Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 2, pp. 439-449
DOI : 10.47974/JDMSC-2060

Domination of zero-divisor graphs

Haneen Al-Janabi *
Zainab Yasir Alrikabi †
Haider Baqer Ameen §
Shaymaa Maki Kadham ‡
Department of Mathematics
Faculty of Computer Science and Mathematics
University of Kufa
Kufa 540011
Al-Najaf
Iraq

Abstract

The graph that has vertices as elements in a commutative ring R, such that u and v are adjacent only if $uv = 0$, is called the zero-divisor graph, $\Pi(R)$. We study the domination of $\Pi(Z_n)$ for all parts of n in this study, since Z_n is one of the well-known rings.

Subject Classification (2010): *05C25, 05C69, 20F65.*

Keywords: *Zero-divisor graph, Domination number.*

1. Introduction

Here, we used some definitions and notation from the Section 2. An unusual but highly important technique in mathematics is to create an item that appears to be too far away from the original in order to obtain much more information about the structure. These include, among other things, grouping representations and finite plane uses in cryptography and coding theory. We discover that the same steps apply for building the $\Pi(R)$ for a ring.

§ *ORCID-ID:* **0000-0002-0324-740X**
‡ *ORCID-ID:* **0000-0003-2664-1669**

* *E-mail:* **haneenk.aljanabi@uokufa.edu.iq (Corresponding Author)**
† *E-mail:* **zainaby.alrikabi@uokufa.edu.iq**
§ *E-mail:* **hayder.ameen@uokufa.edu.iq**
‡ *E-mail:* **shaimam.altaee@uokufa.edu.iq**

In 1988, Beck found the thought of a $\Pi(R)$ for a commutative ring [7]. Anderson and Livingston [6] provided the initial Beck's graph simplification in 1999. Additional changes emerged as a result of Mulay [16]. This history demonstrates that different zero-divisor graph definitions might be helpful depending on the objectives. For instance, in our situation, loops had to be eliminated. One of the primary subjects in the domination graph, which was initially presented by Jsenish in 1862, in order to determine the least number of unique queen pieces required in order to control the chessboard game. In 1958, Claude Berge [8] produced his book on graph theory, in which he proposed the concept of the domination number of a graph, which is the coefficient of an external stability. In 1977, Cockayne and Hedetniemi [9] released a survey and make advantage of $\gamma(G)$ as a symbol on a graph denoting the domination number. In 2013, Rad et al. [17] found the domination number for the product of two commutative rings' zero-divisor graph with 1, and they provide the domination number for each commutative Artinian ring's zero-divisor graph. In 2015, Jafari and Rad [13] examined the domination over for a commutative ring with zero-divisor graphs of matrix rings 1. In 2019, Shipiya Raj Shree et al. [18] provided some findings on the domination number, both total and co-total, for a few zero-divisor graphs on the direct product of commutative rings. Our objective is to provide a comprehensive solution for the domination number of $\Pi(Z_n)$, $\forall\, n$.

2. Definitions and Notion

For all of the graphs in this research, undirected simple graphs $G = (V, E)$, where the edge set of G is denoted by $E = E(G)$, and the vertex set by $V = V(G)$. The number of incident edges to dv, or $d(v)$, equals the vertex's degree of v ($v \in V(G)$) [19-27]. In equivalent terms, the maximum (lowest) degree of the vertex is $\Delta(G)$ ($\delta(G)$). We refer to the complete (null) graph on m vertices as K_m ($\overline{K_m}$).

We emphasize that a clique need not be maximal; it might refer to any arbitrary full subgraph in this context. $\omega(G)$ is the clique number of G, which is the greatest clique of vertices consisting of G. When no edge joins any two of a set's vertices, the set is said to be independent (coclique) [2, 3, 11, 14]. A graph termed $K_{p,q}$ is created with two independent sets of vertices, $|X_1| = p$, and $|X_2| = q$ that can split (X_1, and X_2), so that every vertex in X_1 and every vertex in X_2 has a connection. If G is not *chordal* [10], then G is an induced cycle of length at least four. $G \cup H$ is the *union* of G and H, with the vertex set is $V(G) \cup V(H)$, and the edge set is $E(G) \cup E(H)$ [15].

Definition 1: For a numbers n and t, where $t \leq n$, we say t is a totative, if $g.c.d(n, t) = 1$. In other words, n and t are coprimes.

Definition 2: [12] The totient function of Euler, $\theta(n)$, is the number of totatives of n.

Lemma 3: [12] *The statement below is valid:*

$$\sum \theta(d) = n, d/n.$$

Definition 4: [7] The graph whose vertices are the elements of R such that there is an edge between the vertices u and v if and only if $uv = 0$, is known as a zero-divisor graph of a commutative ring R, $\Pi(R)$.

Notation 5: The ring Z_n of a number n is the ring modulo n, that is created from the congruence classes. It may be understood in a variety of ways, such as by the set $\{0, \dots, n-1\}$. Being identical in Z_n is similar to congruence modulo n. We will utilize a natural number $n = p_1^{r_1} \times p_2^{r_2} \times p_3^{r_3} \times \cdots \times p_\alpha^{r_\alpha}$ for a ring Z_n all along.

Definition 6: [3] We say that n is exponent-free, if $n = p_1 \times p_2 \times \cdots \times p_\alpha$.

3. Auxiliary Results

3.1 *The Equivalence Classes of* $\Pi(R)$

To have a good image of $\Pi(R)$, R is for whatever commutative ring R, we will propose an equivalence relation $\simeq$ on its base set V. Firstly, we define the general concept for this equivalence relation for an arbitrary graph.

Definition 7: [3] For a graph G, ac$\in E \Leftrightarrow bc \in E$; $\forall$ other vertex $z \in R/\{a, b\}$. Thus, a$\simeq b$, where a, b, and c are the vertices belong to $V(G)$.

Remark 8: Proving the transitivity of $\simeq$ is a simple task.

Upon examining the (most fascinating) specific situation in which $\Pi(R)$, we now obtain.

Definition 9: [3] For other element $c \in R/\{a, b\}$, $ac = 0 \Leftrightarrow bc = 0$, $\forall\, a, b \in R$, then $a \simeq b$.

Remark 10: In our work, loops are prohibited in $\Pi(R)$, as we said in the Section 2 above. This explains why ($z = x$, and $z = y$) are not included.

Definition 11: [3] $EC(x)$ is the class of x in $\simeq$ for $x \in R$. A subset of the shape $EC(x)$ will have a name of either class or partial subset. We will utilize the well-known ring, Z_n, where ($n \geq 2$).

Notation 12: [3] Suppose k be any proper divisor, k/n. The auxiliary subset of Z_n will be denoted by $A(k)$, where $A(k) := \{\lambda k : 1 \leq \lambda \leq \frac{n}{k}, (\lambda, \frac{n}{k}) = 1\}$.

Theorem 13: [3] *The sets $A(k)$ where k runs across all proper divisors of n (in other meaning, $EC(k) = A(k)$) are the equivalence classes for the relation $\simeq$. Moreover, the one-element class* $\{0\}$.

Remark 14: Even though zero-divisors are specified as non-zero, practically speaking, it is preferable to retain the zero vertex of the ring in the $\Pi(R)$ and place it next to everything.We acquire the set of all totatives of n for $k = 1$. Actually, the two straightforward claims that follow are really helpful in understanding the composition of our $\Pi(Z_n)$. Take note that they apply to any type of G.

Proposition 15: [3] Given two distinct classes S_1 and S_2 with regard to $\simeq$, let us assume the following: we identify an edge $e = uv$, s.t., $u \in S_1$ and $v \in S_2$. Next, every vertex in S_1 and every vertex in S_2 are join to each other. Conversely, it is evident that any vertex in S_1 and any vertex in S_2 are not neighboring if this uv is not an edge.In every class S and for every simple G, S is either an independent set or a clique.

Definition 16: [3] We say that S_1 is adjacent to S_2 if S_1 and S_2 are two distinct partial subsets with every vertex $v_1 \in S_1$ and, any vertex $v_2 \in S_2$ being neighboring. The Application of Theorem 13 above for any element y in Z_n is now under consideration.

Theorem 17: [3] *Let y be an arbitrary element of Z_n, and $k := (y, n)$. Then $EC(y) = EC(k)$.*

Representation of the zero-divisor graph of the ring Z_n [3]

We will say $\Pi(Z_n)$, for the zero-divisor graph of Z_n, we will explain it more in below: $\theta(\frac{n}{k})$ vertices represents the class $EC(k)$ as a circle, as you can see in Figure 1. This circle a figure only, it is not a subgraph. It illustrates a good picture for $\Pi(Z_n)$, Which makes understanding easier for all readers.

Claim 18: [3] $k = p_1^{g_1} \times p_2^{g_2} \times p_3^{g_3} \times \cdots \times p_\alpha^{g_\alpha}$, $g_j \geq \left\lceil \frac{r_j}{2} \right\rceil$, $1 \leq g_j \leq r_j$, $1 \leq j \leq \alpha$ if and only if $k^2 \cong 0$ modulu n.

Lemma 19: [3] *Consider $n = p_1^{r_1} \times p_2^{r_2} \times p_3^{r_3} \times \cdots \times p_\alpha$, $k = p_1^{g_1} \times p_2^{g_2} \times p_3^{g_3} \times \cdots \times p_\alpha^{g_\alpha}$, and $k|n$. As usual, $S = EC(k)$. A clique is induced in $\Pi(Z_n)$ by S. If $1 \leq g_j \leq r_j$, $1 \leq j \leq \alpha$, then $g_j \geq \left\lceil \frac{r_j}{2} \right\rceil$. In the absence of this, S generates an empty graph and ($g_j < \left\lfloor \frac{r_j}{2} \right\rfloor$).*

Proof: Considering two numbers in S, $v_1 = \lambda_1 \cdot k$ and $v_2 = \lambda_2 \cdot k$, such that $k = p_1^{r_1/2} \times p_2^{r_2/2} \times p_3^{r_3/2} \times \cdots \times p_\alpha^{r_\alpha/2}$, then $v_1 \cdot v_2 = \lambda_1 \cdot \lambda_2 (p_1^{r_1/2} \times p_2^{r_2/2} \times p_3^{r_3/2} \times \cdots \times p_\alpha^{r_\alpha/2})(p_1^{r_1/2} \times p_2^{r_2/2} \times p_3^{r_3/2} \times \cdots \times p_\alpha^{r_\alpha/2}) = 0 \; modulo \; n$. Then, there exists an edge $e = uv$ between any two vertices u and v belong to S.

Remark 20: [3] Our figures do not indicate if a class in $\Pi(Z_n)$ causes an independent set or a clique.

Remark 21: [3] Assume for the moment that n_1 and n_2 have the same number of powers and primes. Then, the two matched graphs on them differ only in class sizes; they have the same number of classes.Moreover, $\Pi(Z_{n_1})$ and $\Pi(Z_{n_2})$ are depicted (drawn) in a same way. Figure 1 represents a way of drawing for $\Pi(Z_n)$, where n is a value of the shape $n = p_1 \times p_2 \times p_3$.

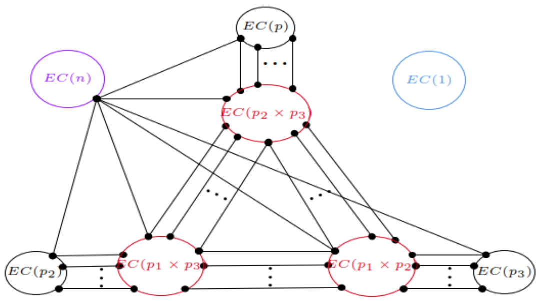

Figure 1
Illustration of Remark 21, $\Pi(Z_n), n = p_1 \times p_2 \times p_3$.

Remark 22: [3] Henceforth, two classes $EC(n)$, and $EC(1)$; where $EC(n) = \{0\}$, it is join with every vertex of $\Pi(Z_n)$, and $EC(1)$ is the set of all totatives of n (isolated vertices) will not be shown in our statements and Figures. Since we will be using them to prove Theorem, we will mention the corollary and its proof below 36.

Corollary 23: [3] *Assume n be exponent-free, then, $\exists$ a clique k in Π, s.t., $|k| = \alpha$ is the cardinality of this clique.*

Proof: We can divide (the product of $\frac{n}{p_i}$ and $\frac{n}{p_j}$) by n. As a result, $T_i = EC(\frac{n}{p_i})$, and $T_j = EC(\frac{n}{p_j})$ are join to each other, where $\forall\, 1 \leq i$ and $j \leq \alpha$. One vertex from each T_i was taken by the authors to create k. α will be a size of it.

Corollary 24: [3] *If $n = p^r$, s.t., the classes will be $EC(p)$, $EC(p^2), \cdots, C(p^{r-1})$, then the statements below are valid:*

(i) let r be an even, the clique number is: $\omega(\Pi) = \sum \, |EC(p^i)|, \left\lceil \frac{r}{2} \right\rceil \leq i \leq r-1$

(ii) *let r is odd, then the clique number is:* $\omega(\Pi) = 1 + \sum \left|EC(p^j)\right|, \left\lceil \frac{r}{2} \right\rceil \leq j \leq r - 1.$

Definition 25: [3] The union of all clique-classes is cl_{Π}.

Corollary 26: [3] *cl_{Π} is a clique.*

Lemma 27: *$\Pi(Z_n)$ is chordal when n is a prime power of p^r.*

Definition 28: [3] Suppose L be a class, let us delete the zero class $EC(n)$, L is join with exactly one class, B. Thus, L is referred to as a leaf-class. Moreover, B is said to be a branch-class.

Lemma 29: [3] *The leaf-classes for Z_n are $L_i := EC(p_i)$, and $T_i := EC(n/p_i)$ are the branch-classes, where $i = 1,2, \ldots, \alpha$.Moreover, for every one branch-class, there is exactly one leaf-class adjacent to it.In particular, if $n = p^r$ and $r \geq 3$. Subsequently, there is one leaf-class, $EC(p)$, and one branch-class, $EC(p^{r-1})$, as Figure 2 illustrates. If $r = 2$, then it is evident that there is no leaf-class.*

Corollary 30: [3] *Now, allow us to examine $\Pi(Z_n)$, where $n \neq p, p^2$. After that, each leaf-class functions independently, and each branch-class has the structure $T_i = EC(\frac{n}{p_i}) = Sub\{p_1^{r_1} \times p_2^{r_2} \times \cdots \times p_i^{r_i-1} \times \cdots \times p_\alpha^{r_\alpha}\}$ is a clique-class, where $p_i^{r_i-1} > 1$. It is easy to observe that the following statement is accurate.*

Observation 31: [3] Every class in $\Pi(Z_n)$ is an independent class if n is exponent-free.

Lemma 32: [3] *Now, allow us to examine $n = p_1^r \times p_2 \times \cdots \times p_\alpha$, where $r = 2$ or $r = 3$. subsequently there exists a clique cl_{Π}, s.t., U_r is the unique clique-class in $\Pi(Z_n)$, so:*

$$|V(cl_{\Pi})| = |U_r| + 1,$$

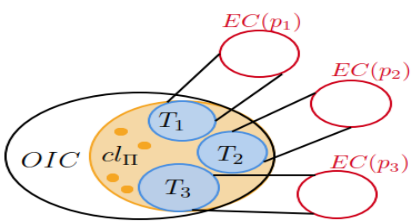

Figure 2
Represents $\Pi(Z_n)$, (OIC) means other independent classes

3.2 *About Domination of* $\mathbf{\Pi}(Z_n)$

This section included a few definitions and findings.

Definition 33: [1] A set $D \subseteq V(G)$ in $G = (V, E)$ is called a dominating set if every vertex in $V(G)$ is either an element of D or is adjacent to the element of D.

Definition 34: [1] The minimal cardinality of a minimal dominating set of G referred to as a γ-set of G is the domination number ($\gamma(G)$) of that G.

Proposition 35: [4, 5] $(i) \forall G$ of order m, $\gamma(G) = 1$ if $\Delta(G) = m - 1$. Especially, $\gamma(K_m) = 1$ (ii) $\gamma(\overline{K_m}) = m$. (iii) If $p = 1$, then $\gamma(K_{p,q}) = 1$; otherwise $\gamma(K_{p,q}) = 2$, where $1 \leq p \leq q$.

4. Main Result

Theorem 36: *Let us* $\Pi = \Pi(Z_n)$, $n \geq 10$. *If* $n = p_1 \times p_2$, *and* $p_1 = 2$, *then* $\gamma(\Pi) = 1$. *Otherwise,* $\gamma(\Pi) = \alpha$.

Proof: Based on the value of n, there are four scenarios that may be used to demonstrate this theorem, as follows:

Case 1: The class $EC(p_2)$ will have one vertex, p_2 (Definition 11, Notation 12, and Theorem 13), if $n = p_1 \times p_2$ and $p_1 = 2$. Then, Π is isomorphic to $K_{1,q}$ since the vertex p_2 is adjacent to every vertex in the class $EC(p_1)$. According to Proposition (35 iii), $\gamma(\Pi) = 1$.

Case 2: There are two sub-cases if $n = p^r$, or $\alpha = 1$. These are indicated tow subcases: (i) $\Pi(Z_{p^2})$ will be complete if $r = 2$, in which case $\gamma(\Pi(Z_{p^2})) = 1$, Proposition 35. (ii) There is one vertex v in clique-class $EC(p^{r-1})$ (Corollary 24 and Lemma 29) that dominates on all the other vertices in the $\Pi(Z_n)$ Π if $r \geq 3$. Therefore, $\Pi(Z_{p^r})$ is a chordal (Lemma 27).

Case 3: According to $n = p_1 \times p_2 \times \cdots \times p_\alpha$, if n is exponent-free. Additionally, there are two sub-cases: $(\boldsymbol{i})$ Π will be isomorphic to $K_{p,q}$ with $p \neq 1$ if $\alpha = 2$, which is $n = p_1 \times p_2$ (here, $p_1 \neq p_2 \neq 1$). Therefore, $\gamma(K_{p,q}) = 2 = \alpha$ (using Proposition (35. iii)). $(\boldsymbol{ii})$ $\alpha \geq 3$ allows us to select one vertex (v_i) from each independent branch-class T_i (see Corollary 23 and its proof), such that the vertex v_i will dominate on all vertices of the leaf-class $EC(p_i)$ (Lemma 29) and on all vertices of the other branch-classes T_j, $j \neq i$ (when $n = p_1 \times p_2 \times \cdots \times p_\alpha$), As can be seen from Observation 31, each class is autonomous. About the other vertices in other independent classes, all classes $EC(d)$ are connected to T_i such that $p_i | d$, where d is a valid divisor of n. Thus, the vertex v_i in T_i may dominate on $EC(d)$ vertices. $\gamma(\Pi) = \alpha$, since the number of branch-classes equals α (Corollary 23).

Case 4: In this instance, where $n = p_1^{r_1} \times p_2^{r_2} \times p_3^{r_3} \times \cdots \times p_\alpha^{r_\alpha}$ (Lemma 29 and Corollary 30) state that we can select one vertex (v_i) from each clique branch-class T_i such that the vertex v_i dominates on all vertices of the leaf-class $EC(p_i)$ (Lemma 29) and on all vertices of the same branch clique-class T_i. Because all classes $EC(d)$ are related to T_i, $p_i|d$, where d is a valid divisor of n, is achieved. Thus, v_i in T_i can dominate on $EC(d)$ vertices. Lemma 29 makes it clear that there are α leaf-classes; so, $\gamma(\Pi) = \alpha$. In particular, $n = p_1^{r_1} \times p_2 \times p_3 \times \cdots \times p_\alpha$, where $2 \leq i \leq \alpha$, Hence, as stated in Lemma 32, there is only one clique branch-class and all other branch-classes are independent.

Example 37: The Graph $\Pi(Z_{200})$ is given by $200 = 2^3 \times 5^2$. This graph's classes are $EC(2)$ and $EC(2^2)$, $EC(2^3)$, $EC(5)$, $EC(5^2)$, $EC(2 \times 5)$, and $EC(2 \times 5^2)$, we have $T_1 = EC(2^2 \times 5^2)$, $T_2 = EC(2^3 \times 5)$, and $EC(2^2 \times 5)$. The leaf-classes in $\Pi(Z_{200})$ are $EC(2)$ and $EC(5)$, whereas the branch-classes are T_1 and T_2. $\gamma(\Pi(Z_{200})) = 2$ because the vertices v_1 and v_2 in the classes T_1 and T_2 dominate on all the vertices in other classes. The domination of vertices v_1 and v_2 alone is depicted in Figure 3; no other connections are shown. However, this image does not depict the additional adjacency between certain other classes in $\Pi(Z_{200})$.

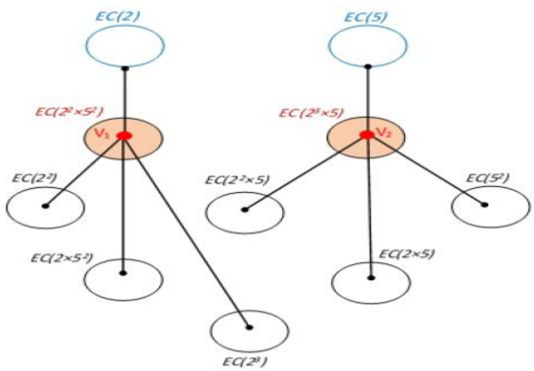

Figure 3
Illustrations the domination of $\Pi(Z_n), \Pi(Z_{200})$.

References

[1] M. N. Al-Harere, A. A. Omran, and A. T. Breesam, "Captive domination in graphs," *Discrete Mathematics, Algorithms and Applications,* vol. 12, no. 6 (2020).

[2] H. Al-Janabi and G. Bacsó, "Integral-root Polynomials and Chromatic Uniqueness of Graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 24, no. 4, pp. 1127–1147 (2021).

[3] H. Al-Janabi and G. Bacsó, "Chromatic Uniqueness of Zero-Divisor Graphs," T*he Art of Discrete and Applied Mathematics,* vol. 6, no. 1 (2022).

[4] Z. Y. Alrikabi and A. A. Omran, "Examining Captive and Inverse Captive Domination in Selected Graphs and Their Complements," *Math Model Eng Probl.,* vol. 10, no. 5 (2023).

[5] Z. Y. Alrikabi and A. A. Omran, "Restrained Captive Domination Number," *Open Engineering,* vol. 14 (2024).

[6] D. F. Anderson and P. S. Livingston, "The Zero-Divisor Graph of a Commutative Ring," *Journal Algebra*, vol. 217, no. 2, pp. 434–447 (1999).

[7] I. Beck, "Coloring of Commutative Rings," *Journal Algebra,* vol. 116, no. 1, pp. 208–226 (1988).

[8] C. Berge, The Theory of Graphs and Its Applications, Methuen and Co. Ltd., London (1962).

[9] E. J. Cockayne and S. T. Hedetniemi, "Towards a Theory of Domination in Graphs," *Networks,* pp. 247–261, vol. 7 (1977).

[10] R. Diestel, Graph Theory, 5th ed. Springer-Verlag (2017).

[11] F. M. Dong, K. M. Koh, and K. L. Teo, Chromatic Polynomial and Chromaticity of Graphs, World Scientific Publishing Co. Pte. Ltd., Singapore (2005).

[12] A. J. Hildebrand, Elementary Number Theory, Definitions and Theorems (Class Notes), University of Illinois at Urbana-Champaign (2011).

[13] S. H. Jafari and N. J. Jafari Rad, "On Domination of Zero-Divisor Graphs of Matrix Rings," *Canadian Mathematical Bulletin*, vol. 58, no. 2, pp. 271–275 (2015). http://dx.doi.org/10.4153/CMB-2015-017-6.

[14] K. M. Koh, F. Dong, K. L. Ng, and E. G. Tay, Graph Theory Undergraduate Mathematics (2015).

[15] K. R. Kumar and J. Mathew, "Hyperstar Decomposition of r-Partite Complete, Knodel and Fibonacci Hypergraphs," *Journal of Physics: Conference Series,* vol. 1850, p. 012-018 (2021).

[16] S. B. Mulay, "Cycles and Symmetries of Zero-Divisors," Communications in Algebra, vol. 30, no. 7, pp. 3533–3558 (2002).

[17] N. J. Rad, S. H. Jafari, and D. A. Mojdeh, “On Domination in Zero-Divisor Graphs,” *Canadian Mathematical Bulletin,* vol. 56, no. 2, pp. 407–411 (2013).

[18] S. Shipiya Raj Shree, V. Maheswari, and R. Rajeswari, “Total and Co-total Domination Number of Some Zero Divisor Graph,” *International Journal of Recent Technology and Engineering* (IJRTE), vol. 8, no. 3S2 (2019).

[19] S. Ahmad, H. M. A. Siddiqui, A. Ali, M. R. Farahani, M. Imran, and I. N. Cangul, “On Wiener Index and Wiener Polarity Index of Some Polyomino Chains,” *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 22, no. 7, pp. 1151–1164 (2019). [Online]. Available: https://doi.org/10.1080/09720529.2019.1688965.

[20] M. Alaeiyan, F. Afzal, M. R. Farahani, and M. A. Rostami, “An Exact Formula for the Wiener Polarity Index of Nanostar Dendrimers,” *Journal of Information and Optimization Sciences,* vol. 41, no. 4, pp. 933–939 (2020). [Online]. Available: https://doi.org/10.1080/02522667.2020.1748274.

[21] S. Ediz, İ. Çiftçi, M. Cancan, and M. R. Farahani, “On k-Total Distance Degrees and k-Total Wiener Polarity Index,” *Journal of Information and Optimization Sciences,* vol. 42, no. 7, pp. 1469–1477 (2021). [Online]. Available: https://doi.org/10.1080/02522667.2021.1896652.

[22] W. Gao and M. R. Farahani, “The Hyper-Zagreb Index for an Infinite Family of Nanostar Dendrimers,” *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 20, no. 2, pp. 515–523 (2017). [Online]. Available: https://doi.org/10.1080/09720529.2016.1220088.

[23] M. Nadeem, S. Ahmad, M. K. Siddiqui, M. A. Ali, M. R. Farahani, and A. J. M. Khalaf, “On Some Applications Related with Algebraic Structures Through Different Well-Known Graphs,” *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 24, no. 2, pp. 451–471 (2021). [Online]. Available: https://doi.org/10.1080/09720529.2021.1885806.

[24] A. H. Alwan, “Small Intersection Graph of Subsemimodules of a Semimodule,” *Communications in Combinatorics, Cryptography & Computer Science*, vol. 2022, no. 1, pp. 15–22 (2022).

[25] S. Ediz, M. Cancan, and M. R. Farahani, “Eccentric Connectivity and Connective Eccentricity Indices of Generalized Petersen Graphs,” *Communications in Combinatorics, Cryptography & Computer Science*, vol. 2021, no. 1, pp. 1–6 (2021).

[26] R. Ponraj, S. Prabhu, and M. Sivakumar, "Pair Mean Cordial Labeling of Total Graph of Some Graphs and Prism Related Graphs," *Communications in Combinatorics, Cryptography & Computer Science*, vol. 2024, no. 2, pp. 181–192 (2024).

[27] F. O. Oduol and I. O. Okoth, "Counting Vertices Among All Non-Crossing Trees by Levels and Degrees," *Communications in Combinatorics, Cryptography & Computer Science,* vol. 2024, no. 2, pp. 193–212 (2024).

Received May, 2024