

2J-ideals in commutative rings

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Abstract

The notion of 2J-ideals in commutative rings is presented in this article. The symbols $N(A)$ and $J(A)$ stand for A 's nil-radical and Jacobson radicals, respectively. In this paper, we present and investigate a new Extension of J-ideals named 2J-ideals. A proper ideal I_1 of a ring A is named a 2J-ideal if every time $a, b \in A$ with $a, b \in I_1$ and $a \notin J(A)$, then $b^2 \in I_1$. We investigate the relationships between this class of ideals and several other classes of ideals, as well as many of its characteristics and instances.

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1. Introduction

Every ring in this paper is commutative with identity. When considering a ring A , its units, radical, Jacobson radical, and set of zero divisors are all considered by $U(A), N(A), J(A)$ and $Z(A)$ respectively. We recommend the reader to [1] for definitions of any notations or terminology used in commutative ring theory. These kinds include prime, primary, and maximal ideals, which are important concepts in the theory of commutative algebra. Numerous generalizations and associated kinds of these ideals, nearly prime (primary),

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weakly prime (primary), and semi-prime, have been investigated over the last three decades see [2-6, 12-22].

We carry out this strategy in the second part of this paper, where we introduce 2J-ideals: an extension of J-ideals. A proper ideal L of ring A is named a J-ideal if when $a, b \in L$ and $a \notin J(A)$, then $b \in L$ [7]. A proper ideal L of a ring A is named a r-ideal if when $a, b \in A$ with $ab \in L$ and $\text{Ann}(a) = 0$, then $b \in L$ [8]. A n-ideal is a proper ideal L of a ring A if, for any $a, b \in A$ with $ab \in L$ and $a \notin N(A)$, then $b \in L$ [9].

By providing instances of 2j-ideals that are not J-ideals in a given ring, we shall first demonstrate the validity of this generalization. In addition, We present several characterizations and results of 2J-ideals. An appropriate ideal L of a ring A is named a 2J-ideal if whenever $a, b \in L$ and $a \notin J(A)$, then $b^2 \in L$. Additionally, we defend a few conditions under which the opposite is true.

2. Commutative rings with 2J-ideals

The class of 2J-ideals is introduced as well as some of their properties are examined in this section. Additionally, their relationship to certain other categories of ideals is clarified.

Definition 2.1: Let ring $A \neq I_1$, I_1 is known as a 2J-ideal If when $a, b \in A$ with $a, b \in I_1$ and $a \notin J(A)$, then $b^2 \in I_1$.

Remark and Example 2.2: 1-If I_1 is A 's J-ideal, then I_1 is also A 's 2J-ideal.

Proof: Let $a, b \in A$ with $ab \in I_1$ if $a \notin J(A)$ then $b \in I_1$ so $b^2 \in I_1$.

2. Let B be an A -module and A a ring. The set $\{(a, b): a \in A, b \in B\}$ is the idealization ring $A(B)$ of B in A . Its componentwise addition and multiplication are defined as $(a, b)(s, n) = (as, bn + sb)$. It is easily verifiable that $A(B)$ is an identity commutative ring $(1_A, 0_B)$. If K were A 's ideal, then U is a submodule of B , then an ideal of $A(B)$ is given by $K(U) = \{(a, b): a \in K, b \in U\}$ [10].

But the converse of 1 is not true in general for example consider the idealization ring $A = \mathbb{Z}(+)2\mathbb{Z}$ let $I_1 = 0(+)2\mathbb{Z}$, I_1 is not J-ideal since $(0, 1), (2, 0) \in R$ with $(2, 0) \cdot (0, 1) = (2 \cdot 0, 0 \cdot 0 + 2 \cdot 1) = (0, 2) \in 0(+)2\mathbb{Z}$ and $(2, 0) \notin J(R) = J(\mathbb{Z}) + \mathbb{Z} = 0(+)2\mathbb{Z}$ but $(0, 1) \notin I_1$ but if $(a, b)(c, d) \in I_1 = 0(+)2\mathbb{Z}$ and $(c, d) \notin J(A) = 0(+)2\mathbb{Z}$ hence $(ac, bc + ad) \in I_1 = 0(+)2\mathbb{Z}$ and $c \neq 0, d \in \mathbb{Z}$ then $ac = 0$ and $bc + ad \in 2\mathbb{Z}$ since $c \neq 0$ we have $a = 0$ and so $(a, b)^2 = (a, b)(a, b) = (0, b \cdot 0 + 0 \cdot b) = (0, 0) \in I_1 = 0(+)2\mathbb{Z}$ thus I_1 is 2J-ideal, but not J-ideal.

3. If I_1 is 2J-ideal of a ring A , then $I_1 \subseteq J(A)$.

Proof: Suppose $I_1 \not\subseteq J(A)$. then $\exists a \in I_1$ and $a \notin J(A)$ Now $a \cdot 1 \in I_1$ and $a \notin J(A)$ indicates that $1^2 = 1 \in I_1$ and so $I_1 = A$ which is contradiction this $I_1 \subseteq J(A)$.

But we can identify an ideal of a ring A with $I_1 \subseteq J(A)$. That is not a 2J-ideal. For example, consider the ring Z_6 , then $J(Z_6) = \langle 0 \rangle$ and $I_1 = \langle 0 \rangle \subseteq J(Z_6)$ but I_1 is not a 2J-ideal since $2 \cdot 3 \in I_1$ with $2 \notin J(Z_6)$ and $3^2 \notin \langle 0 \rangle$.

Proposition 2.3: Consider A to be a ring. Below are identical concepts

- (1) The ring A is quasi-local.
- (2) All of A 's proper ideals are 2J – ideals.
- (3) A has 2J-ideals for each of its principal ideal.

Proof: (1) $\Rightarrow$ (2) By proposition (2.3,[7])

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Assume that A has each suitable principle ideal of A is a 2J-ideal. Consider K is A 's highest ideal and let $a \in K$. Now, $\langle a \rangle$ is a 2J – ideal and $1 \cdot a \in \langle a \rangle$. If $a \notin J(A)$, then $1 \in \langle a \rangle$, a contradiction. Thus, $a \in J(A)$ and $K=J(A)$. The one and only maximal ideal of A is K . A is a quasi-local ring as a result.

Corollary 2.4: Let A represent a ring. The phrases that follow are equivalent:

- (1) A is local ring.
- (2) Each proper ideal is a J-ideal.
- (3) The J-ideal can be found in every principal ideal.
- (4) 2J is the ideal for every proper ideal.
- (5) A 2J-ideal can be found in every proper principal ideal.

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ 3 (by proposition 2.3).

(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) (by proposition 2.3).

Remark 2.5: All n-ideals of ring A are 2J-ideals.

Proof: Let $a, b \in A$ such that $ab \in I_1$ and $a \notin J(A)$. Due to $N(A) \subseteq J(A)$, $a \notin N(A)$ and so $b \in I_1$, so $b^2 \in I_1$. Therefore, I_1 is a 2J-ideal of A .

The following example shows that, in general, the opposite of statement 2.5 is not true.

Instance 2.6: Consider the quasi-local ring $Z_{\langle 2 \rangle} = \{ \frac{c}{d} : c, d \in Z, 2 \nmid d \}$. Because of each ideal in $Z_{\langle 2 \rangle}$ is by Proposition 2.4, is a 2 J-ideal, $\langle 6 \rangle_{\langle 2 \rangle} = \{ \frac{c}{d} : a \in \langle 6 \rangle, 2 \nmid b \}$ is a 2J-ideal of $Z_{\langle 2 \rangle}$ that isn't an n-ideal. In fact, for instance, $\frac{2}{3}, \frac{3}{5} \in Z_{\langle 2 \rangle}$ and $\frac{2}{3} \cdot \frac{3}{5} = \frac{6}{15} \in \langle 6 \rangle_{\langle 2 \rangle}$ but $\frac{2}{3} \notin N(Z_{\langle 2 \rangle}) = \{0\}$ and $\frac{3}{5} \notin \langle 6 \rangle_{\langle 2 \rangle}$.

Remember that $Z(A) \subseteq J(A)$ is the presimplifiable property of a ring A . Domainlike rings in the class where $Z(A) \subseteq N(A)$ is a subclass of presimplifiable rings [10, 11].

Proposition 2.7: Assume that ring A is presimplifiable (Domainlike). After then, every r -ideal in A is a $2J$ -ideal.

Proof: Let I be any r -ideal of A , and assume that A is presimplifiable. Let $a, b \in A$ such that $ab \in I_1$ where $a \notin J(A)$. Then $a \notin Z(A)$ and quite obviously, $Ann(a) = 0$. Because of I_1 is an r -ideal, $b \in I_1$ and so $b^2 \in I_1$ therefore, I_1 is a $2J$ -ideal.

A equals L , the I_1 appropriate ideal, whenever L is an ideal of A and $I_1 + L = A$. Superfluous refers to ring A .

Proposition 2.8: In a ring A , every $2J$ -ideal is superfluous.

Proof: Presume L is an ideal of A the extent that $I_1 + L = A$. Then $1 = i + l$ to certain $i \in I_1$ and $l \in L$ and then $1 - l = i \in I_1 \subseteq J(A)$ it follows that $l \in U(A)$ thus $L = A$ as needed.

Proposition 2.9: Assuming I_1 to be a semi-prime ideal of A , I_1 can only be considered a J -ideal of A if it also happens to be a $2J$ -ideal.

Proof: $\Rightarrow$ It's obvious.

$\Leftarrow$ Let $ab \in I_1, a \notin J(A)$ then $b^2 \in I_1$ But I_1 semi prime so $b \in I_1$ and I_1 is a J -ideal of A .

Proposition 2.10: Let A P.I.R. and $I_1 \subsetneq P$, I_1 is $2J$ -ideal $\forall$ ideal C, B of $A, CB \subseteq I_1, C \not\subseteq J(A) \Rightarrow B^2 \subseteq I_1$.

Proof: $\Leftarrow$ Let $a, b \in I_1$ and $a \notin J(A)$ put $C = \langle a \rangle, B = \langle b \rangle$ so $\langle a \rangle \langle b \rangle \subseteq I_1$ and $C \not\subseteq J(A)$ $B^2 \subseteq I_1$ that is $b^2 \in I_1$

$\Rightarrow CB \subseteq I_1$ and $C \not\subseteq J(A), \exists c \in C$ and $c \notin J(A)$ since A is P.I.R, $B = \langle b \rangle$ for some $b \in A$ now $a \cdot b \in I_1, a \notin J(A)$ so $b^2 \in I_1$ thus $B^2 = \langle b^2 \rangle \subseteq I_1$.

Proposition 2.11: Let $I_1 \subseteq A, I_1$ is $2J$ -ideal $\leftrightarrow (I_1 : a^2) \subseteq J(A) \forall a^2 \notin I_1$.

Proof: $\Rightarrow$ Let $b \in (I_1 : a^2)$ then $ba^2 \in I_1$ since $a^2 \notin I_1$ and I_1 is $2J$ -ideal then $b \in J(A)$.

$\Leftarrow$ Let $cd \in I_1$ and $c \notin J(A)$. T.P. $d^2 \in I_1$ suppose $d^2 \notin I_1$ hence by hyp. $(I_1 : d^2) \subseteq J(A)$ but $c \in (I_1 : d^2), cd^2 \in I_1$ hence $c \notin J(A)$ contradiction.

If an ideal K of ring A is the intersection of maximum ideals, then it is referred to be Jacobson. For instance, any ideal generated by multiplying different prime integers is Jacobson in the domain of integers. Now, if I_1 is a Jacobson $2J$ -ideal, then it is evident that $J(A) \subseteq I_1$. Since also $I_1 \subseteq J(A)$ by Remark and Examples 2.2.(3) We infer that the only Jacobson $2J$ -ideal, if there is any, is $J(A)$. Let A be a ring and let S be a non-empty subset of A . For an ideal I_1 of A , the set $(I_1 : S^2) = \{a \in A : aS^2 \subseteq I_1\}$ is clearly an ideal of A .

Lemma 2.12: Let S be a nonempty subset of A and let A be a ring. If I_1 is a 2J-ideal of A with $S^2 \not\subseteq I_1$, then $(I_1 : S^2)$ is 2J-ideal of A .

Proof: If $(I_1 : S^2) = A$, then $1 \in (I_1 : S^2)$ and so $S^2 \subseteq I_1$, a contradiction. Hence, $(I_1 : S^2) \neq A$. Let $ab \in (I_1 : S^2)$ and $a \notin J(A)$. Then we have $abs \in I_1$ for each $s \in S$. Because of I_1 is a 2J-ideal of A we draw the conclusion that $b^2s^2 \in I_1$ and so $b^2 \in (I_1 : S^2)$.

Lemma 2.13: If A and S are two rings, and f is an epimorphism of rings from A onto S , then $f(J(A)) \subseteq J(S)$. [7]

Proposition 2.14: If $f: A_1 \rightarrow A_2$ is an epimorphism of rings, then the subsequent statement holds.

- (1) If $\ker(f) \subseteq I_1$, and I_1 is a 2J-ideal of A_1 , then $f(I_1)$ is a 2J-ideal of A_2 .
- (2) If $\ker(f) \subseteq J(A_1)$, and L is a 2J-ideal of A_2 , then $f^{-1}(L)$ is a 2J-ideal of A_1 .

Proof: (1) Let $y_1, y_2 \in f(I_1)$ and $y_1 \notin J(A_2)$ since f is epimorphism, $y_1 = f(a)$, $y_2 = f(b)$, $a, b \in A$ and $y_1y_2 = f(ab) \in f(I_1)$. $\exists c \in I_1$ s.t. $f(ab) = f(c)$. thus $ab - c \in \ker f \subseteq I_1$ hence $ab \in I_1$, note that if $a \in J(A_1)$ then $f(a) = y_1 \in J(A_2)$ by Lemma 2.13, that is paradoxical.

(2) Let $a_1, b_1 \in A_1$ where $a_1b_1 \in f^{-1}(L)$ and $a_1 \notin J(A_1)$. Then $f(a_1)f(b_1) = f(a_1b_1) \in L$. We demonstrate that $f(a_1) \notin J(A_2)$, presume $f(a_1) \in J(A_2)$ and let T be maximal ideal of A_1 then $f(T)$ is a maximal ideal of A_2 due to $\ker f \subseteq J(A_1) \subseteq T$. Hence, $f(a_1) \in f(T)$ and so $a_1 \in M$ as $\ker f \subseteq T$, hence, $a_1 \in J(A_1)$ it is paradoxical, due to L is a 2J-ideal $f(b_1) \in L$ i.e. $f(b_1)^2 \in L$. Therefore $b_1^2 \in f^{-1}(L)$.

Corollary 2.15: Let I_1, L be two ideals of ring A , with $L \subseteq I_1$. Next, it is true what follows.

- (1) In case I_1 is a 2J-ideal of A , then A/L 's I_1/L is also a 2J-ideal.
- (2) If $L \subseteq J(A)$ and I_1/L is a 2J-ideal of A/L , then I_1 is a 2J-ideal of A .
- (3) If L is a 2J-ideal of A and I_1/L is a 2J-ideal of A/L , then I_1 is a 2J-ideal of A .

Proof:

- (1) Presume that I_1 is a 2J-ideal of A with $L \subseteq I_1$. Let $\pi: A \rightarrow A/L$ be the natural epimorphism described by $\pi(A) = a + L$. Observe that $\ker \pi = L \subseteq I_1$. Therefore, via Proposition 2.14 (1), it follows that $\pi(I_1) = I_1/L$ is a 2J-ideal of A/L .
- (2) Think about the natural epimorphism once more. $\pi: A \rightarrow A/L$. Due to $L \subseteq J(A)$, via Proposition 2.14 (2), $I_1 = \pi^{-1}(I_1/L)$ is a 2J-ideal of A .
- (3) This is evident by (2) and remark and Examples 2.2.

For corollary 2.15, the opposite of (1) is in generally not true. As an illustration

$\langle 5 \rangle / \langle 5 \rangle$ is a $2J$ -ideal in the field $\mathbb{Z} / \langle 5 \rangle \cong \mathbb{Z}_5$ but $\langle 5 \rangle$ is not a $2J$ -ideal of $\mathbb{Z}$ as $\langle 5 \rangle \not\subseteq J(\mathbb{Z}) = \langle 0 \rangle$.

Proposition 2.16: Let A be a ring and $\{I_i : i \in \Delta\}$ be a family that is not empty of $2J$ -ideals of A . Then, $\bigcap_{i \in \Delta} I_i$ is a $2J$ -ideal of A .

Proof: Let $a, b \in A$ such that $ab \in \bigcap_{i \in \Delta} I_i$ and $a \notin J(A)$. Then $ab \in I_i$ for all $i \in \Delta$. Because of this I_i is a $2J$ -ideal for all $i \in \Delta$, we conclude that $b^2 \in I_i$ for all i and so $b^2 \in \bigcap_{i \in \Delta} I_i$. Thus, $\bigcap_{i \in \Delta} I_i$ is a $2J$ -ideal.

Proposition 2.17: Presume that S is a multiplicatively closed subset of ring A such that $J(S^{-1}A) = S^{-1}J(A)$. If I_1 is a $2J$ -ideal of A with $I_1 \cap S = \emptyset$, then $S^{-1}I_1$ is a $2J$ -ideal of $S^{-1}A$.

Proof: Let $a, b \in A$ and $s, t \in S$ such that $\frac{a}{s} \cdot \frac{b}{t} \in S^{-1}I_1$. Then $\frac{a}{s} \notin J(S^{-1}A)$ and so clearly $a \notin J(A)$. Now, $\frac{ab}{st} = \frac{x}{v}$ for some $x \in I_1$ and $v \in S$. Thus, there exist $t_1 \in S, s, t, t_1 vab = t_1 tsx \in I_1$, then $t_1 vab \in I_1$ put $t_1 v = u \in S, uab \in I_1$ since I_1 is $2J$ -ideal of A and $a \notin J(A)$ so $(ub)^2 \in I_1$ i.e. $u^2 b^2 \in I_1$ and hence $(\frac{b}{t})^2 = \frac{ub^2}{ut^2} = \frac{b^2}{t^2} \in S^{-1}(I_1)$.

We see that Proposition 2.17.'s converse is generally untrue. For instance though $\langle 2 \rangle / \langle 2 \rangle$ a $2J$ -ideal in the ring $\mathbb{Z} / \langle 2 \rangle$ the ideal $\langle 2 \rangle$ is not a $2J$ -ideal of $\mathbb{Z}$ as $\langle 2 \rangle \not\subseteq J(\mathbb{Z})$.

Proposition 2.18: Let A_1 and A_2 be two rings. Then $A_1 \times A_2$ has no $2J$ -ideals.

Proof: Assume that $I = I_1 \times I_2$ is a $2J$ -ideal of $A_1 \times A_2$ where I_1, I_2 are ideals of A_1, A_2 respectively. Since $(0, 1), (1, 0) \in I = I_1 \times I_2$ with $(0, 1), (1, 0) \notin J(A_1 \times A_2)$, we get $(0, 1), (1, 0) \in I_1 \times I_2 = I$ and so $I = A_1 \times A_2$, a contradiction.

Definition 2.19: Given a ring A , let S be a nonempty subset such that $A - J(A) \subseteq S$. If ab is within S , for all $a \in A - J(A)$ and all $b^2 \in S$. Then S is considered a $2J$ -multiplicatively closed subset of A .

We establish the relationship between $2J$ -ideals and $2J$ -multiplicatively closed subsets of A in the following proposition.

Proposition 2.20: I_1 is a $2J$ -ideal of A if and only if $A - I_1$ is a $2J$ -multiplicatively closed subset of A for a proper ideal I_1 of A .

Proof: Presume that I_1 is a $2J$ -ideal of A . Therefore via Remarks and Examples 2.2(3) we have $I_1 \subseteq J(A)$ thus $A - J(A) \subseteq A - I_1$. Let $a \in A - J(A)$ and $b^2 \in A - I_1$. Presume that $a_1 b_1 \in I_1$. Since $a_1 \notin J(A)$ and I_1 is a $2J$ -ideal, we get $b_1^2 \in I_1$,

a contradiction. Consequently, we get the conclusion that $a_1 b_1 \in A - I_1$ and so $A - I_1$ is a 2J-multiplicatively closed subset of A .

Conversely, let's say that I_1 is an ideal and $A - I_1$ is a 2J-multiplicatively closed subset of A . We show that I_1 is a 2J-ideal. Let $c, d \in A$ and $cd \in I_1$ with $c \notin J(A)$. Then we have $d^2 \in I_1$ since otherwise we would have $cd \in A - I_1$ which is contradiction. Thus, I_1 is a 2J-ideal of A .

Remember that there is a prime ideal B of A containing I_1 such that $B \cap S = \emptyset$ in the event that I_1 is a disjoint ideal from a subset S of A that is multiply closed. A similar finding for 2J-ideals is stated in the following proposition.

Proposition 2.21: Let I_1 be an ideal of A s.t $I_1 \cap S = \emptyset$ where S is a 2J-closed portion of multiplicatively A . Then $\exists$ a 2J-ideal L including I_1 such that $L \cap S = \emptyset$.

Proof: Let $\Omega = \{Q \mid Q \text{ is an ideal of } A \text{ and } Q \cap S = \emptyset\}$. Then, inclusion partially orders Ω . Since $I \in \Omega$, we have $\Omega \neq \emptyset$. If $Q_1 \subseteq Q_2 \subseteq \dots$ is any chain in Ω , then $\bigcup_{i=1}^{\infty} Q_i$ is an ideal of A with $\bigcup_{i=1}^{\infty} Q_i \cap S = \emptyset$. Hence, $\bigcup_{i=1}^{\infty} Q_i$ is the chain's upper bounds. Applying Zorn's Lemma, and the maximum element L of Ω is obtained. Now, we show that L is a 2J-ideal of A , presumably not. Then we have $xy \in L$ for some $x \notin J(A)$ and $y^2 \notin L$. Thus, we get $y^2 \in (L : x)$ and $L \subsetneq (L : x)$. According to L maximality, we get $(L : x) \cap S \neq \emptyset$ thus, there is $s \in S$ such that $s \in (L : x)$. So, we have $xs \in L$. As well, $sx \in S$ because $x \in A - J(A)$, $s^2 \in S$ and S is a 2J-multiplicatively-closed subset of A . Therefore, we have $S \cap L \neq \emptyset$ and this contradicts that $L \in \Omega$. Hence, L is a 2J-ideal of A .

Proposition 2.22: Consider an ideal L of A is P.I.R such that $L \not\subseteq J(A)$, and let A be a ring. Next, the subsequent hold.

- (1) If I_1 and I_2 are 2J-ideals of A with $I_1 L = I_2 L$, then $I_1^2 = I_2^2$.
- (2) If I_1 is an ideal in which $I_1^2 L^2$ is a 2J-ideal, then $I_1^2 L^2 = I_1^2$.

Proof:

- (1) We have $I_2 L = I_1 L \subseteq I_1$. Since I_1 is a 2J-ideal, $I_2^2 \subseteq I_1$ by Proposition 2.10. Similarly, as I_2 is a 2J-ideal, we get $I_1^2 \subseteq I_2^2$ and there is equality.
- (2) Due to $I_1^2 L^2$ is a 2J-ideal, $I_1 L^2 \subseteq I_1^2 L^2$ and $L \not\subseteq J(A)$, we get $I_1^2 L^2 \subseteq I_1^2$.

Definition 2.23: If there isn't a 2J-ideal that contains I_1 , then I_1 is 2J-ideal of A that referred to as a maximal 2J-ideal.

It is obvious that an ideal I of a ring A is a 2J-ideal if it is prime and $I_1 \subseteq J(A)$. We ascertain a condition on I_1 when the opposite is true in the next.

Proposition 2.24: Let I_1 represent a maximal 2J-ideal of A . Hence, I_1 is a prime ideal. In particular, $I_1 = J(A)$.

Proof: Suppose I_1 is a maximal 2J-ideal. Let $a, b \in A$ such that $ab \in I_1$ and $a \notin I_1$. Then $(I_1 : a^2)$ is a 2J-ideal by Lemma 1.2.14. and $I_1 \subseteq (I_1 : a^2)$. By maximality of I_1 , we have $b^2 \in (I_1 : a^2) = I_1$ and so I_1 is a prime ideal of A . Now, presume $I_1 = J(A)$ is a prime ideal of A . Let $a, b \in A$ such that $ab \in I_1$ and $a \notin J(A)$. then $b^2 \in J(A) = I_1$ and so I_1 is a 2J-ideal. Moreover, I_1 is maximal by remark and examples 2.2.

After that, we find all integers $n \geq 2$ for which Z_n is a ring with 2J ideals.

Corollary 2.25: For any integer $n \geq 2$, is a prime integer power if and only if the ring Z_n has 2J-ideal.

Proof: $\Leftarrow$ via Proposition 2.3 indicates that Z_n is quasi-local if n is a power of a prime integer.

$\Rightarrow$ Supposing Z_n has a 2J-ideal. Presume that $n = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ is a distinct prime decomposition of n where $n \geq 2$. Then $J(Z_n) = \langle p_1 p_2 \dots p_k \rangle$ is not a 2J-ideal of Z_n because it is not prime. Let I_1 be a maximal 2J-ideal of Z_n . Then $I_1 \subseteq J(Z_n)$ and by Proposition 2.24 I_1 is also a prime ideal. Thus, $I_1 = J(Z_n)$ which is a contradiction ring

If $J(A) = 0$, ring A is considered semiprimitive. Moreover, A is called a von Neumann regular ring if for every $a \in A$, there exists $x \in A$ such that $a = a^2 x$.

Examples 2.26:

- (1) Von Neumann's regular ring consenses are all semiprimitive.
- (2) Every semisimple ring is semiprimitive.
- (3) The ring Z is semiprimitive.

Proposition 2.27: Let A represent a ring. Next, the subsequent hold.

1. If I_1 is a 2J-ideal of A then $\sqrt{I_1}$ is a J-ideal.
2. If I_1 and I_2 are J-ideal of A then $I_1 I_2$ is a 2J-ideal.
3. If I_1 and I_2 be coprime ideal (I_1 and I_2 are proper ideal and $I_1 + I_2 = A$) if $I_1 I_2 = I_1 \cap I_2$ is a 2J-ideal, then I_1 and I_2 .
4. Every maximal 2J-ideal of A is prime.

Proof:

- (1) Let $ab \in \sqrt{I_1}$ and $a \notin J(A)$ then $a^n b^n \in I_1$ for some $n \in \mathbb{Z}^+$ since $a \notin J(A)$ so $\exists T$ (maximal ideal of A) such that $a \notin T$ hence $a^n \notin T$ (because if $a^n \in T$, then $a \in \sqrt{T} = T$ thus $a^n \notin J(A)$ as I_1 is a 2J-ideal and $a^n \notin J(A)$ we have $(b^n)^2 \in I_1$ it follows that $b \in \sqrt{I_1}$.
- (2) Let $a, b \in A$ with $ab \in I_1 I_2$ and $a \notin J(A)$. We have $ab \in I_1$ and $ab \in I_2$. Then, $b \in I_1$ and $b \in I_2$. Therefore, $b^2 \in I_1 I_2$. Hence, $I_1 I_2$ is a 2J-ideal.

- (3) Since I_1 and I_2 are coprime ideal, $I_1 + I_2 = A$ hence $1 = x + y$ for some $x \in I_1$ and $y \in I_2$ let $ab \in A$ with $ab \in I_1$ and $b \notin J(A)$ then $(ab)y \in I_1 I_2$ but $I_1 I_2$ is a 2J-ideal and $b \notin J(A)$ so that $(ay)^2 \in I_1 I_2 \subseteq I_1$ thus $a^2 = a^2 \cdot 1 = a^2(x + y)^2 = a^2 x^2 + 2a^2 xy + a^2 y^2 \in I_1$ hence I_1 is 2J-ideal, similary I_2 is a 2J-ideal.
- (4) Let T be a maximal 2J – ideal of A , and let $a, b \in I_1$ with $ab \in T$ and $a \notin T$. We have to show that $b \in T$. By (1), $\sqrt{T}$ is an J -ideal, and so a 2J –ideal. By the maximality of T , we obtain $T = \sqrt{T}$. Thus, T is an J –ideal to show $(T : a) < A$ then $1 \in (T : a) \Rightarrow a \in T$ contraiction let $cd \in (P : a)$ and $c \notin J(A)$ then $ad \in T$ then $d \in (T : a)$ that is $(T : a)$ is J -ideal $T \leq (T : a)$ it is clear, Again, by the maximality of P , we obtain that $P = (P : a)$. Hence, $b \in (P : a) = P$.

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