

Approximately 2-absorbing and weakly approximately 2-absorbing sub-modules

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Abstract

We outline the ideas of approximately 2-absorbing and weakly approximately 2-absorbing submodules and list the fundamental characteristics of these submodule classes. This is a generalization of prime and weakly prime sub-modules. Let A be a T -module and T a commutative ring. Weakly prime is the state in which a proper submodule U of A is when $a, b \in T$ and $m \in A$, $bam \in U$ implies that $am \in U$ or $bm \in U$. When a, b within T , m in A and $abm \in U$ then a proper submodule U of A is said to be approximately 2-absorbing (or weakly approximately 2-absorbing sub-modules). If $a, b, \in T$ A $m \in A$, and $abm \in U$ (resp. $0 \neq abm \in U$), then then $a^2 m \in N$ or $b^2 m \in N$ or $a^2 b^2 \in (U:A)$.

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1. Introduction

In this article, all modules are regarded as unitary, and all rings are commutative with nonzero identities. Algebra has established it self as an important mathematical tool in wide variety subjects, see[1-3, 14-17].

A key concept in commutative ring theory is prime ideals. Let T be a ring that commutes. Naturally, if $ab \in P$ indicates that $a \in P$ or $b \in P$ where $a, b \in T$, then a proper ideal P of R is considered to be a prime ideal. The notion of weakly prime ideals of a commutative ring was examined by Anderson and Smith [4].

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Weakly prime is a proper ideal P of T such that if $a, b \in T$ and $ab \in P$, it entails that one of a or $b \in P$. The idea of almost prime ideals is another way to generalize prime ideals. Consider the integral domain T . If $a, b \in T$ and $ab \in I - I^2$, it implies that either $a \in I$ or $b \in I$, making a valid ideal I of T almost prime (see [5]). This formulation may clearly be established for any ring T that is commutative. Prime ideals in commutative rings are generalized in a number of ways in [6]. In module theory over commutative rings, prime submodules are crucial. Let T be a commutative ring and A a module over it. If $a \in N$ implies that $m \in U$ or $a \in (U :_R A)$ (resp. $a^k \in (U :_R A)$ for some positive integer k), the prime (resp. primary) submodule U of A is a proper submodule.

The concept of the 2-absorbing ideal was introduced by Badawi in 2007. A proper ideal I of T is called a 2-absorbing ideal if whenever $abx \in I, a, b, x \in T$ implies that either $ax \in I$ or $bx \in I$ or $ab \in I$ [7]. This concept was generalized to a 2-absorbing submodules, a proper submodule U of A is said to be 2-absorbing sub-modules. If whenever $a, b \in T, m \in A$ and $abm \in U$, then $am \in U$ or $bm \in U$ or $ab \in (U : A)$, by Darani and Soheilire in 2011 in [8], 2-absorbing modules have been extensively studied, and many generalizations for these concepts have been obtained by many authors; see, for example, [9-11]. Additionally, Kadhim and Shyaa generalized the concept of a 2-absorbing submodule to approximately a 2-absorbing submodule. A proper submodule U of A is said to be approximately a 2-absorbing sub-module if and whenever $a, b \in T, m \in A$ and $abm \in U$, then $a^2m \in U$ or $b^2m \in U$ or $a^2b^2 \in (U : A)$ [12]. We present an additional prime submodule generalization. A proper sub-module U of A is said to be weakly approximately 2-absorbing if whenever $a, b \in R, m \in A$ and $(0 \neq abm \in U)$, then $a^2m \in U$ or $b^2m \in U$ or $a^2b^2 \in (U : M)$. We look at a few attributes approximately 2-absorbing and weakly approximately 2-absorbing sub-modules. Also, get a few fundamental outcomes. The paper includes several results.

2. Weakly approximately 2-absorbing Submodules.

We present various characterizations of this idea in this section.

Definition 2.1: A proper sub-module U of A is said to be approximately 2-absorbing (resp. weakly approximately 2-absorbing) sub-modules. If whenever $a, b \in T, m \in M$ and $abm \in U$ (resp. $0 \neq abm \in U$), then $a^2m \in U$ or $b^2m \in U$ or $a^2b^2 \in (U : M)$.

a nonzero proper ideal I of T to be approximately-2 absorbing (resp. weakly 2-approximately-2 absorbing) ideals if whenever $a, b, c \in T$ and $abc \in I$ (resp. $0 \neq abc \in I$) then $a^2c \in I$ or $b^2c \in I$ or $a^2b^2 \in I$.

Remarks and Examples 2.2: Let U be a submodule of A . Let A be a module over a ring T that is commutative.

- 1) Submodules of primes indicate weak primes, implying approximately 2-absorbing submodules.
- 2) Submodules that are weakly prime are simply approximately 2-absorbing

- 3) approximately 2-absorbing submodules implies weakly approximately 2-absorbing submodules.
- 4) weakly 2-absorbing submodules implies weakly approximately 2-absorbing submodules.

Proof:

- 1) Let $0 \neq am \in U$ when $a \in T$ and $m \in A$, since U is a Prime submodule, we get $m \in U$ or $a \in (U:_T A)$, so U is a weak prime submodule. Let $abm \in U$ when $a, b \in T$ and $m \in A$, since U is a weak prime submodule, so $am \in U$ or $bm \in U$ by U submodule, we get $a^2m \in U$ or $b^2m \in U$ so U approximately 2-absorbing submodule.
- 2) Let $0 \neq abm \in U$ when $a, b \in T$ and $m \in M$, due to U being a weakly prime submodule, so $am \in N$ or $bm \in N$ by N submodule we get $a^2m \in N$ or $b^2m \in U$ so U is a weakly approximately 2-absorbing submodule.
- 3) Let $0 \neq abm \in U$ when $a, b \in T$ and $m \in A$, since U is a weakly prime submodule, we get $a^2m \in U$ or $b^2m \in U$ so U is approximately 2-absorbing submodules.
- 4) Let $0 \neq abm \in U$ when $a, b \in T$ and $m \in A$, since U is a weakly 2-absorbing submodules, we get $am \in U$ or $bm \in U$ or $ab \in (U:_T A)$, by U is a submodule of A , so $a^2m \in U$ or $b^2m \in U$ or $(ab)^2 \in (U:_T A)$, so U is a weakly approximately 2-absorbing submodules.
- 5) Let A be a T -module and T a commutative ring. By Remarks and Examples 2.2. Although the opposite is not always true, every approximately 2-absorbing sub-module is a weakly approximately 2-absorbing

Take, for instance, the situation where $T = \mathbb{Z}$, $A = \mathbb{Z} / 30\mathbb{Z}$ and $U = 0$. Then $2 \cdot 3 \cdot (5 + 30\mathbb{Z}) = 0 \in U$ although $(2 \cdot 3)^2 \notin (U:_T A)$, $2^2 \cdot (5 + 30\mathbb{Z}) \notin U$ and $3^2 \cdot (5 + 30\mathbb{Z}) \notin U$, U is not approximately 2-absorbing as it is a weakly approximately 2-absorbing.

Theorem 2.3: Let A be a T -module and T be a commutative ring. Then the intersection of any two distinct weak prime submodules of a module A is weakly approximately 2-absorbing.

Proof: Suppose that A has two distinct weak prime submodules, U and K . presume that $0 \neq abm \in U \cap K$ for all $a, b \in T$ and $m \in M$ with $a^2m \notin U \cap K$ and $b^2m \notin U \cap K$. The case $a^2m \notin U$ and $b^2m \notin U$ by U is weak prime so is approximately 2-absorbing submodule, then $(ab)^2 \in (U:_T A)$, and $a^2m \notin K$ and $b^2m \notin K$ due to K is weak prime. Thus, U is approximately 2-absorbing and Then $(ab)^2 \in (K:_T A)$ we have $(ab)^2 \in (U:_T A)$ and $(ab)^2 \in (K:_T A)$. Hence $(ab)^2 \in (U:_T A) \cap (K:_T A) = (U \cap K:_T A)$. $U \cap K$ is a weakly approximately 2-absorbing

Proposition 2.4: Let N be a weak approximately 2-absorbing, $K \leq N$. Then U/K is a weakly approximately 2-absorbing in A/K .

Proof: Let $a, b \in T, m + K \in A/K$ with $0 \neq ab(m + K \in U/K$. Hence $abm \in U$. (Note that if $abm = 0$, then $ab(m + K = abm + K = 0 + K = K = 0_{\frac{M}{K}}$, which is a contradiction. As U is weakly approximately 2-absorbing submodule. So, either $a^2m \in N$ or $b^2m \in N$ or $a^2b^2 \in (U:_T M)$. This implies either $a^2(m + K) \in \frac{U}{K}$ or $b^2(m + K) \in \frac{U}{K}$ or $a^2b^2 \in (\frac{U}{K} :_R \frac{A}{K})$.

Remark 2.5: Proposition 2.4 opposite may not be true. For instance,

Example 2.6: Consider Z -module Z , let $U = 60Z$. U is not weakly approximately 2-absorbing, since $0 \neq 60 = 4.3.5 \in U$, but $4^2.5 = 80 \notin U$, $3^2.5 = 45 \notin U$ and $4^2.3^2 = 144 \notin (U:_R Z) = 60Z$. Take $K = 60Z \leq U$, $\frac{U}{K} = \frac{60Z}{60Z} \cong \langle 0 \rangle$ which is weakly approximately 2-absorbing.

Theorem 2.7: Let A be an T -module and T a commutative ring. and U weakly approximately 2-absorbing sub modules of M . In the event that U is not approximately 2-absorbing, $(U:_T A)^2 U = 0$

Proof: Presume that $(U:_R A)^2 U \neq 0$. We are going to demonstrate that N is approximately 2-absorbing. Assume that $abm \in U$ for all $a, b \in T$ and $m \in A$ if $abm \neq 0$ then $(ab)^2 \in (U:_T A)$ or $a^2m \in U$ or $b^2m \in U$, due to weakly approximately 2-absorbing. As a result. presume that $bam = 0$. First, suppose that $abU \neq 0$ say $abn_0 \neq 0$ for some $n_0 \in U$ then $ab(m + n_0) \in U = abn_0 \neq 0$. since U is a weakly approximately 2-absorbing, we get $(ab)^2 \in (U:_T A)$ or $a^2(m + n_0) \in U$ or $b^2(m + n_0) \in U$. Hence $a^2m \in U$ or $b^2m \in U$ or $(ab)^2 \in (U:_T A)$. Consequently, $abU = 0$ may be assumed. If $am(U:_T A) \neq 0$, then there exists $r_0 \in (U:_T A)$ such that $ar_0m \neq 0$. Then $0 \neq ar_0m = a(b + r_0)m \in U$. Since U is weakly approximately 2-absorbing we $a^2(b^2 + r_0) \in (U:_T A)$ or $a^2m \in U$ or $(b^2 + r_0)m \in U$. Thusly $(ab)^2 \in (U:_T A)$ or $a^2m \in U$ or $b^2m \in U$. So It can be inferred that $am(U:_T A) = 0$. Furthermore, we can presume that $bm(U:_T A) = 0$. Since $(U:_T A)^2 U \neq 0$, there exist $a_0, b_0 \in (U:_T A)$ and $x_0 \in U$ with $a_0b_0x_0 \neq 0$. If $ab_0x_0 \neq 0$, then $0 \neq ab_0x_0 = a(b + b_0)(m + x_0) \in U$ implies that $(ab)^2 + b_0 \in (U:_T A)$ or $(a^2m) + x_0 \in U$ or $(b^2 + b_0)(m + x_0) \in U$. Hence $(ab)^2 \in (U:_T A)$ or $a^2m \in U$ or $b^2m \in U$. Thus, we may presume that $am(N:_T M) = 0$ or $ab_0x_0 = 0$. Likewise, we can assume that $bm(N:_T M) = 0$. Since $(N:_T M)^2 N \neq 0$, there exist $a_0, b_0 \in (N:_T M)$ and $x_0 \in N$ with $a_0b_0x_0 \neq 0$. If $ab_0x_0 \neq 0$ a. Then $0 \neq ab_0x_0 = a(b + b_0)(m + x_0) \in N$ implies that $a^2(b^2 + b_0) \in (N:_R M)$ or $a^2(m + x_0) \in N$ or $b^2 + b_0)(m + x_0) \in N$. Hence $(ab)^2 \in (U:_T A)$ or $a^2m \in U$ or $b^2m \in U$. So we can assume that $ab_0x_0 = 0$. In the same way, we can presume that $a_0b_0m = 0$ and $a_0bx_0 = 0$. Then, from $0 \neq a_0b_0x_0 = (a + a_0)(b + b_0)(m + x_0) \in U$ we get $((a^2 + a_0)(b^2 + b_0)) \in (U:_T M)$ or $(a^2 + a_0)(m + x_0) \in U$ or $(b^2 + b_0)(m + x_0) \in U$

As a result $(ab)^2 \in (U :_T A)$ or $a^2 m \in U$ or $m b^2 \in U$, and so U is approximately 2-absorbing. An T -module's submodule U nilpotent submodule if $(U :_T A)kU = 0$ for a certain positive number k .

Corollary 2.8: *Let A be an T -module and let T be a commutative ring. If U is a submodule of A that is not approximately 2-absorbing but is weakly approximately 2-absorbing, then*

- 1) U is nilpotent.
- 2) $U^3 = 0$ if A is a module for multiplication.

Proof: Since U is nilpotent then $(U :_T A)^2 U = 0$, A is a multiplication, so $m(I :_T A)^2 U = 0$ for some ideal I of T . $(rA \subseteq IA)^2 U = 0$; we get $(U :_T A)^2 U = 0$ so $U^3 = 0$.

Proposition 2.9: *Let A be a faithful multiplication and T a ring. The T -module presupposes U is a sub-module of A that is weakly roughly 2-absorbing. $U \subseteq A - \text{rad}0$ if U is not approximately 2-absorbing.*

Proof: In the event that U is not approximately 2-absorbing. According to Theorem 2.7, $(U :_T A)^2 N = 0$. Currently, we have $(U :_T A)^3 \subseteq ((U :_T A)^2 U :_T A) = (0 :_T A) = 0$. Given that A is faithful, this means

$$(U :_T A)^3 = 0 \text{ If } a \in (U :_T A), \text{ then } a^3 = 0 \text{ and hence } a \in \sqrt[3]{0}$$

$$\text{So that } (U :_T A) \subseteq \sqrt{0} \text{ Consequently } U = (U :_T A)A \subseteq \sqrt{0} = A - \text{rad}0.$$

Theorem 2.10: *Let A be a faithful multiplication T -module with finite generation and T a commutative ring. If U is a projective submodule of A that is nonzero and weakly roughly 2-absorbing. Then, U is never approximately 2-absorbing.*

Proof: First, we demonstrate that $(0 :_T (U :_T A)^2 U) = (0 :_T U)$. Clearly $(0 :_T U) \subseteq (0 :_T (U :_T A)^2 U)$. Choose an element for reversible containment. $a \in (0 :_T (U :_T A)^2 U)$. Given that A is a module for multiplication, $U = (U :_T A)A$. Thus, it is evident from $a(U :_T A)^2 U = 0$ that $a(U :_T A)^3 \subseteq (0 :_T A) = 0$. So $a(U :_T A)^3 = 0$. Therefore $(0 :_T A) \cap (U :_T A) \supseteq a(U :_T A)^2$. Let $\text{Tr}(U) = \sum \phi \in \text{HoAT}(U, T) \phi(U)$ represent U 's trace ideal. Due to U is projective, $\text{Tr}(U)$ is a pure ideal of T , $(0 :_T U) = (0 :_T \text{Tr}(U))$ and $U = \text{Tr}(U)U$ via [13, Proposition 3.30]. Henceforth $(U :_T A) = (\text{Tr}(U)U :_T A) = \text{Tr}(U)(U :_T A) \subseteq \text{Tr}(U)$.

Thus, $a(U :_T A)^2 \subseteq \text{Tr}(U) \cap (0 :_T \text{Tr}(U))$. $\text{Tr}(U)$ is a pure ideal of T , therefore using [13], we obtain $\text{Tr}(U) \cap (0 :_T \text{Tr}(U)) = 0$. As a result $a(U :_T A)^2 = 0$. By using this procedure again, we'll have $a(U :_T A) = 0$, and as a result, $aU = 0$, that is $a \in (0 :_T U)$. Thus $(0 :_T (U :_T A)^2 U) \subseteq (0 :_T U)$. Thusly $(0 :_T (U :_T A)^2 U) = (0 :_T U)$. Now if U is not approximately 2-absorbing, then $U^3 = 0$. Then $(0 :_T U) = (0 :_T U^3) = T$. This contradicts itself.

Proposition 2.11: *Let A be a cyclic T -module, U be a submodule of A , and T be a commutative ring.*

$(U;_R A)$ is an approximately 2-absorbing ideal of T , iff U is an approximately 2-absorbing sub-module of A .

Proof: (1) Assume that U is an approximately 2-absorbing submodule of A . Presume that $A=Um$ for a given m in A , and let $abc \in (U;_T A)$ given some a, b , and $c \in T$, but $(ab)^2 \notin (U;_T A)$ and $b^2c \notin (U;_T A)$. Here are r and s in T such that $(ab)^2(rm) \notin U$ and $cb^2(sm) \notin U$. From above, it follows that $abcm \in U, b^2cm \notin U$ that $a^2c \in (U;_T A)$ due to U is approximately 2-absorbing.

On the other hand, suppose $(U;_T A)$ is an approximately 2-absorbing ideal of T , and let $abx \in U$ for some $a, b \in T$ & $m \in A$. let $abx \in U$. $x=rm$ and $c \in T$ are present. After that, $abc \in (U;_T A) = (U;_T m)$ and $(U;_T A)$ weakly approximately 2-absorbing ideal gives $(ab)^2 \in (U;_T A)$ or $a^2c \in (U;_T A)$ or $b^2c \in (U;_T A)$. Therefore, $(ab)^2 \in (U;_T A)$ or $a^2x \in U$ or $b^2x \in U$. Hence, U is an approximately 2-absorbing ideal.

A submodule U of A is considered weakly roughly 2-absorbing if and only if the quotient $(U;_T A)$ is a weakly approximately 2-absorbing ideal of T , with the condition that A is also faithful.

Consider three elements a, b , and c belonging to the set T . Let $0 \neq abc$ be an element of $(U;_T A)$, but $(ab)^2$ is not an element of $(U;_T A)$ and b^2c is not an element of $(U;_T A)$. Consequently, $abcm \neq 0 \in U$ in any case $abcm=0$ means $abc \in (0;_T A)=0$, which is contradictory. Now, an analogous disagreement as in part 1's proof.

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