

Reduced Sombor index of graphs

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Abstract

Gutman developed the concept of Sombor indices and it was shown that any vertex degree-based graph invariant can be viewed as a reduced Sombor-type index. In the paper, we investigate extremal graphs concerning the reduced Sombor index for trees, unicyclic graphs, and the class of graphs with a given number of vertices and cut edges (pendent vertices).

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1. Introduction

The notation $G=(V,E)$ denotes a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex v is the number of all neighbors of the vertex v in G and denoted by $\deg_G(v)$. If $\deg_G(u)=1$, then the vertex u is termed *pendent*, and a *pendent edge* is an edge where one of its end vertices is pendent. In a connected graph G , a *cut edge* is an edge that, when removed, causes the graph G to become disconnected. For a vertex w in G , $\mathfrak{M}_G(w)$ denotes the set of all neighbors of w . Additionally, denote $\mathfrak{M}_G[w]$ as the set $\mathfrak{M}_G(w) \cup \{w\}$.

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A novel graph invariant based on vertex degree, introduced by Gutman [9] and called the ‘Sombor index’ of a graph G , denoted by $SO(G)$, is defined as follows:

$$SO(G) = \sum_{xy \in E(G)} \sqrt{\deg_G(x)^2 + \deg_G(y)^2}.$$

After the introduction of the graph invariant, researchers promptly investigated its fundamental properties and applications in [3, 4, 13, 15, 16, 17, 18]: Cruz et al. [3] determined the extremal graphs concerning the Sombor index for hexagon systems, trees and molecular graphs. In [13] and [14], sharp upper and lower bounds on the Sombor index were obtained for some classes of graphs. Das et al. [4] established relations between the classical Zagreb indices of graphs and the Sombor index.

The reduced Sombor index [9] is described as

$$SO_{red}(G) = \sum_{xy \in E(G)} \sqrt{(\deg_G(x) - 1)^2 + (\deg_G(y) - 1)^2}.$$

In [9], it was demonstrated that the irregularity (Albertson index) is related to the distance between the degree-points and their duals. In [5], authors investigated certain vertex degree-based topological indices of an m^k -graph, where k is a positive integer. Gutman [10] showed that any graph invariant based on vertex degree can be viewed as a reduced Sombor-type index, i.e., as being related to the distance between the degree-points and some other elements of the degree-space. Kulli and Gutman [8] established precise formulas of the reduced Sombor index for certain networks. The study of [6] focused on exploring some extremal properties of SO_{red} , and it characterized bicyclic graphs attaining extremal SO_{red} values.

In the paper, we establish some extremal properties of SO_{red} index and identify extremal graphs concerning SO_{red} over trees and unicyclic graphs. For the class of graphs with the given order and cut edges (pendent vertices), the problems have been studied of characterizing graphs that attain the extremal first and second Zagreb indices [7], reduced second Zagreb index [12], Sombor index [13], multiplicative sum Zagreb index [14], Wiener, Hyper-Wiener and Harary indices [2, 19], and F -index [1]. Furthermore, we explore this problem for SO_{red} index.

2. Graphs with maximum SO_{red} -value

The main purpose of the section is to characterize the graphs that have maximal SO_{red} index. Firstly, we present three lemmas that will play a crucial role in our main results.

Lemma 2.1: *Let x and y be vertices in G such that $\deg_G(x) \geq \deg_G(y)$ and $\mathfrak{M}_G(y) \setminus \mathfrak{M}_G[x] = \{y_1, y_2, \dots, y_t\} \neq \emptyset$. If $G' = G - \{yy_1, yy_2, \dots, yy_t\} + \{xy_1, xy_2, \dots, xy_t\}$ then $SO_{red}(G') > SO_{red}(G)$.*

Proof: One can easily see that $\deg_{G'}(z) = \deg_G(z)$ for $z \neq x, y$ whereas $\deg_{G'}(x) = \deg_G(x) + t$ and $\deg_{G'}(y) = \deg_G(y) - t$. For convenience, set $d_z = \deg_G(z) - 1$. for all $z \in V(G)$. It can be easily seen that

$$\sqrt{(d_x + t)^2 + (d_y - t)^2} > \sqrt{d_x^2 + d_y^2}, \quad (1)$$

since $d_x \geq d_y$ and $t \geq 1$. For fixed positive integers p, q ($p \geq q$) and c , consider a function $h(s) = \sqrt{(p+s)^2 + c^2} + \sqrt{(q-s)^2 + c^2}$. Then

$$h'(t) = \frac{p+s}{\sqrt{(p+s)^2 + c^2}} - \frac{q-s}{\sqrt{(q-s)^2 + c^2}},$$

and one can easily see that $h'(s) > 0$ for $s > 0$, since $p \geq q$. Hence, we get $h(t) > h(0)$ because h is continuous, and it follows that

$$\sqrt{(d_x + t)^2 + d_v^2} + \sqrt{(d_y - t)^2 + d_v^2} > \sqrt{d_x^2 + d_v^2} + \sqrt{d_y^2 + d_v^2}. \quad (2)$$

Let $\phi = \begin{cases} 1, & \text{if } xy \in E(G) \\ 0, & \text{if } xy \notin E(G) \end{cases}$. Then from the definition of SO_{red} , we have

$$\begin{aligned} SO_{red}(G') - SO_{red}(G) &= \sum_{u \in \mathfrak{M}_G(x) \setminus \mathfrak{M}_G[y]} \sqrt{(d_x + t)^2 + d_u^2} + \sum_{i=1}^t \sqrt{(d_x + t)^2 + d_{y_i}^2} \\ &\quad + \sum_{v \in \mathfrak{M}_G(x) \cap \mathfrak{M}_G(y)} \sqrt{(d_x + t)^2 + d_v^2} + \sum_{v \in \mathfrak{M}_G(x) \cap \mathfrak{M}_G(y)} \sqrt{(d_y - t)^2 + d_v^2} \\ &\quad + \phi \cdot \sqrt{(d_x + t)^2 + (d_y - t)^2} - \sum_{u \in \mathfrak{M}_G(x) \setminus \mathfrak{M}_G[y]} \sqrt{d_x^2 + d_u^2} - \sum_{i=1}^t \sqrt{d_x^2 + d_{y_i}^2} \\ &\quad - \sum_{v \in \mathfrak{M}_G(x) \cap \mathfrak{M}_G(y)} \sqrt{d_x^2 + d_v^2} - \sum_{v \in \mathfrak{M}_G(x) \cap \mathfrak{M}_G(y)} \sqrt{d_y^2 + d_v^2} - \phi \cdot \sqrt{d_x^2 + d_y^2} \\ &> \sum_{v \in \mathfrak{M}_G(x) \cap \mathfrak{M}_G(y)} (\sqrt{(d_x + t)^2 + d_v^2} + \sqrt{(d_y - t)^2 + d_v^2} - \sqrt{d_x^2 + d_v^2} - \sqrt{d_y^2 + d_v^2}) \\ &\quad + \phi \cdot [\sqrt{(d_x + t)^2 + (d_y - t)^2} - \sqrt{d_x^2 + d_y^2}] > 0 \end{aligned}$$

by using inequalities (1) and (2). This finishes the proof.

In the statement of Lemma 2.1, it is worth noting that if G is connected, then G' is connected if and only if $xy \in E(G)$ or $\mathfrak{M}_G(x) \cap \mathfrak{M}_G(y) \neq \emptyset$. Let $\mathcal{G}_n^{k+}$ be the class of connected graphs with n vertices and at least k cut edges.

Lemma 2.2: [11] *Let G be a connected graph, and xy be an edge in G such that $\mathfrak{M}_G(y) \setminus \mathfrak{M}_G[x] = \{y_1, y_2, \dots, y_t\}$. For the graph $G' = G - \{yy_1, yy_2, \dots, yy_t\} + \{xy_1, xy_2, \dots, xy_t\}$, the cardinality of the set of cut edges in G' is greater than or equal to the cardinality of the set of cut edges in G .*

Proposition 2.3: Let G be a connected graph in $\mathcal{G}_n^{k+}$ with SO_{red} being maximum in $\mathcal{G}_n^{k+}$. Then the maximum degree $\Delta(G)$ is $n-1$.

Proof: On the contrary, assume that x is a vertex such that $\deg_G(x) = \Delta(G)$, and $\deg_G(x) < n-1$. Since G is connected, one can find vertices y and y_1 in G such that xy and yy_1 are edges in G , while xy_1 is not, where y_1 is the vertex at a distance of two from x . Clearly, $y_1 \in \mathfrak{M}_G(y) \setminus \mathfrak{M}_G[x]$ and let $\mathfrak{M}_G(y) \setminus \mathfrak{M}_G[x] = \{y_1, y_2, \dots, y_t\}$. Consider the graph

$$G' = G - \{yy_1, yy_2, \dots, yy_t\} + \{xy_1, xy_2, \dots, xy_t\}.$$

Then $G' \in \mathcal{G}_n^{k+}$ by Lemma 2.2.

Also, we get $SO_{red}(G) < SO_{red}(G')$ since Lemma 2.1, $\deg_G(u) = \Delta(G)$ and $t \geq 1$. Hence, we encounter a contradiction since $SO_{red}(G)$ is maximum in $\mathcal{G}_n^{k+}$. Consequently, we deduce that $\Delta(G) = n-1$.

Denote by $\mathcal{H}_n^{p+}$ the class of connected graphs with n vertices and at least p pendent vertices.

Proposition 2.4: Let G be a graph in $\mathcal{H}_n^{p+}$ with SO_{red} being maximum in $\mathcal{H}_n^{p+}$. Then the maximum degree $\Delta(G)$ is $n-1$.

Proof: On the contrary, assume that x is a vertex such that $\deg_G(x) = \Delta(G)$, and $\deg_G(x) < n-1$. Since G is connected, one can find vertices y and y_1 in G such that xy and yy_1 are edges in G , while xy_1 is not, where y_1 is the vertex at a distance of two from x . Clearly, $y_1 \in \mathfrak{M}_G(y) \setminus \mathfrak{M}_G[x]$ and let $\mathfrak{M}_G(y) \setminus \mathfrak{M}_G[x] = \{y_1, y_2, \dots, y_t\}$. Consider the graph

$$G' = G - \{yy_1, yy_2, \dots, yy_t\} + \{xy_1, xy_2, \dots, xy_t\}.$$

Then, for all vertices z in G except for x and y , $\deg_G(z) = \deg_{G'}(z)$. On the other hand we have $\deg_G(x) \geq 2$ and $\deg_G(y) \geq 2$. Therefore all pendent vertices in G are also pendent in G' and we get $G' \in \mathcal{H}_n^{p+}$. Additionally,

according to Lemma 2.1, we observe that $SO_{\text{red}}(G) < SO_{\text{red}}(G')$, leading to a contradiction. Therefore, we obtain the desired result.

Building on the above proposition, we readily derive the following two theorems.

Theorem 2.5: *For any tree T with n vertices, we have*

$$SO_{\text{red}}(T) \leq (n-2)(n-1)$$

with equality holding if and only if T is isomorphic to the star graph S_n .

Theorem 2.6: *For any unicyclic graph $\mathcal{U}$ with n vertices, we have*

$$SO_{\text{red}}(\mathcal{U}) \leq 2\sqrt{(n-2)^2 + 1} + (n-3)(n-2) + \sqrt{2}$$

with equality holding if and only if $\mathcal{U}$ is isomorphic to $S_n + e$.

Let K_n^p be the obtained graph from a null graph pK_1 by joining each vertex of pK_1 to one vertex of a complete graph K_{n-p} . Note that if $k \geq n-2$ then $\mathcal{G}_n^{k+}$ is the same as the set of all trees with n vertices. Consequently, when $k \geq n-2$, $SO_{\text{red}}(S_n)$ attains its maximum in $\mathcal{G}_n^{k+}$, since Theorem 2.5.

Proposition 2.7: Let n and k be non-negative integers such that $k \leq n-3$. If $SO_{\text{red}}(G)$ attains its maximum in $\mathcal{G}_n^{k+}$, then G is isomorphic to K_n^k .

Proof: Let w be a vertex in G such that $\deg_G(w) = \Delta(G)$. Then we have $\deg_G(w) = n-1$ from Proposition 2.3. Therefore, each cut edge of G is a pendent edge incident to w . Let l ($l \geq k$) be the number of cut edges in G . If $l \geq n-2$, then by Theorem 2.5, G is isomorphic to S_n . According to the definition of SO_{red} , if $xy \notin E(G)$, then we have:

$$SO_{\text{red}}(G) < SO_{\text{red}}(G + xy). \quad (3)$$

On the other hand by $k \leq n-3$ and the inequality (3), we have $S_n + e \in \mathcal{G}_n^{k+}$ and $SO_{\text{red}}(S_n) < SO_{\text{red}}(S_n + e)$. This is a contradiction. Let now $l \leq n-3$. Since $SO_{\text{red}}(G)$ is maximum, we have $G \cong K_n^l$ by the inequality (3). Suppose that $l > k$ and let G' be a obtained graph from G by connecting a non-pendent vertex to a pendent vertex of G . Then $G' \in \mathcal{G}_n^{k+}$ and

$$SO_{\text{red}}(G') > SO_{\text{red}}(G)$$

by the inequality (3). However, this contradicts the fact that $SO_{\text{red}}(G)$ attains its maximum in $\mathcal{G}_n^{k+}$. Thus, we conclude that $l = k$, completing the proof.

Theorem 2.8: For any graph G in $\mathcal{G}_n^k$ distinct from K_n^k , we have

$$SO_{red}(G) < SO_{red}(K_n^k) = k(n-2) + \binom{q}{2}(q-1)\sqrt{2} + q\sqrt{(q-1)^2 + (n-2)^2},$$

where $q = n - k - 1$.

Proof: From Proposition 2.7, we have

$$SO_{red}(G) < SO_{red}(K_n^k)$$

since G is distinct from K_n^k . Because $K_n^k \in \mathcal{G}_n^k \subseteq \mathcal{G}_n^{k+}$. Moreover, it can be readily verified that

$$SO_{red}(K_n^k) = k(n-2) + \binom{q}{2}(q-1)\sqrt{2} + q\sqrt{(q-1)^2 + (n-2)^2}.$$

Hence, we get the required result.

Denote by $\mathcal{H}_n^p$ the class of connected graphs with n vertices and p pendent vertices. Applying the same argument as in the proof of Theorem 2.8, we obtain the following result.

Theorem 2.9: For any graph G in $\mathcal{H}_n^p$, we have

$$SO_{red}(G) \leq p(n-2) + \binom{n-p-1}{2}(n-p-2)\sqrt{2} + (n-p-1)\sqrt{(n-2)^2 + (n-p-2)^2}$$

with equality holding if and only if $G \cong K_n^p$.

3. Graphs with minimum SO_{red} -value

The main purpose of this section is to identify graphs with the minimum reduced Sombor index. Initially, our focus is on determining the extremal trees of order n with the minimum reduced Sombor index.

Theorem 3.1: For any tree T with n vertices, we have

$$SO_{red}(T) \geq (n-3)\sqrt{2} + 2 \tag{4}$$

with equality holding if and only if $T \cong P_n$.

Proof: Clearly, T has at least two pendent edges, as expected for a tree. Let e_1 and e_2 be pendent edges in T . Denote by u_1 and u_2 the non-pendent

end vertices of e_1 and e_2 , respectively. Also we denote $\deg_G(u_1) = d_1$ and $\deg_G(u_2) = d_2$. Then we have

$$\begin{aligned} SO_{red}(T) &= (d_1 - 1) + (d_2 - 1) + \sum_{xy \in E(T) \setminus \{e_1, e_2\}} \sqrt{(\deg_G(x) - 1)^2 + (\deg_G(y) - 1)^2} \\ &\geq d_1 + d_1 - 2 + \sum_{xy \in E(T) \setminus \{e_1, e_2\}} \frac{\deg_G(x) - 1 + \deg_G(y) - 1}{\sqrt{2}} \\ &= (d_1 + d_2 - 2) \cdot \frac{\sqrt{2} - 1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot M_1(T) - (n - 1)\sqrt{2} \\ &\geq \sqrt{2}(\sqrt{2} - 1) + \frac{1}{\sqrt{2}} \cdot (4n - 6) - (n - 1)\sqrt{2} = (n - 3)\sqrt{2} + 2 \end{aligned}$$

since $d_1 = \deg_G(u_1) \geq 2$, $d_2 = \deg_G(v_1) \geq 2$ and $M_1(T) \geq 4n - 6$, where M_1 is the well-known first Zagreb index and $M_1(T) = 4n - 6$ if and only if $T \cong P_n$. Therefore, we easily conclude that equality holds in (4) if and only if $T \cong P_n$.

A pendent path in G is a path where one terminal vertex, known as the origin, has a degree of at least three, the other terminal vertex is pendent, and each internal vertex (if it exists) has a degree of two in G . Denote by U_n^k the unicyclic graph with n vertices, exactly one pendent vertex and girth $n - k$. Let $\mathcal{U}_n^{k+} = \{U_n^s, s \geq k\}$.

Theorem 3.2: Let k and n be non-negative integers so that $5 \leq k + 3 \leq n$. Consider a cyclic graph G satisfying $SO_{red}(G)$ attains its minimum in $\mathcal{G}_n^{k+}$. Then $G \in \mathcal{U}_n^{k+}$.

Proof: If the size of G exceeds n , then there must be a non-cut edge e in G . Consequently, $SO_{red}(G) > SO_{red}(G - e)$ and $G - e \in \mathcal{G}_n^{k+}$, leading to a contradiction. Therefore, G must be unicyclic, since $k \leq n - 3$.

Suppose there exists two pendent paths in G , denoted as Q and P with origins v and u , respectively. Let x be a vertex on P such that $xu \in E(G)$. Also let zy be the pendent edge on Q and $\deg_G(y) = 1$. Recall that $d_z = \deg_G(z) - 1$ for all vertex in G . If $G' = G - ux + xy$ and $u \neq v$ then

$$\begin{aligned} SO_{red}(G) - SO_{red}(G') &= \sum_{w \in \mathfrak{M}_G(u) \setminus x} \sqrt{d_u^2 + d_w^2} + \sqrt{d_u^2 + d_x^2} \\ &\quad + d_z - \sum_{w \in \mathfrak{M}_G(u) \setminus x} \sqrt{(d_u - 1)^2 + d_w^2} - \sqrt{1^2 + d_x^2} - \sqrt{1^2 + d_z^2} \\ &> \sqrt{d_u^2 + d_x^2} - \sqrt{1 + d_x^2} + d_z - \sqrt{1 + d_z^2}. \end{aligned}$$

Since $d_x \leq 1$, $d_u \geq 2$ and $d_z \geq 1$, we get $SO_{red}(G) > SO_{red}(G')$.

Let now $G' = G - ux + xy$ and $v \equiv u$. If $u \equiv z$ then

$$\begin{aligned}
SO_{red}(G) - SO_{red}(G') &= \sum_{w \in \mathfrak{M}_G(u) \setminus \{x, y\}} \sqrt{d_u^2 + d_w^2} + d_u + \sqrt{d_u^2 + d_x^2} \\
&\quad - \sum_{w \in \mathfrak{M}_{G'}(u) \setminus \{x, y\}} \sqrt{(d_u - 1)^2 + d_w^2} - \sqrt{(d_u - 1)^2 + 1^2} - \sqrt{1^2 + d_x^2} \\
&> \sqrt{d_u^2 + d_x^2} - \sqrt{1^2 + d_x^2} \geq 0
\end{aligned}$$

by $d_u \geq 1$.

Let $u \neq z$ and x' be the vertex on Q adjacent to u . Then

$$\begin{aligned}
SO_{red}(G) - SO_{red}(G') &= \sum_{w \in \mathfrak{M}_G(u) \setminus \{x, x'\}} \sqrt{d_u^2 + d_w^2} + \sqrt{d_u^2 + d_x^2} \\
&\quad + \sqrt{d_u^2 + d_x^2} + d_z - \sqrt{1^2 + d_x^2} - \sqrt{1^2 + d_z^2} \\
&\quad - \sum_{w \in \mathfrak{M}_{G'}(u) \setminus \{x, x'\}} \sqrt{(d_u - 1)^2 + d_w^2} - \sqrt{(d_u - 1)^2 + d_x^2} \\
&> \sqrt{d_u^2 + d_x^2} + d_z - \sqrt{1^2 + d_x^2} - \sqrt{1^2 + d_z^2}.
\end{aligned}$$

Therefore, based on the above inequalities, it follows that

$$SO_{red}(G) > SO_{red}(G')$$

since $d_u \geq 2$, $d_z \geq 1$ and $d_x \leq 1$. This is a contradiction, because $G' \in \mathcal{G}_n^{k+}$.

Hence G is isomorphic to U_n^s for some positive integer s . Thus, since $G \in \mathcal{G}_n^{k+}$ and $s \geq k$, we conclude that $G \in \mathcal{U}_n^{k+}$.

Theorem 3.3: Let $SO_{red}(G)$ attain its minimum in $\mathcal{G}_n^k$.

- (i) If $k = n - 1$ then $G \cong P_n$.
- (ii) If $k < n - 1$ then $G \cong U_n^k$.

Proof:

- (i) As $k = n - 1$, $\mathcal{G}_n^k$ becomes the class of trees with n vertices, leading to the conclusion that G is isomorphic to P_n by Theorem 3.1.
- (ii) When $k = n - 2$, $\mathcal{G}_n^k$ is an empty set. Therefore $k \leq n - 3$. If $k = 0$ or $k = 1$ then there is the unique unicyclic graph C_n or U_n^1 in $\mathcal{G}_n^k$, respectively. Otherwise, by $\mathcal{G}_n^k \subset \mathcal{G}_n^{k+}$ and Theorem 3.2, we have $G \in \mathcal{G}_n^k \cap \mathcal{U}_n^{k+}$. This completes the proof and establishes the required result.

Theorem 3.4: Let g and n be non-negative integers satisfying $3 \leq g \leq n - 2$. If $SO_{red}(G)$ attains its minimum in the set of unicyclic graphs with n vertices and girth g , then G is isomorphic to U_n^{n-g} .

Proof: Every unicyclic graph with n vertices and girth g has precisely $n - g$ cut edges. Therefore, by Theorem 3.3, we readily conclude that G is isomorphic to U_n^{n-g} .

Theorem 3.5: For a unicyclic graph G with n vertices, we have

$$SO_{red}(G) \geq n\sqrt{2}$$

with equality holding if and only if G is isomorphic to C_n .

Proof: Let g be the girth of G . If $g \leq n - 2$ then we have $SO_{red}(G) \geq (n - 4)\sqrt{2} + 3\sqrt{5} + 1 > n\sqrt{2}$. If $g \geq n - 1$ then $G \cong U_n^1$ or $G \cong C_n$. An elementary calculation gives that $SO_{red}(U_n^1) = (n - 3)\sqrt{2} + 2\sqrt{5} + 2 > n\sqrt{2} = SO_{red}(C_n)$. This finishes the proof and establishes the required result.

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