

Diagonal matrix of the tensor product ($\otimes^* D_p$), $p \geq 3$ is prime number

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Abstract

Several concepts are presented within this article that will be utilized in the diagonal matrix calculation and compute the $(\otimes^{\omega} (\otimes^* D_p))$, $p \geq 3$ is prime number. Where $\otimes^{\omega} (\otimes^* D_p)$ is indicate for the tensor product of the rational character table matrix of dihedral group D_p by itself.

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1. Introduction

As well as Dunya M.H and Intider Z.M in [3,4] they computed the invariant factors of the tensor product of the $\otimes^n (\otimes^* C_{\alpha^3})$. Also authors in [5-8] studied the same idea for $PSL(2,p)$ and $SL(2,p)$ where $p = 31, 37$ and $SL(2,49)$. Finding the general form of $(\otimes^{\omega} (\otimes^* D_p))$, $p \geq 3$ prime number is the paper's purpose.

2. Preliminaries

Definition 2.1: [5] A ring R is known as principal ideal domain (P.I.D) if R integral domain and each ideal is principal ideal.

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Definition 2.2: [1] The tensor product, or Kronecker Product of two matrices

$$Q = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1s} \\ q_{21} & q_{22} & \dots & q_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ q_{vs} & q_{v1} & \dots & q_{vs} \end{bmatrix}_{v \times s} \quad \text{and} \quad K = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1t} \\ k_{21} & k_{22} & \dots & k_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ k_{wt} & k_{w2} & \dots & k_{wt} \end{bmatrix}_{w \times t} \quad \text{over a field } R$$

is symbols by $Q \otimes K$ and is define by,

$$Q \otimes K = \begin{bmatrix} q_{11}K & q_{12}K & \dots & q_{1n}K \\ q_{21}K & q_{22}K & \dots & q_{2n}K \\ \vdots & \vdots & \ddots & \vdots \\ q_{vs}K & q_{v1}K & \dots & q_{vs}K \end{bmatrix}_{vw \times st}$$

Theorem 2.1: [5] Let $\mathbb{N}$ be $w \times w$ matrix over R , where R is P. I. D, then $\exists$ non-singular matrices S, C and D such that.

- $S \mathbb{N} C^{-1} = D$ is known as diagonal matrix
- If we refer D_u by d_u , then $\exists \sigma \in \mathbb{Z}^+$ such that $0 \leq \sigma \leq w$ and $\ell > \sigma$, then $d_\ell = 0$ and if $\ell \leq \sigma$ hence $d_\ell \neq 0$ and $1 \leq \ell \leq \sigma$ implies $d_\ell | d_{\ell-1}$.

Definition 2.3:[5] Let R be a P.I.D, if $\mathbb{N}$ is a matrix over R and $\mathbb{N}$ equivalent to a matrix $D = \text{Diag} \{ d_1, d_2, \dots, d_\sigma, 0, 0, \dots, 0 \}$ such that $d_\ell | d_{\ell+1}$ for $1 \leq \ell \leq \sigma$, then D is known as invariant factor matrix of $\mathbb{N}$.

Theorem 2.2:[2] If $p \neq 2$ is a prime number and I_{r-1} is identity matrix of size $r-1$ and let A and B be two matrices

$$A = \begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 1 & 1 \\ \begin{bmatrix} I_{r-1} \end{bmatrix} & 0 & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 & 1 \end{pmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ (p-1) & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ p(p-1) & (p-1) & 1 & \dots & 0 & 0 & 0 & 0 \\ p^2(p-1) & p(p-1) & (p-1) & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ p^{r-2}(p-1) & p^{r-3}(p-1) & p^{r-4}(p-1) & \dots & (p-1) & 1 & 0 & 0 \\ -p^{r-1}(p-1) & -p^{r-2} & -p^{r-3} & \dots & -p & -1 & 1 & 0 \\ p^r & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $A (\equiv D_p) B = \text{Diag} \{ 2p^r, p^{r-1}, p^{r-2}, \dots, p, 1, 1 \}$.

Theorem 2.3: [5] Let S, C be matrices such that $|S| \neq 0$ and $|C| \neq 0$ over R such that R is P.I.D and if $Y_1 S X_1 = \text{Diag} \{d_1(S), d_2(S), \dots, d_n(S)\} = D(S)$ and $Y_2 C X_2 = \text{Diag} \{d_1(C), d_2(C), \dots, d_n(C)\} = D(C)$. Hence, $(Y_1 \otimes Y_2).(S \otimes C).(X_1 \otimes X_2) = D(S) \otimes D(C)$.

Theorem 2.4: [2] The rational character table of $D_p (\equiv^* D_p)$ is given by

QCT(D_p)	1	R	s
$\emptyset_1$	$p-1$	-1	0
Ψ_2	1	1	1
Ψ_3	1	1	-1

3. Diagonal Matrix of The Tensor Product ($\equiv^* D_p$), $p \geq 3$ is prime number

Everywhere in this essay we are using Th.(2.3) to calculate the diagonal matrix of $(\otimes (\equiv^* D_p))$, s.t $\otimes (\equiv^* D_p) = \underbrace{(\equiv^* D_p) \otimes \dots \otimes (\equiv^* D_p)}_{\leftarrow \omega\text{-times} \rightarrow}$. By Theorem

(2.4) we have, $(\equiv^* D_p) = \begin{pmatrix} p-1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. And by utilizing Theorem (2.2)

and Th(2.3), we can found two non-singular matrices $S = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ -(p-1) & 1 & 0 \\ p & 0 & -1 \end{pmatrix},$$

Hence, $S(\equiv^* D_p)C = \begin{pmatrix} 2p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{Diag} \{2p, 1, 1\}$. Now, we take the

case where $\omega=2$, hence

$$(S \otimes S).(\otimes (\equiv^* D_p)) \otimes (\otimes (\equiv^* D_p)).(C \otimes C) = (\otimes^2 S).(\otimes^2 ((\equiv^* D_p))).(\otimes^2 C) = D(\otimes^2 (\equiv^* D_p))$$

$$\begin{pmatrix} 4p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{9 \times 9} = \text{Diag} \{4p^2, 2p, 2p, 2p, 2p, 1, 1, 1, 1\}$$

If we take $\omega=3$, hence

$$(\otimes^3 S).(\otimes^3 (\equiv^* D_p)).(\otimes^3 C) = \text{Diag} \{ \underbrace{2^3 p^3}_{\leftarrow 6 \rightarrow}, \underbrace{2^2 p^2, \dots, 2^2 p^2}_{\leftarrow 12 \rightarrow}, \underbrace{2p, \dots, 2p}_{\leftarrow 8 \rightarrow}, \underbrace{1, \dots, 1}_{\leftarrow 8 \rightarrow} \} = D(\otimes^3 (\equiv^* D_p)).$$

$$\begin{aligned}
& \text{In case } \omega=4 \Rightarrow (\otimes^4 S).(\otimes^4 (\equiv^* D_p)).(\otimes^4 C)=\text{Diag}\{2^4 p^4, \\
& \underbrace{2^3 p^3, \dots, 2^3 p^3}_{\leftarrow 8 \rightarrow}, \underbrace{2^2 p^2, \dots, 2^2 p^2}_{\leftarrow 24 \rightarrow}, \underbrace{2p, \dots, 2p}_{\leftarrow 32 \rightarrow}, \underbrace{1, \dots, 1}_{\leftarrow 16 \rightarrow}\} = \mathcal{D}(\otimes^4 (\equiv^* D_p)) \\
& \text{If } \omega=5 \Rightarrow (\otimes^5 S).(\otimes^5 (\equiv^* D_p)).(\otimes^5 C)=\text{Diag}\{2^5 p^5, \underbrace{2^4 p^4, \dots, 2^4 p^4}_{\leftarrow 10 \rightarrow}, \\
& \underbrace{2^3 p^3, \dots, 2^3 p^3}_{\leftarrow 40 \rightarrow}, \underbrace{2^2 p^2, \dots, 2^2 p^2}_{\leftarrow 80 \rightarrow}, \underbrace{2p, \dots, 2p}_{\leftarrow 80 \rightarrow}, \underbrace{1, \dots, 1}_{\leftarrow 32 \rightarrow}\} = \mathcal{D}(\otimes^5 (\equiv^* D_p)) \\
& \text{If } \omega=6 \Rightarrow (\otimes^6 S).(\otimes^6 (\equiv^* D_p)).(\otimes^6 C)=\text{Diag}\{2^6 p^6, \underbrace{2^5 p^5, \dots, 2^5 p^5}_{\leftarrow 12 \rightarrow}, \\
& \underbrace{2^4 p^4, \dots, 2^4 p^4}_{\leftarrow 60 \rightarrow}, \underbrace{2^3 p^3, \dots, 2^3 p^3}_{\leftarrow 160 \rightarrow}, \underbrace{2^2 p^2, \dots, 2^2 p^2}_{\leftarrow 240 \rightarrow}, \underbrace{2p, \dots, 2p}_{\leftarrow 192 \rightarrow}, \underbrace{1, \dots, 1}_{\leftarrow 64 \rightarrow}\} = \mathcal{D} \\
& (\otimes^6 (\equiv^* D_p)).
\end{aligned}$$

The general of the whole of $\mathcal{D}(\otimes^\omega (\equiv^* D_p))$, where $p \geq 3$ is prime number, $\omega \in \mathbb{Z}^+$ is to offered in below theorem:

Theorem 3.1: If $p \geq 3$ is prime number and D_p is dihedral group, $\omega \in \mathbb{Z}^+$ then

$$\begin{aligned}
& \mathcal{D}(\otimes^\omega (\equiv^* D_p)) = \text{Diag}\{ \underbrace{(2p)^\omega}_{\leftarrow 2 \cdot \left(\frac{\omega!}{(\omega-1)!}\right) - \text{times} \rightarrow}, \underbrace{(2p)^{\omega-1}, \dots, (2p)^{\omega-1}}_{\leftarrow 2^2 \cdot \left(\frac{\omega!}{2!(\omega-2)!}\right) - \text{times} \rightarrow}, \underbrace{(2p)^{\omega-2}, \dots, (2p)^{\omega-2}}_{\leftarrow 2^3 \cdot \left(\frac{\omega!}{3!(\omega-3)!}\right) - \text{times} \rightarrow}, \\
& \underbrace{(2p)^{\omega-3}, \dots, (2p)^{\omega-3}}_{\leftarrow 2^4 \cdot \left(\frac{\omega!}{4!(\omega-4)!}\right) - \text{times} \rightarrow}, \dots, \underbrace{(2p)^{\omega-4}, \dots, (2p)^{\omega-4}}_{\leftarrow 2^{\omega-1} \cdot \left(\frac{\omega!}{(\omega-1)!(\omega-(\omega-1)!}\right) - \text{times} \rightarrow}, \\
& \underbrace{(2p), \dots, (2p)}_{\leftarrow 2^{\omega-1} \cdot \left(\frac{\omega!}{(\omega-1)!(\omega-(\omega-1)!}\right) - \text{times} \rightarrow}, \underbrace{1, \dots, 1}_{\leftarrow 2^\omega - \text{times} \rightarrow} \} = (\otimes^\omega S).(\otimes^\omega (\equiv^* D_p)).(\otimes^\omega C)
\end{aligned}$$

Proof: The phrase is undoubtedly correct for $\omega = 1$, according to an inductive argument assuming it is valid for an arbitrarily chosen ω

$$\begin{aligned}
& \mathcal{D}(\otimes^\omega (\equiv^* D_p)) = \text{Diag}\{ \underbrace{(2p)^\omega}_{\leftarrow 2 \cdot \left(\frac{\omega!}{(\omega-1)!}\right) - \text{times} \rightarrow}, \underbrace{(2p)^{\omega-1}, \dots, (2p)^{\omega-1}}_{\leftarrow 2^2 \cdot \left(\frac{\omega!}{2!(\omega-2)!}\right) - \text{times} \rightarrow}, \\
& \underbrace{(2p)^{\omega-2}, \dots, (2p)^{\omega-2}}_{\leftarrow 2^3 \cdot \left(\frac{\omega!}{3!(\omega-3)!}\right) - \text{times} \rightarrow}, \underbrace{(2p)^{\omega-3}, \dots, (2p)^{\omega-3}}_{\leftarrow 2^4 \cdot \left(\frac{\omega!}{4!(\omega-4)!}\right) - \text{times} \rightarrow}, \\
& \dots, \underbrace{(2p), \dots, (2p)}_{\leftarrow 2^{\omega-1} \cdot \left(\frac{\omega!}{(\omega-1)!(\omega-(\omega-1)!}\right) - \text{times} \rightarrow}, \underbrace{\{1, 1, \dots, 1\}}_{\leftarrow 2^\omega - \text{times} \rightarrow} \} = (\otimes^\omega S).(\otimes^\omega (\equiv^* D_p)).(\otimes^\omega C)
\end{aligned}$$

Now, it must be proven that it correct for $\omega + 1$. However, we immediately obtain this from Th.(2.3), hence $(\otimes^{\omega+1} (\equiv^* D_p)) = (\otimes^\omega (\equiv^* D_p)) \otimes (\equiv^* D_p)$. Therefore $\mathcal{D}(\otimes^{\omega+1} (\equiv^* D_p)) = \mathcal{D}(\otimes^\omega (\equiv^* D_p)) \otimes (\mathcal{D}(\equiv^* D_p))$.

Conclusion

This paper is to study and submit the general formula of the diagonal matrix of the tensor product of the $(\equiv^* D_p)$.

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