Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 2, pp. 415-421
DOI : 10.47974/JDMSC-2281

Essential-small projective semimodule

Baraa Abd Alhussain *
Asaad Mohammed Ali Alhossaini †
Department of Mathematics
College of Education for Pure Sciences
University of Babylon
Babylon 51001
Iraq

Abstract

Previously, some authors have presented a study on the e-small projective Modules. The aim of this paper provides as comprehensive study as possible about e-small Projective semimodules and related concepts to access some new features and results of semimodule.

Subject Classification (2020): *17A01, 17A05, 17A30.*

Keywords: *Small subsemimodule, Essential semimodule, E-small subsemimodule, Projective semimodule.*

1. Introduction

The concept of projective semi modules was studied by many authors one of them [1]. We discuss in this paper a new generalization of projective semimodule (it has been studied previously for modules [2]). The idea is to introduce and study e-small projective semimodule. The study of superfluous subsemimodule has been extensively considered, a subsemimodule $\mathcal{F}$ of a semimodule $\mathcal{M}$ is said to be "small", "denoted by $\mathcal{F} \ll \mathcal{M}$", if for each "$j \leq \mathcal{M}$" such that "$\mathcal{F} + j = \mathcal{M}$", implies "$j = \mathcal{M}$". A surjective homomorphism f: $\mathcal{A}ń\mathcal{B}$,where $\mathcal{A}, \mathcal{B}$ are semimodules is called "small surjective" if Ker$f \ll \mathcal{A}$" [3]. "Pawar studied some useful results on essential subsemimodules" [4, 12-20]. Ali and Alhussaini are submitted a comprehensive study of small

* *ORCID-ID:* 0000-0002-0588-2159
† *ORCID-ID:* 0000-0001-9300-8834

* *E-mail:* baraa abd968@gmail.com (Corresponding Author)
† *E-mail:* asaad-hosain@itnel.uobabylon.edu.iq

subsemimodule and some results [5]. We study in this research the concept of essential-small subsemimodule, written (e-s), as it is a subsemimodule $\mathfrak{B}$, where $\mathcal{M}$is semimodule is said to be e-s subsemimodule of $\mathcal{M}$, written $\mathfrak{B} \ll_e \mathcal{M}$. If for each essential subsemimdule $\mathcal{J}$ of $\mathcal{M}$ with $\mathfrak{B} + \mathcal{J} = \mathcal{M}$ implies $\mathcal{J} = \mathcal{M}$. A semimodule $\mathcal{M}$ is e-s projective if for each e-s surjective homomorphism $g: \mathcal{A}\acute{n}\mathfrak{B}$, and any $f: \mathcal{M}\acute{n}\mathfrak{B}$, there exists$k: \mathcal{M}\acute{n}\mathcal{A}$ is a homomorphism such that $gk = f$.

2. Preliminaries

Remark: The definitions: "semimodule, subsemimodule, subtractive semimodule, projective semimodule, cancellative semimodule, direct sum, direct summand and retract subsemimodule are found in" [6].

Definition 2.1: [7] "A map $f: \mathcal{M}\acute{n}\mathcal{N}$ is called a morphism of a semimodule if "$f(\sigma + \acute{\sigma}) = f(\sigma) + f(\acute{\sigma})$", and "$f(r\sigma) = rf(\sigma)$ "for all " $\sigma, \acute{\sigma} \in \mathcal{M}$"and "$r \in \mathfrak{R}$".

For a morphism of a semimodule $f: \mathcal{M}\acute{n}\mathcal{N}$ we define:

(1) "Ker(f)=$\{v \in M | f(v) = 0\}$;

(2) $Im(f) = \{\sigma \in \mathcal{N}, \sigma + f(v) = f(\acute{v})$ for some $v, \acute{v} \in \mathcal{M}\}$".

Definition 2.2: [8] "let $\mathcal{M}$ and $\mathcal{N}$semimodules". "A homomorphism $f: \mathcal{M}\acute{n}\mathcal{N}$" is said to be "k-regular homomorphism", if for all $\sigma, \acute{\sigma} \in \mathcal{M}$. So that"$f(\sigma) = f(\acute{\sigma}) \Rightarrow \sigma + k = \acute{\sigma} + \acute{k}$", for some "$k, \acute{k} \in$ Ker(f)".

Definition 2.3: [3] "A homomorphism $g: \mathcal{M}\acute{n}\mathcal{N}$of semimodules is considered k-quasiregular, if k belong to $Ker(g)$ occurs $\mathcal{T} \leq \mathcal{M}$, b belong to $\mathcal{M}/\mathcal{T}$, and $\acute{b}$ belong to $\mathcal{T}$, if $g(b) = g(\acute{b})$ consequently $b = \acute{b} + k$.

Definition 2.7: [8] "A semimodule $\mathcal{M}$ is called completely subtractive, if all of its subsemimodules are subtractive".

Definition 2.8: [9] "A Modular Law for semimodules" is defined by if $\mathcal{M}$ is a semimodule, $\mathcal{J}, \mathcal{K}$, and $\mathcal{H}$ are semimodules of$\mathcal{M}$ with "$\mathcal{J}$ is a subtractive" and "$\mathcal{K} \leq \mathcal{J}$", then " $\mathcal{J} \cap (\mathcal{K} + \mathcal{H}) = \mathcal{K} + (\mathcal{J} \cap \mathcal{H})$".

3. Essential-Small

Definition 3.1*:* A subsemimodule$\mathcal{H}$ of a semimodule $\mathcal{M}$ is called essential-small or (e-s), written $\mathcal{H} \ll_e \mathcal{M}$, if for each $\mathcal{K} \leq_e \mathcal{M}$, such that $\mathcal{H} + \mathcal{K} = \mathcal{M}$. Implies $\mathcal{K} = \mathcal{M}$.

Definition 3.2*:* Let $\mathcal{A}, \mathcal{B}$ be a semimodules, A surjective homomorphism $g: \mathcal{A}\acute{n}\mathcal{B}$ is said to be e-s surjective homomorphism if $Ker(g) \ll_e \mathcal{A}$.

Lemma 3.3: *If $\vartheta: \mathcal{M}\acute{n}\mathcal{T}$ is a homomorphism, and $\mathcal{K} \leq_e \mathcal{T}$, then $\vartheta^{-1}(\mathcal{K}) \leq_e \mathcal{M}$.*

Proof: The method of proof is similar to the proof in modules [10].

Lemma 3.4*:* Let $\vartheta: \mathcal{M} \acute{n} \mathcal{P}$ be a homomorphism of a semimodules and $\mathcal{H} \leq \vartheta(\mathrm{M})$, then $\vartheta(\vartheta^{-1}(\mathcal{H})) = \mathcal{H} \cap \vartheta(M) = \mathcal{H}$.

Proof: The method of proof is similar to the proof in modules [10].

Proposition 3.5: If $\mathcal{N}$ is an e-s subsemimodule of $\mathcal{M}$ and $\vartheta: \mathcal{M} \acute{n} \mathcal{T}$ is a k-quasi-regular homomorphism, then $\vartheta(\mathcal{N})$ is an e-s subsemimodule of $\vartheta(\mathcal{M})$.

Proof: Let $\mathcal{H} \leq_e \vartheta(\mathcal{M})$, and $\vartheta(\mathcal{N}) + \mathcal{H} = \vartheta(\mathcal{M})$, since $\mathcal{H} \leq \vartheta(\mathcal{M})$, then $\vartheta(\vartheta^{-1}(\mathcal{H})) = \mathcal{H}$. Implies that $\vartheta(\mathcal{N}) + \vartheta(\vartheta^{-1}(\mathcal{H})) = \vartheta(\mathcal{M})$, claim $\mathcal{M} = \mathcal{N} + \vartheta^{-1}(\mathcal{H})$, let $m \in \mathcal{M}$, then $\vartheta(m) \in \vartheta(\mathcal{N} + \vartheta^{-1}(\mathcal{H}))$, $\vartheta(m) = \vartheta(n + \hbar)$, where $n \in \mathcal{N}$ and $\hbar \in \vartheta^{-1}(\mathcal{H})$, implies that $m = n + \hbar + k, k \in \mathrm{Ker}(\vartheta) \subseteq \vartheta^{-1}(\mathcal{H})$, $\mathcal{M} = \mathcal{N} + \vartheta^{-1}(\mathcal{H})$, then $\vartheta^{-1}(\mathcal{H}) = \mathcal{M}$. Hence $\mathcal{H} = \vartheta(\mathcal{M})$, then $\vartheta(\mathcal{N}) \ll_e \vartheta(\mathcal{M})$.

Lemma 3.6*:* Assume that $\mathcal{H}$ and $\mathcal{K}$ are subsemimodules of $\mathcal{M}$. If $\mathcal{H}$ is an essential subsemimodule of $\mathcal{K}$ and $\mathcal{N}$ is a subsemimodule of $\mathcal{K}$, then $\mathcal{H} \cap \mathcal{N}$ is an essential of $\mathcal{K}$.

Proof: since $\mathcal{N}$ subsemimodule of $\mathcal{K}$, such that $\mathcal{H} \cap \mathcal{K}) \cap \mathcal{N} = 0$, then $\mathcal{H} \cap (\mathcal{K} \cap \mathcal{N}) = 0$, implies that $\mathcal{H} \cap \mathcal{N} = 0$, then $\mathcal{N} = 0$.

Lemma 3.7: *If $\mathcal{N}$ be an e-s subsemimodule of $\mathcal{K}$, K is a subsemimodule of $\mathcal{M}$, and $\mathcal{K}$ be a subtractive, then $\mathcal{N}$ is an e-s subsemimodule of $\mathcal{M}$.*

Proof: Assume that $\mathcal{N} \leq \mathcal{M}$ and for each $\mathcal{H} \leq_e \mathcal{M}$. Let $\mathcal{N} + \mathcal{H} = \mathcal{M}$, since $\mathcal{N} \ll_e \mathcal{K}$, implies that for each non-zero sub semimodule $\mathcal{H}$ of $\mathcal{K}$, such that $\mathcal{N} + \mathcal{H} = \mathcal{K}$, then $\mathcal{H} = \mathcal{K}$. By Modular Law of semimodule we have $(\mathcal{N} + \mathcal{H}) \cap K = \mathcal{N} + (\mathcal{H} \cap \mathcal{K}) = \mathcal{K}$, since $\mathcal{K}$ is a subtractive, then $\mathcal{H} \cap \mathcal{K} = \mathcal{K}$, since $\mathcal{H} \cap \mathcal{K} \leq_e \mathcal{K}$ [by **Lemma** 3.6] and $\mathcal{N} \ll_e \mathcal{K}$, then $\mathcal{K} \leq \mathcal{H}$, such that $\mathcal{H} \leq_e \mathcal{M}, \mathcal{H} \cap \mathcal{K} \leq_e \mathcal{K}$. So $\mathcal{N} + \mathcal{H} = \mathcal{H}$, then $\mathcal{H} = \mathcal{M}$. Hence $\mathcal{N} \ll_e \mathcal{M}$.

Lemma 3.8: *If $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$, where $\mathcal{L}_1$ is subsemimodule of $\mathcal{P}_1$ and $\mathcal{L}_2$ is subsemimodule of $\mathcal{P}_2$, hence $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}_1 \oplus \mathcal{L}_2$.*

Proof: It is clear.

Proposition 3.9: $\mathcal{H} + \mathcal{K} \ll_e \mathcal{T}$ if and only if $\mathcal{H} \ll_e \mathcal{T}$ and $\mathcal{K} \ll_e \mathcal{T}$, where $\mathcal{T}$ is a semimodule $\mathcal{K} \leq_e \mathcal{N} \leq \mathcal{T}$, and $\mathcal{H} \leq_e \mathcal{T}$.

Proof: ($\Rightarrow$) Assume that $\mathcal{K} + \mathcal{H} \ll_e \mathcal{M}$. Let $\mathcal{K} + \mathcal{J} = \mathcal{T}$, where $\mathcal{J} \leq_e \mathcal{T}$, then $\mathcal{K} + \mathcal{H} + \mathcal{J} = \mathcal{T}$, implies that $\mathcal{J} = \mathcal{T}$, hence $\mathcal{K} \ll_e \mathcal{T}$. Let $\mathcal{H} + \mathcal{J} = \mathcal{T}$, where $\mathcal{J} \leq_e \mathcal{T}$, then $\mathcal{K} + \mathcal{H} + \mathcal{J} = \mathcal{T}$, implies that $\mathcal{J} = \mathcal{T}$. Hence $\mathcal{H} \ll_e \mathcal{T}$.

($\Leftarrow$): Assume that $\mathcal{K} \ll_e \mathcal{T}$ and $\mathcal{H} \ll_e \mathcal{T}$, let $\mathcal{K}+\mathcal{H}+\mathcal{J}=\mathcal{T}$, where $\mathcal{J} \leq_e \mathcal{T}$, then $\mathcal{H}+\mathcal{J} \leq_e \mathcal{T}$, $\mathcal{K} \ll_e \mathcal{T}$, implies that $\mathcal{H}+\mathcal{J}=\mathcal{T}$, since$\mathcal{H} \ll_e \mathcal{T}$, then $\mathcal{J}=\mathcal{T}$. Hence $\mathcal{K}+\mathcal{H} \ll_e \mathcal{T}$.

Lemma 3.10: *Let $\mathcal{A}_1 \leq \mathcal{M}_1, \mathcal{A}_2 \leq \mathcal{M}_2, f\colon A_1 \to A_2$ be an isomorphism and $g\colon M_1 \to M_2$ be an isomorphism such that $g|A_1 = f$. If $A_1 \leq^{\oplus} M_1$ then $A_2 \leq^{\oplus} M_2$.*

Proof: It is clear.

4. E-Small projective

Definition 4.1*:* A semimodule $\mathcal{P}$ is e-s projective semimodule, if every e-s k-regular surjective homomorphism $g\colon \mathcal{A}$ń$\mathcal{B}$ and any $f\colon \mathcal{P}$ń$\mathcal{B}$ is homomorphism, there exists $k\colon \mathcal{P}$ń$\mathcal{A}$ is homomorphism such that $gk = f$.

Proposition 4.2: Let $\mathcal{P}$ be an $\mathfrak{R}$-semimodule, then the following statements are equivalent:

(1) $\mathcal{P}$ is e-s projective semimodule.

(2) For each semimodule $\mathcal{A}$ and each $f\colon \mathcal{P}$ń $\mathcal{A}/\mathcal{H}$ is a homomorphism, where $\mathcal{H} \ll_e \mathcal{A}$ there exists $h\colon \mathcal{P}$ń$\mathcal{A}$ such that $gh = f$, where g is the natural map of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{H}$.

(3) For each e-s surjective homomorphism $g\colon \mathcal{A}$ń$\mathcal{B}$ and any $f\colon \mathcal{P}$ń$\mathcal{B}$ is a homomorphism, there exists $k\colon \mathcal{P}$ń$\mathcal{A}$ such that $gk = f$.

Remark 4.3*:* If $\mathcal{A}$ is fixed in Definition 4.1 we say $\mathcal{P}$ is e-s projective relative to $\mathcal{A}$.

Proposition 4.4: Let $\mathcal{P}$ be an $\mathfrak{R}$ $-$semimodule which is e-s projective relative to $\mathcal{A}$. If $\mathcal{B}$ is a subtractive subsemimodule of $\mathcal{A}$ then $\mathcal{P}$ is e-s projective relative to $\mathcal{B}$.

Proof: Let $\mathcal{E}$ be an $\mathfrak{R}$-semimodule such that $\mathcal{A} \leq \mathcal{E}$. Consider the diagram

Where i is the inclusion*:* $\mathcal{A}$ń$\mathcal{E}$ and ℓ is defined by $\ell\big(g(a)\big) = a\,/\,(Ker(g))$ [Note that Ker(g) $\leq \mathcal{A} \leq \mathcal{E}$, ℓ is well defined since $g(a_1) = g(a_2)$, then $a_1 + k_1 = a_2 + k_2$ for some $k_1, k_2 \in$ Ker(g), which implies $a_1\,/\,Ker(g) = a_2\,/\,Ker(g)$]. By assumption there is $h\colon \mathcal{P}$ń$\mathcal{E}$ a homomorphism so that $\pi h = \ell f$. Note that for $x \in \mathcal{P}, \pi\big(h(x)\big) = \ell(f(x))$, then $h(x)/Ker(g) = a/Ker(g)$, where $g(a) = f(x)$ for some $a \in \mathcal{A}$, $h(x) + k_1 = a + k_2$ for some $k_1, k_2 \in Ker(g)$. so $\ell\big(f(x)\big) = a\,/\,Ker(g)$, if and only if $f(x) = g(a)$. $\ell\big(f(\mathcal{P})\big) = \{a/Ker(g)\,|\,g(a) = f(x),$ for some $a \in g^{-1}(f(\mathcal{P}))\} = g^{-1}(f(\mathcal{P}))\,/\,Ker(g)$, hence $\pi\big(h(\mathcal{P})\big) = g^{-1}(f(\mathcal{P}))\,/\,Ker(g)$, $h(\mathcal{P}) + Ker(g) = g^{-1}\big(f(\mathcal{P})\big) + Ker(g)$, then $h(\mathcal{P}) + Ker(g) = g^{-1}(f(\mathcal{P}))$. Therefore $h(\mathcal{P}) = g^{-1}(f(\mathcal{P})) \leq \mathcal{A}$.

Definition 4.5: " $\mathcal{P}$-cyclic subsemimodule*:* Similar to the define in modules". [11]

Definition 4.6: " $\mathcal{P}$-principally injective*:* Similar to the define in modules". [11]

Definition 4.7: A semimodule $\mathcal{P}$ is called e-small factor of a semimodule $\mathcal{K}$, if there is an e-s surjective homomorphism $\mathscr{k}: \mathcal{K}\acute{n}\mathcal{P}$.

Proposition 4.8: Let $\mathcal{P}$ be an e-s projective semimodule such that every $\mathcal{P} -$ cyiclic subsemimodule of $\mathcal{P}$ is e-s projective, then every e-s factor semimodule of a $\mathcal{P}$-principally injective semimodule is $\mathcal{P} -$ principally injective semimodule.

Proof: Let $\mathscr{g}: \mathcal{A}\acute{n}\mathcal{B}$ be an e-s surjective homomorphism, where $\mathcal{A}$ is $\mathcal{P}$-principally injective semimodule*,* where $\mathscr{f}: \varphi(\mathcal{P})\acute{n}\mathcal{B}$ is a homomorphism, $\varphi \in End(\mathcal{P})$, and $\alpha: \varphi((\mathcal{P})\acute{n}\mathcal{P}$ is inclusion homomorphism. By assumption $\varphi(\mathcal{P})$ is an e-s projective semimodule, so there exists $\mathscr{h}: \varphi(\mathcal{P})\acute{n}\mathcal{A}$ is a homomorphism so that $\mathscr{g}\mathscr{h} = \mathscr{f}$. Since $\mathcal{A}$ is $\mathcal{P}$-principlly injective, there is a homomorphism $\mathscr{h}_1: \mathcal{P}\acute{n}\mathcal{A}$ such that $\mathscr{h}_1\alpha = \mathscr{h}$. Define $\mathscr{h}_2: \mathcal{P}\acute{n}\mathcal{B}$ by $\mathscr{h}_2 = \mathscr{g}\mathscr{h}_1$. $\mathscr{h}_2\alpha = \mathscr{g}\mathscr{h}_1\alpha = \mathscr{g}\mathscr{h} = \mathscr{f}$.

Corollary 4.9: *If $\mathcal{P}$ is an e-s projective semimodule such that every $\mathcal{P}$ −cyclic subsemimodule of $\mathcal{P}$ is e-s projective, then every e-s projective, then every e-s factor semimodule of an injective semimodule of an injective semimodule is $\mathcal{P}$ −principally injective.*

References

[1] H. M. J. Al-Thani, Projective and Injective Semimodules over Semirings (Doctoral dissertation, University of East London) (1998).

[2] M. F. Manhal and M. S. Abbas, "Essential-Small M-Projective Modules," *Iraqi J. Sci.*, vol. 59, no. 1B (2018). [Online] Available: https://doi.org/10.24996/ijs.2018.59.1b.16.

[3] N. X. Tuyen and H. X. Thang, "On Superfluous Subsemimodules," *Georgian Math. J.*, vol. 10, no. 4, pp. 763-770 (2003). [Online] Available: https://doi.org/10.1515/gmj.2003.763.

[4] K. Pawar, "Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon-425 001 (M.S.) India," *New Trends in Mathematical Sciences*, vol. 1, no. 2, pp. 18-21 (2013).

[5] H. A. Ali and A. M. Alhossaini, "Small Subsemimodule and Some Related Concept," *J. Discrete Math. Sci. Cryptogr.*, to appear (2025).

[6] J. S. Golan, Semirings and Their Applications, Kluwer Academic Publishers, Dordrecht, Springer Nature (1999). [Online] Available: https://doi.org/10.1007/978-94-015-9333-5.

[7] J. R. Tsiba and D. Sow, "On Generators and Projective Semimodules," *International Journal of Algebra,* vol. 4, no. 24, pp. 1153-1167 (2010). [Online] Available: https://www.m-hikari.com/ija/ija-2010/ija-21-24-2010/sowIJA21-24-2010.pdf.

[8] M. Takahashi, "On the Bordism Categories II," *Mathematics Seminar Notes*, vol. 9, no. 2, pp. 495-530 (1981).

[9] S. Alhashemi and A. M. Alhossaini, "Extending Semimodule and Singularity," The Seventh Scientific Conference of the Al-Khwarizmi Society, in cooperation with the Technical Institute-Mosul, pp. 330-340 (2021).

[10] F. Kasch, Modules and Rings, L.M.S. Monograph No. 17, Academic Press, New York (1982).

[11] N. V. Sanh, K. P. Shum, S. Dhompong, and S. Wongwai, "On Quasi-principally Injective Modules," *Algebra Colloquium*, vol. 6, no. 3, pp. 269-276 (1999).

[12] A. H. Alwan, "Small Intersection Graph of Subsemimodules of a Semimodule," *Commun. Combinatorics Cryptogr. Comput. Sci.,* vol. 2022, no. 1, pp. 15-22 (2022).

[13] S. Ediz, M. Cancan, and M. R. Farahani, "Eccentric Connectivity and Connective Eccentricity Indices of Generalized Petersen Graphs," *Commun. Combinatorics Cryptogr. Comput. Sci*., vol. 2021, no. 1, pp. 1-6 (2021).

[14] D. Diallo and P. C. Diop, "Modules Whose t-closed Submodules Are Their Homomorphic Images," *J. Discrete Math. Sci. Cryptogr*., vol. 23, no. 6, pp. 1297-1314 (2020). [Online] Available: https://doi.org/10.1080/09720529.2020.1815339.

[15] A. Jabbar and A. H. Hussain, "A Q-Smarandache Pseudo Clean Ideals of BH-Algebra," *J. Discrete Math. Sci. Cryptogr.*, vol. 25, no. 6, pp. 1945-1950 (2022). [Online] Available: https://doi.org/10.1080/09720529.2022.2099060.

[16] E. A. Naim and A. R. Mehdi, "On gs-pseudo-injective modules," *J. Discrete Math. Sci. Cryptogr.,* vol. 25, no. 5, pp. 1535-1545 (2022). [Online] Available: https://doi.org/10.1080/09720529.2022.2082724.

[17] R. N. Mirza and T. Y. Ghawi, "Radg-lifting Modules," *J. Interdiscip. Math.,* vol. 26, no. 6, pp. 1001-1013 (2023). [Online] Available: https://doi.org/10.47974/jim-1600.

[18] W. H. Hanoon and A. A. Abboodi, "Large-maximal Small Submodules," *J. Interdiscip. Math.,* vol. 26, no. 6, pp. 1043-1052 (2023). [Online] Available: https://doi.org/10.47974/jim-160.

[19] M. W. Allami and W. H. Hanoon, "Soc-T-ABSO Submodules and Related Concepts," *J. Interdiscip. Math.,* vol. 26, no. 6, pp. 1077-1088 (2023). [Online] Available: https://doi.org/10.47974/jim-1607.

[20] F. D. Shyaa, "Small Approximately 2-absorbing Submodules and Modules," *J. Interdiscip. Math.,* vol. 28, no. 1, pp. 43-49 (2025). [Online] Available: https://doi.org/10.47974/jim-1774.

Received November, 2024