,17,....

For m is odd

2nd row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} b_\zeta - 2, k_{\eta(n)}) & \text{forall} \end{cases} \quad (13)$$

5th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 5 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (14)$$

8th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 5, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (15)$$

11th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 6, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 4, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ (a_{\xi(i)} - 2, k_{\eta(n)}) & \text{for} i \geq m, m = 9,11 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (16)$$

14th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 7, k_{\eta(n)}) & \text{for} i \geq m, m = 1,3 \\ (a_{\xi(i)} - 5, k_{\eta(n)}) & \text{for} i \geq m, m = 5,7 \\ (a_{\xi(i)} - 3, k_{\eta(n)}) & \text{for} i \geq m, m = 9,11 \\ (a_{\xi(i)} - 1, k_{\eta(n)}) & \text{for} i \geq m, m = 13 \\ b_\zeta - 2, k_{\eta(n)}) & \text{for} i < m \end{cases} \quad (17)$$

and so on....For m is even

2nd row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 2,4n + i - 2m) & \text{for} m = 2 \\ (a_{\xi(i)} + 2,4n + i - 2m) & \text{for} m = 4 \\ (a_{\xi(i)} + 4,4n + i - 2m) & \text{for} m = 6 \\ (a_{\xi(i)} + 6,4n + i - 2m) & \text{for} m = 8 \\ (a_{\xi(i)} + 8,4n + i - 2m) & \text{for} m = 10 \\ so on \ldots \end{cases} \quad (18)$$

5th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 3,4n + i - 2m) & \text{for} m = 2 \\ (a_{\xi(i)} - 1,4n + i - 2m) & \text{for} m = 4 \\ (a_{\xi(i)} + 3,4n + i - 2m) & \text{for} m = 6 \\ (a_{\xi(i)} + 7,4n + i - 2m) & \text{for} m = 8 \\ (a_{\xi(i)} + 11,4n + i - 2m) & \text{for} m = 10 \\ (a_{\xi(i)} + 15,4n + i - 2m) & \text{for} m = 12 \\ so on \ldots \end{cases} \quad (19)$$

8th row: $$r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 4,4n + i - 2m) & \text{for} m = 2 \\ (a_{\xi(i)} - 2,4n + i - 2m) & \text{for} m = 4 \\ (a_{\xi(i)}, 4n + i - 2m) & \text{for} m = 6 \\ (a_{\xi(i)} + 4,4n + i - 2m) & \text{for} m = 8 \\ (a_{\xi(i)} + 8,4n + i - 2m) & \text{for} m = 10 \\ (a_{\xi(i)} + 12,4n + i - 2m) & \text{for} m = 12 \\ so on \ldots \end{cases} \quad (20)$$

$$\text{11th row:} \quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 5, 4n + i - 2m) & \text{for} m = 2 \\ (a_{\xi(i)} - 3, 4n + i - 2m) & \text{for} m = 4 \\ (a_{\xi(i)} - 1, 4n + i - 2m) & \text{for} m = 6 \\ (a_{\xi(i)} + 3, 4n + i - 2m) & \text{for} m = 8 \\ (a_{\xi(i)} + 5, 4n + i - 2m) & \text{for} m = 10 \\ (a_{\xi(i)} + 9, 4n + i - 2m) & \text{for} m = 12 \\ (a_{\xi(i)} + 13, 4n + i - 2m) & \text{for} m = 14 \\ so on \ldots \end{cases} \tag{21}$$

$$\text{14th row:} \quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 6, 4n + i - 2m) & \text{for} m = 2 \\ (a_{\xi(i)} - 4, 4n + i - 2m) & \text{for} m = 4 \\ (a_{\xi(i)} - 2, 4n + i - 2m) & \text{for} m = 6 \\ (a_{\xi(i)}, 4n + i - 2m) & \text{for} m = 8 \\ (a_{\xi(i)} + 4, 4n + i - 2m) & \text{for} m = 10 \\ (a_{\xi(i)} + 6, 4n + i - 2m) & \text{for} m = 12 \\ (a_{\xi(i)} + 10, 4n + i - 2m) & \text{for} m = 14 \\ (a_{\xi(i)} + 12, 4n + i - 2m) & \text{for} m = 16 \\ so on \ldots \end{cases} \tag{22}$$

$$\text{17th row:} \quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 7, 4n + i - 2m) & \text{for} m = 2 \\ (a_{\xi(i)} - 5, 4n + i - 2m) & \text{for} m = 4 \\ (a_{\xi(i)} - 3, 4n + i - 2m) & \text{for} m = 6 \\ (a_{\xi(i)} - 1, 4n + i - 2m) & \text{for} m = 8 \\ (a_{\xi(i)} + 1, 4n + i - 2m) & \text{for} m = 10 \\ (a_{\xi(i)} + 5, 4n + i - 2m) & \text{for} m = 12 \\ (a_{\xi(i)} + 9, 4n + i - 2m) & \text{for} m = 14 \\ so on \ldots \end{cases} \tag{23}$$

By investigating the rows of the Bakelite network, different patterns based on whether m is odd or even. For odd values of m, the function $r(y_{\xi(i,m)}|Z)$ adhare a pattern where each subsequent row shows increasing subtractions from $a_{\xi(i)}$, creating a linear progression. For example, in the 2nd row, the series starts with $b_\zeta - 2$. As m increases, terms like $a_{\xi(i)} - 4$ and $a_{\xi(i)} - 2$ appear in the 5th row, with the differences between terms growing evenly through rows.

For even values of m, the series alternates between adding and subtracting from $a_{\xi(i)}$, representing in a more different sequence. Starting with $a_{\xi(i)} - 2$ in the 2nd row, the pattern includes shifts that increase in steps, such as $a_{\xi(i)} - 3$ and $a_{\xi(i)} - 1$ in the 5th row for $m = 4$. As m increases, the resolving function becomes more complecated, representing the opposit nature of operations in even rows. These sequences shows the structure of the Bakelite network's MD, where the complexity of the resolving function grows with both the row number and the parity of m.

Now we discuss the rows 3,6,9,12,

$$\text{3rd row:}\quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 3, 4n - 1) & \text{for} m = 1 \\ (a_{\xi(i)} + 1, 4n - 5) & \text{for} m = 2 \\ (a_{\xi(i)} + 5, 4n - 9) & \text{for} m = 3 \\ (a_{\xi(i)} + 9, 4n - 13) & \text{for} m = 4 \\ (a_{\xi(i)} + 13, 4n - 17) & \text{for} m = 5 \\ soon \ldots \end{cases} \tag{24}$$

$$\text{6th row:}\quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 4, 4n + 2) & \text{for} m = 1 \\ (a_{\xi(i)} - 2, 4n - 2) & \text{for} m = 2 \\ (a_{\xi(i)} + 2, 4n - 6) & \text{for} m = 3 \\ (a_{\xi(i)} + 6, 4n - 10) & \text{for} m = 4 \\ (a_{\xi(i)} + 10, 4n - 14) & \text{for} m = 5 \\ (a_{\xi(i)} + 14, 4n - 18) & \text{for} m = 5 \\ soon \ldots \end{cases} \tag{25}$$

$$\text{9th row:}\quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 5, 4n + 5) & \text{for} m = 1 \\ (a_{\xi(i)} - 3, 4n + 1) & \text{for} m = 2 \\ (a_{\xi(i)} - 1, 4n - 3) & \text{for} m = 3 \\ (a_{\xi(i)} + 3, 4n - 7) & \text{for} m = 4 \\ (a_{\xi(i)} + 7, 4n - 11) & \text{for} m = 5 \\ (a_{\xi(i)} + 11, 4n - 15) & \text{for} m = 5 \\ soon \ldots \end{cases} \tag{26}$$

$$\text{12th row:}\quad r(y_{\xi(i,m)}|Z) = \begin{cases} (a_{\xi(i)} - 6, 4n + 8) & \text{for} m = 1 \\ (a_{\xi(i)} - 4, 4n + 4) & \text{for} m = 2 \\ (a_{\xi(i)} - 2, 4n) & \text{for} m = 3 \\ (a_{\xi(i)} + 2, 4n - 4) & \text{for} m = 4 \\ (a_{\xi(i)} + 6, 4n - 8) & \text{for} m = 5 \\ soon \ldots \end{cases} \tag{27}$$

Let's evaluate the patterns shown by rows 3, 6, 9, and 12 of the Bakelite network. Each of these rows reveals a unique sequence depending on the index m.

In the 3rd row, the function $r(y_{\xi(i,m)}|Z)$ begins with values decreasing from $a_{\xi(i)} - 3$ for $m = 1$, and then progresses with increasing increments of 4 for subsequent m-values. This creates a sequence that evolves as $a_{\xi(i)} + 1$, $a_{\xi(i)} + 5$, and so on.

On the other hand 6th row exhibits a different pattern. Here, the values alternate between decreasing and increasing by fixed values, starting from $a_{\xi(i)} - 4$ and transitioning to positive increments like $a_{\xi(i)} + 2$ and $a_{\xi(i)} + 6$.

In the 9th row, the pattern involves a mix of subtractions and additions, starting with $a_{\xi(i)} - 5$ and progressing to positive increments such as $a_{\xi(i)} + 3$.

Finally, the 12th row follows a similar structure, beginning with $a_{\xi(i)} - 6$ and increasing in steps of 2, leading to values like $a_{\xi(i)} + 2$.

These patterns showes change in the network's MD and highlight the varied behavior of the Bakelite network's structural patterns.

For the Bakelite network candidate, it is found by ranking 2nd on MD. In other words, given the distances over all pairs of vertices, any coordinate may be unambiguously expressed by picking only 2 cleverly chosen coordinates. This result admits a structural observation of the network: strong level of symmetry and regularity. But the layout of the entire network is efficiently resolved using just two reference points. Essentially this property is emphasizing the geometric and combinatorial characteristic of Bakelite network to be compact with an organized structure.

6. Significance of the Study

It is imperative to research the MDs of lattice structures such as Polythiophene network, Backbone DNA network and Bakelite Network because it carries so much importance in both science as well industry. Determine the smallest set of points which need to be specified in order that everything else mayl be uniquely point located within these networks. This insight also may open the door to developing superior, less expensive systems in such divergent areas as organic electronics,bioengineer networks and materials science. The fact that is coincidence with all these networks /n show a MD of 2 suggests as possible explanation some innate structural property. The discovery provides a powerful instru- ment for researchers and practitioners to improve their network designs, as discussed in the sequel.

7. Conclusion

Here, we comprehensively scrutinize the MD of three different lattice models: Polythiophene network, Backbone DNA-network and Bakelite-network. Our examination reveals that all these networks are with a MD of 2. In other words, any two peculiar vertices uniquely identify the rest of (either) vertex within these networks.

This result further underscores the intrinsic order and economy of these lattice geometries, showing that their metric properties are well-preserved across even chemically dissimilar CRN systems for different operating regimes. Both the Polythiophene network, a model system for organic electronics and the Backbone DNA network that is fundamental to many biological processes, are found to have low dimensional metric spaces an expected result in which simple analysis or design can be conducted efficiently. This result could apply to the Bakelite 100 network (used in this study here as industrial application) proving that also for networks operating at other than room temperature MD is meaningful tool of network characterization.

Our results are going to be a strong area for outlining the member density within these lattice structures, an obvious demand if one desires deploying them in various technical and experimental situations. More generally, this work paves the way for future research into edge metric and fault tolerant MD on more complex or different lattice networks which may also be able to uncover any new structures within lattices. This could in turn improve current understanding of graph theory both from a theoretical perspective and for real-world applications.

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