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Co-regular captive domination number γ_{crg} in graphs

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Abstract

This paper initiated the new domination number, which is named the co-regular captive, it is linked to two numbers of dominations they are captive domination number (CDN) and r-regular captive domination number. Many of the bounded, properties, and theorems of this domination are determined.

Subject Classification: *68Rxx, 68-XX, 68Uxx.*

Keywords: *Co-regular captive domination number, Co-regular captive dominating set.*

1. Introduction

Consider G to be a undirected, simple, and finite graph [2-3]. A subset D of the vertex set V in a graph $G = (V, E)$ is named a dominating set (DS) if every vertex $v \in V - D$ belongs to at least one neighborhood of a vertex in the set D. Furthermore, the minimum cardinality of all DSs is namely the domination number denoted by $\gamma(G)$. A DS has $\gamma(G)$ belongs to the γ −set of G[2]. Also, a DS D is called a captive (CDS) if each vertex in the set D has two neighborhoods at least one of them in the set D and the others in the set $V - D$. A gain the minimum cardinality of all CDSs is namely the captive domination number denoted by $\gamma_{ca}(G)$ [1]. After this Z. Y. Alrikabi, A.A. Omran presented new conditions of captive domination [7-17].The co-regular domination number is presented by linking two concepts: the captive domination number and r-r-regular domination number. For a new concept, many bounded, properties, examples, and theorems are been proved. There are many fields in mathematics studied with graph theory, especially with domination as topological graph, fuzzy graph [4-6], and general graph. We have come up with new Observation, every support vertex $v, v \in D$, if G contains a component that isomorphic to a path graph of order two or three, then this graph has no CDS, and a star graph has no CDS for all its order.

2. Co-regular Captive domination number

Definition 3.1: Consider that G be a graph, and a subset $D \subset V(G)$ is named co-regular CDS (CO-RCDS) in G if $\langle V - D \rangle$ is regular and D is CDS.A subset $D \subset V(G)$ is a minimal CO-RCDS if it has no proper CO-RCDS. And the minimum cardinality of all minimal CO-RCDS of G denoted by $\gamma_{cgr}(G)$ is called the CO-RC domination number.

Proposition 3.2: If G has co-regular captive domination number γ_{crg} and G has pendent vertex then $\langle V - D \rangle$ is 0-regular induced subgraph.

Proof: Since every support vertex must be in D this means every pendent vertex must be in set $V - D$ and since G has co-regular captive domination number γ_{crg} then $\langle V - D \rangle$ is null induced subgraph, therefore $r = 0$.(As an example, see Figure 1 and Figure 2)

Proposition 3.5: If G is a graph isomorphic to path. Then $\gamma_{crg}(P_n) = \left\{ \begin{matrix} 2, if\ m = 4 \\ \left\lfloor \frac{2n}{3} \right\rfloor, if\ n \equiv 1(mod\ 3) \end{matrix} \right\}$. Otherwise G has no captive co-regular dominating set.

Proof: By definition of a captive dominating set, the order of the graph is greater than or equal to three. Furthermore, if $n = 3$, then it is obvious that the graph P_3 has no captive dominating set. Now, let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of the path graph. To create a co-regular captive dominating set, the vertex v_1 cannot belong to a co-regular captive dominating set, if not the vertex v_2 must belong to this dominating to keep the total property. In this case the vertex v_1 not dominates any vertex in $V - D$. Thus, the set D is not captive. Therefore, the vertex v_2 is the first vertex belonging to the set D, also the vertex v_3 belongs to D to maintain the total property and leave the vertex v_4 to maintain the condition of captive. Now, since the vertex v_1 belongs to $V - D$ and it is isolated in $V - D$, this dominating is 0-regular to maintain r-regular. Therefore, the strategy for choosing the co-regular captive dominating set is to take two vertices and leave one and this is the only strategy to create D. So, four parts classification:

I: If $n = 4$, so the co-regular captive dominating set is $D = \{v_2, v_3\}$. Thus, the result is obtained.

II: Suppose that $n \equiv 1(mod\ 3)$, so let $D = \left\{v_{2+3i}, v_{2+3i}; i = 0,1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$, this set is dominating to all other vertices in V. Also, maintain the above strategy, so it is co-regular captive dominating set of 0-regular and it is the minimum. Thus, $\gamma_{crg}(P_n) = \lfloor D \rfloor = \left\lfloor \frac{2n}{3} \right\rfloor$.

III: Suppose that $n \equiv 0(mod\ 3)$, so by following the previous strategy, the last two adjacent in the set D are v_{n-1}, v_n. Thus, the vertex v_n does not dominate each vertex in the set $V - D$, so the set D is not captive. Thus, the result is obtained.

V: Suppose that $n \equiv 2(mod\ 3)$, so by following the previous strategy, the last two adjacent in the set D are v_{n-2}, v_{n-3}.Therefore. there is no vertex in the set D is neighborhood to the vertex v_n, so the set D is not DS.

According to all cases above, the results are determined.

Figure 1
A path P_7

Figure 2
$\langle V - D \rangle$ is 0- regular

Proposition 3.6: $\gamma_{crg}(C_n) = \begin{Bmatrix} \frac{n}{2} \text{ if } n \equiv 0(mod\ 4), r = 1 \\ \frac{2n}{3} \text{ if } n \equiv 0(mod\ 3), r = 0 and\ n \not\equiv 0(mod\ 4) \end{Bmatrix}$.

Proof: Three parts classification: **I:** Suppose that $n \equiv 0(mod\ 4)$, so there is no isolated vertex in the set D according to the definition of CO-RCDS, so each adjacent two vertices in the set D dominate four vertices as a maximum. Thus, let $D = \{v_{1+4i}, v_{2+4i}, i = 0,1, \dots, \frac{n}{4} - 1\}$, it's obvious that this set is CO-RCDS, since each vertex in D has the maximum neighborhood. So, there is no CO-RCDS has the cardinality less than of $|D|$, then D is the minimum CO-RCDS, so $\gamma_{crg}(C_n) = 2\left(\frac{n}{4}\right) = \frac{n}{2}$, and since $\langle V - D \rangle$ is perfect matching then $r = 1$. **I:** Suppose that $n \equiv 0(mod\ 3)$, so let $D = \{v_{1+3i}, v_{2+3i}, i = 0,1, \dots, \frac{n}{3} - 1\}$ represent CO-RCDS. In the same manner in the Case 1 the set D is the minimum CO-RCDS, so $\gamma_{crg}(C_n) = 2\left(\frac{n}{3}\right)$, and since $\langle V - D \rangle$ is null then $r = 0$. **I:** Suppose that $n = 5$, so D is not CDS. According to all cases above, the results are determined.

Proposition 3.9: $\gamma_{crg}(F_n) = \begin{Bmatrix} 2\ if\ n = 2,4 \\ \left\lceil \frac{n}{3} \right\rceil\ if\ n \equiv 2(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1\ if\ n \equiv 0,1(mod\ 3) \end{Bmatrix}$.

Proof: Let $\{v_1, v_2, \dots, v_n\} \cup \{u\}$ is vertex set of a fan graph $F_n \equiv P_n + K_1$ of order n such that $\{v_1, v_2, \dots, v_n\}$ be the vertex set of the path P_n and $\{u\}$ is a vertex of K_1 and let $D \subset V(G)$ defined as follows:

$$D = \begin{Bmatrix} \left\{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 2\right\} \cup \{u\}\ if\ n \equiv 2\ (mod\ 3) \\ \left\{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 2\right\} \cup \{u, v_n\}\ if\ n \equiv 1(mod\ 3) \\ \left\{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\} \cup \{u\}\ if\ n \equiv 0\ (mod\ 3) \end{Bmatrix}$$ Four parts classification:

I: Suppose that $n = 2,4$ so two sub-classifications as follow. **Subcase 1.** If $n = 2$, then $F_n \equiv C_3$, so $\gamma_{crg}(F_2) = \gamma_{crg}(C_3) = 2$, the result is obtained. **Subcase 2.** Suppose that $n = 4$, so let $D_1 = \{v_3, v_4\}$ represent co-regular captive dominating set since all two adjacent vertices in the set D_1 have the maximum neighborhood and since $\langle V - D \rangle \equiv C_3$ then $r = 2$ and $\gamma_{crg}(F_4) = 2$. **II:** Suppose that $n \equiv 2(mod\ 3)$, so let $D_2 = \{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 2\}$ represent captive dominating set to vertices $\{v_1, v_2, \dots, v_n\}$ but not co-regular captive dominating set we must take $\{u\}$ then $\langle V - D \rangle$ is regular and $D = D_2 \cup \{u\} = \{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 2\} \cup \{u\}$ is minimum co-regular captive dominating set, therefore $\gamma_{crg}(F_n) = \left\lceil \frac{n}{3} \right\rceil\ and\ r = 1$. **III:** Suppose that $n \equiv 1(mod\ 3)$, so let $D_3 =$

$\{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 2\}$ represent captive dominating set to vertices $\{v_1, v_2, \dots, v_{n-1}\}$ but not co-regular captive dominating set we must take $\{u, v_n\}$, now $\langle V - D\rangle$ is regular and $D = D_2 \cup \{u, v_n\} = \{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 2\} \cup \{u, v_n\}$ is minimum co-regular captive dominating set, therefore $\gamma_{crg}(F_n) = \left\lceil \frac{n}{3} \right\rceil + 1 \; and \; r = 1$. **V:** Suppose that $n \equiv 0(mod\ 3)$, so let $D_3 = \{v_{3+3i}, i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 1\}$ represent CDS so, by the same technique in the Case 2 and Case 3, then $\gamma_{crg}(F_n) = \left\lceil \frac{n}{3} \right\rceil + 1 \; and \; r = 1$. According to above, the result is done.

Proposition 3.10: $\gamma_{crg}(W_n) = \left\{ \begin{array}{c} 2 \; if \; n-1 = 4 \\ 3 \; if \; n-1 = 5 \\ \left\lceil \frac{n-1}{3} \right\rceil + 1 \; if \; n-1 \equiv 0,1(mod\ 3) \\ \left\lceil \frac{n-1}{3} \right\rceil + 2 \; if \; n-1 \equiv 2(mod\ 3) \end{array} \right\}$

Proof: The wheel graph $W_n \equiv K_1 + C_{n-1}$, then there are two sets $V_1 and\ V_2$; $V_1 = \{v_n\}$ is the vertex set of K_1 and $V_2 = \{v_1, v_2, \dots, v_{n-1}\}$ are the vertices of C_{n-1}. let

$$D = \left\{ \begin{array}{c} \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil \frac{n-1}{3} \right\rceil - 1\right\} \cup \{v_n\} \; if \; n-1 \equiv 0,1 \; (mod\ 3) \\ \left(\left\{v_{1+3i}, i = 0,1, \dots, \left\lceil \frac{n-1}{3} \right\rceil - 2\right\} \cup \{v_{n-4}, v_{n-1}, v_n\} \; if \; n-1 \equiv 2 \; (mod\ 3)\right) \end{array} \right\}$$

Four parts classification: **I:** Suppose that $n-1 = 4$, so let $D_1 = \{v_1, v_2\}$ represent co-regular captive dominating set to all vertices $\{v_3, v_4, v_5\}$, so the set D_1 have maximum neighborhood and D_1 is minimum co-regular captive dominating set, therefore $\langle V - D_1\rangle \equiv C_3$ then $r = 2$ and $\gamma_{crg}(W_n) = 2$. **II:** Suppose that $n-1 = 5$, so let $D_2 = \{v_1, v_2, v_3\}$ represent co-regular captive dominating set to all vertices $\{v_4, v_5, v_6\}$, so by the same way in Case 1. Then $\gamma_{crg}(W_n) = 3$. **III:** Suppose that $n-1 \equiv 0,1(mod\ 3)$, so let $D_3 = \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil \frac{n-1}{3} \right\rceil - 1\right\}$ represent dominating set to vertices $\{v_1, v_2, \dots, v_{n-1}\}$ but not captive dominating set we must take $\{v_n\}$ to maintain on captive and co regular then $\langle V - D\rangle$ is regular and $D = D_3 \cup \{v_n\} = \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil \frac{n-1}{3} \right\rceil - 1\right\} \cup \{v_n\}$ is minimum co-regular captive dominating set, thus $\gamma_{crg}(W_n) = |D_3 \cup \{v_n\}| = \left\lceil \frac{n-1}{3} \right\rceil + 1$. **V:** Suppose that $n-1 \equiv 2(mod\ 3)$, so let $D_4 = \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil \frac{n-1}{3} \right\rceil - 2\right\}$ represent DS to vertices $\{v_1, v_2, \dots, v_{n-4}\}$ but not CDS, so the remaining vertices which are not dominated byD_4 and to maintain on captive and co-regular are $\{v_{n-4}, v_{n-1}, v_n\}$ then $\langle V - D\rangle$ is regular and $D = D_4 \cup \{v_{n-4}, v_{n-1}, v_n\} = \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil \frac{n-1}{3} \right\rceil - 1\right\} \cup \{v_{n-4}, v_{n-1}, v_n\}$ is minimum co-regular captive dominating set, thus $\gamma_{crg}(W_n) = |D_4 \cup \{v_{n-4}, v_{n-1}, v_n\}| = \left\lceil\frac{n-1}{3}\right\rceil + 2$. According to above, the result is done.

3. Conclusions

Depending on the above results, new definition, many properties, and boundaries have been discovered. Along the process of arriving at the results described in this article. Specially identified above are several graphs' operations.

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