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On the central resolver set of the edge coronation graphs

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Abstract

Graph theory as part of mathematics, has experienced significant development from the theoretical aspect, one of it is the theory of metric dimension. The concept of metric dimensions has evolved greatly, including the dominant metric dimension and the complement metric dimension. In this study, the concept of the central metric dimension is introduced, that is a combination of the metric dimension and central of a graph. The minimum number of vertices of a resolver set that contains a central set is called the central metric dimension of graph G, and denoted by $dim_{cen}(G)$. Several characterizations of a graph having a certain central metric dimension are yielded in this study. Several relations are obtained between the central metric dimension and the metric dimension. Furthermore, the central metric dimension is applied on edge coronation graphs. The edge coronation of graphs G and H is denoted by $(G \lozenge H)$. The results of the study show that the central metric dimension of edge coronation of G and H are influenced by the central set and the order of graph G and the metric dimension of graph H.

Subject Classification: *(2020) 05C12, 05C75, 05C38.*

Keywords: *Graph, Metric dimension, Central metric dimension, Accessible transportation, Edge coronation graphs.*

Introduction

Graph theory as part of mathematics, is experiencing rapid development from the theoretical aspect, one of the concepts that continues to develop is the metric dimension. The concept of metric dimensions was firstly introduced by Slater in 1975 [10], he defined the resolver set. Separately, in 1976 [7], Harary also constructed a similar concept. Several studies related to metric dimension have been carried out on generalized wheel graph [11] and k −subdivisin graphs [13], corona product graph [1], [2], [9] and graph characterization with certain metric dimension [4].

The next development is the emergence of the concept of the complement metric dimension by Susilowati et al. [14] and the dominant metric dimension, which was initiated by Brigham et al. [3] and continued by [8] and [12]. The vertex set on a graph is called the domination set if every vertex outside the set is adjacent to at least one element of it [12]. Meanwhile, the resolver set on a graph is a set that if every vertex on the graph has a different representation to the set [4].

Harary in [6] has been introduced the concepts of eccentricity, radius, and diameter on graphs. The eccentricity is the farthest distance from one vertex to another on graph G. The radius is the smallest eccentricity among the vertices of graph G, while the diameter is the highest. The vertex which has the minimum eccentricity on graph G is called the central vertex. Meanwhile, the central set of a graph G is the set whose elements are all central vertices [11]. The subgraph constructed by the central set of G is called the center of G [5]. Some of the results of previous studies are presented as follows.

Lemma 1.1: [12] *Let G be a connected graph and $W \subseteq V(G)$ is a resolver set of G, then for any $u_i, u_j \in W$ hold $r(u_i|W) \neq r(u_j|W)$ where $i \neq j$.*

Lemma 1.2: [4] *Let G be a connected graph, then $dim(G) = 1$ if and only if G is a path graph.*

Lemma 1.3: [5] *Every graph is the center of some graphs.*

Theorem 1.4: [15] If $n \geq 7$ then $dim(K_1 + C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

Theorem 1.5: [16] If $n \neq 1,2,3,6$ then $dim(K_1 + P_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$

Result and Discussion

In this study, the concept of a central metric dimension was developed, which is a combination of the concept of a metric dimension and a central set.

Definition 2.1: For the connected graph G, an ordered set $W \subseteq V(G)$ where $W \neq \emptyset$ is called a **central resolver set** of G if W is a resolver set and contains a central set. A central resolver set with minimum cardinality is called a **central basis**. Meanwhile, the cardinality of central basis of G is called a **central metric dimension**, denoted by $dim_{cen}(G)$.

Since central metric dimension involve a central set, then the properties of central set are described as follows.

Lemma 2.2: *For the connected graph G, the central set of G is $V(G)\ iff\ rad(G) = diam(G)$.*

Proof: Let $S = V(G)$ and suppose $rad(G) < diam(G)$. There is a vertex $u \in V(G)$ such that $e(u) > rad(G)$. Consequently u is not a central vertex or $u \notin S$, contrary to $S = V(G)$. Conversely, let $rad(G) = diam(G)$ and suppose $S \neq V(G)$, then there is $v \notin V(G)$ where $v \notin S$. Consequently $e(v) > rad(G) = diam(G)$,which is a contradiction.

Corollary 2.3: *For the connected graph G, $dim_{cen}(G) = |V(G)|\ iff\ rad(G) = diam(G)$.*

Next discussion is presented the central metric dimension of special graph, such as path graph P_n, cycle graph C_n, star graph S_n and complete graph K_n. Firstly, the central set of path and cycle graph are presented as follows.

Lemma 2.4: *Let P_m be a path graph with order $m \geq 2$, then the central set of P_m is $\left\{v_{\left\lceil \frac{m}{2} \right\rceil}\right\}$ for m is an odd and $\{v_{\left\lceil \frac{m}{2} \right\rceil}, v_{\left\lceil \frac{m}{2} \right\rceil+1}\}$ for m is an even.*

Proof: Let P_m be a path graph with order $m \geq 2$, with the vertex set $V(P_m) = \{v_k | k = 1,2,3, \dots, m\}$ and the edge set $E(P_m) = \{v_k v_{k+1} | k = 1,2,3, \dots, m-1\}$.

1. For m is an odd, then the eccentricity of any vertices on P_m are $(v_1) = e(v_m) = m-1, e(v_2) = e(v_{m-1}) = m-2, \dots, e\left(v_{\left\lceil \frac{m}{2} \right\rceil -1}\right) = e\left(v_{\left\lceil \frac{m}{2} \right\rceil +1}\right) = \left\lceil \frac{m}{2} \right\rceil, e\left(v_{\left\lceil \frac{m}{2} \right\rceil}\right) = \left\lceil \frac{m}{2} \right\rceil - 1$. Hence, vertex $v_{\left\lceil \frac{m}{2} \right\rceil}$ is a vertex that has minimum eccentricity, therefore the vertex $v_{\left\lceil \frac{m}{2} \right\rceil}$ is a central vertex of P_m. Thus, the central set of P_m where m odd is $\left\{v_{\left\lceil \frac{m}{2} \right\rceil}\right\}$.
2. For m is an even, then the eccentricity of any vertices on P_m are $(v_1) = e(v_m) = m-1, e(v_2) = e(v_{m-1}) = m-2, \dots, e\left(v_{\left\lceil \frac{m}{2} \right\rceil -1}\right) = e\left(v_{\left\lceil \frac{m}{2} \right\rceil +2}\right) = \frac{m}{2} + 1, e\left(v_{\left\lceil \frac{m}{2} \right\rceil}\right) = e\left(v_{\left\lceil \frac{m}{2} \right\rceil +1}\right) = \left\lceil \frac{m}{2} \right\rceil$. Hence, vertices $v_{\frac{m}{2}}$ and $v_{\frac{m}{2}+1}$ are vertices that has minimum eccentricity, therefore the vertices $v_{\frac{m}{2}}$ and $v_{\frac{m}{2}+1}$ are central vertices of P_m. Thus, the central set of P_m where m even is $\left\{v_{\left\lceil \frac{m}{2} \right\rceil}, v_{\left\lceil \frac{m}{2} \right\rceil +1}\right\}$.

Theorem 2.5: $\dim_{\mathrm{cen}}(\mathrm{P_n}) = 2$ where $\mathrm{n} \geq 2$.

Proof: Let P_n be a path graph with order $n \geq 2$ with the vertex set $V(P_n) = \{v_i | i = 1,2,3, \dots, n\}$ and the edge set $E(P_n) = \{v_i v_{i+1} | i = 1,2,3, \dots, n-1\}$. According to **Lemma 2.4,** the central set of P_n is $S_1 = \left\{v_{\left\lceil \frac{n}{2} \right\rceil}\right\}$ for n odd and $S_2 = \left\{v_{\left\lceil \frac{n}{2} \right\rceil}, v_{\left\lceil \frac{n}{2} \right\rceil +1}\right\}$ for n even. Let $W = \left\{v_{\left\lceil \frac{n}{2} \right\rceil}, v_{\left\lceil \frac{n}{2} \right\rceil +1}\right\} \subseteq V(P_n)$, then the representation of any vertices $v_i \in V(P_n)$ to W, i.e.

$$r(v_i|W) = \left(d\left(v_i, v_{\left\lceil \frac{n}{2} \right\rceil}\right), d\left(v_i, v_{\left\lceil \frac{n}{2} \right\rceil +1}\right)\right) = \left(\left|-i + \left\lceil \frac{n}{2} \right\rceil\right|, \left|1 - i + \left\lceil \frac{n}{2} \right\rceil\right|\right).$$

Hence, by **Lemma 1.1**, for any vertices $v_i, v_j \in V(P_n)$ hold $r(v_i|W) \neq r(v_j|W)$, where $i \neq j$. Therefore, W is a resolver set of P_n. Since $S_1 \subseteq W$ and $S_2 \subseteq W$, then W is a central resolver set of P_n.

Furthermore, showing that the set W is a central resolver set with the minimum cardinality. Let $R \subseteq V(G)$ where $S_1 \subseteq R$ and $|R| < |W|$, i.e. $|R| = 1$,

(i). For n is an odd, then $R = S_1 = \left\{v_{\left\lceil \frac{n}{2} \right\rceil}\right\}$, consequently $r(v_{i-1}|R) = r(v_{i+1}|R) = (1)$, therefore R is not a resolver set. Thus R is not a central resolver set.

(ii). For n is an even, then there is no $R \subseteq V(G)$ where $S_2 \subseteq R$ and $|R| = 1$.

According to (i) and (ii), it is concluded that W is a central resolver set with the minimum number of vertices. Thus $dim_{cen}(P_n) = 2$.

Lemma 2.6: *The central set of C_n is $V(C_n)$.*

Proof: Let the vertex set $V(C_n) = \{v_i | i = 1,2, \dots, n\}$ and the edge set $E(C_n) = \{v_i v_{i+1} | i = 1,2, \dots, n-1\} \cup \{v_1 v_n\}$, then

$$e(v_1) = e(v_2) = \cdots = e(v_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

Hence, for any vertices $v_i \in V(C_n), i = 1,2, \dots, n$ has the same eccentricity, i.e. $e(v_i) = \left\lfloor \frac{n}{2} \right\rfloor$. Consequently, for any vertices on $C_n$is a central vertex. Thus, the central set of C_n is $V(C_n)$. ■

Below are the effects of **Lemma 2.2** and **Corollary 2.3**.

Corollary 2.7: $dim_{cen}(C_n) = n.$

Corollary 2.8: $dim_{cen}(K_n) = n$ *where* $n \geq 3$.

Corollary 2.9: $dim_{cen}(K_{n,m}) = n + m$ *where* $n, m \geq 2$

Corollary 2.10: $dim_{cen}(W_{n,m}) = n + m$, *where* $n > 1$.

Further results, some graphs characterization with certain central metric dimension are presented as follows.

Theorem 2.11: *Let H be a connected graph. Then* $dim_{cen}(H) = |V(H)| - 1$ $iff\ H \approx K_{1,n}$.

Theorem 2.12: $dim_{cen}(G) = 1\ iff\ G \cong K_1$.

Here are some results that show the relationship of the central metric dimension to the metric dimension of a graph.

Theorem 2.13: *For the connected graph G,* $dim(G) = dim_{cen}(G)\ iff\ G \cong K_1$.

Proof: Let $G \cong K_1$, then it is clear that $dim(G) = dim_{cen}(G) = 1$. Conversely, let $dim(G) = dim_{cen}(G)$. Suppose $G \ncong K_1$, then $|V(G)| \geq 2$ and $|E(G)| \geq 1$. Let $V(G) = \{v_1, \dots, v_n\}$, then $e(v_i) \geq 1$.

a. For $n = 2$, then $e(v_1) = e(v_2) = 1$. Hence $dim(G) = 1$ and $dim_{cen}(G) = 2$, which is a contradiction.
b. For $n > 2$, let $B = \{v_1, \dots, v_n\}$ be a basis of G. Since $dim(G) = dim_{cen}(G)$, then $B \neq V(G)$ and B is a central set of G. Consequently, according to **Lemma 1.3**, there will be a couple vertices $u, v \in V(G)$ such that $d(v_i, u) = d(v_i, v) = 1$ and $d(u, v) = 2$. Hence $r(u|B) = r(v|B)$, which is a contradiction.

In line with the two cases above, it can be concluded that $G \cong K_1$.

Theorem 2.14: *Let C_n be a cycle with order $n > 3$. Then* $dim_{cen}(K_1 + C_n) \leq dim_{cen}(C_n)$.

Proof: According to **Corollary 2.7,** $dim_{cen}(C_n) = n$. Suppose $dim_{cen}(K_1 + C_n) > n$, then $dim_{cen}(K_1 + C_n) = n + 1.$ Consequently, the central set of $K_1 + C_n$ is $V(K_1 + C_n)$. According to **Corollary 2.3**, $diam(K_1 + C_n) = rad(K_1 + C_n)$, meanwhile $rad(K_1 + C_n) = 1$ and $diam(K_1 + C_n) = 2 > 1$, which is a contradiction.

The next part presents the central metric dimension of the edge coronation graph of graph G and H, denoted by $G \lozenge H$. Let G be a connected graph with order m and size p and H be a connected graph with order n. Let $V(G) = \{v_i | \ i = 1,2, \dots, m\}$, $E(G) = \{e_j \big| \ j = 1,2,, \dots, p\}$, $V(H) = \{u_i | \ i = 1,2, \dots, n\}$. A copy$-j$ of H is denoted H_{e_j}, with $j = 1,2, \dots, p$ and $V\left(H_{e_j}\right) = \{u_i^{e_j} | u_i \in V(H), e_j \in E(G)\}$ for $j = 1,2, \dots, p$. The vertex set of $G \lozenge H$ is $V(G \lozenge H) = V(G_0) \cup_{j=1}^{p} V\left(H_{e_j}\right)$, with $V(G_0) = \{v_{0i} | v_i \in V(G), i = 1,2,, \dots, m\}$ and $E(G_0) = \{v_{0i}v_{0j} | v_i v_j \in E(G)\}$. Edge set $G \lozenge H$ is $E(G \lozenge H) = E(G_0) \cup \left\{u_i^{e_j} u_k^{e_j} | \in E\left(H_{e_j}\right)\right\} \cup \left\{v_{0i} u_k^{e_j} | v_{0i} \in V(G_0), u_k^{e_j} \in V\left(H_{e_j}\right), 1 \le j \le p\right\}$.

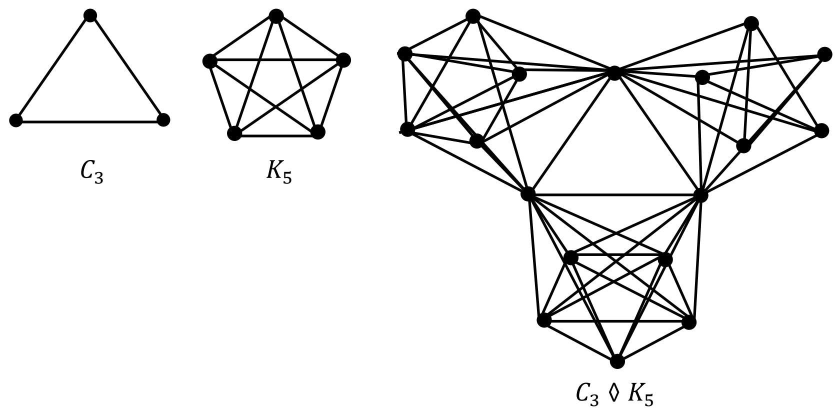

Figure 1

For example, given the cycle C_3 and the complete graph K_5, then the graph $C_3 \lozenge K_5$ is shown in **Figure 1.**

Lemma 2.15: *Let C_d be a cycle graph with $d \ge 3$ and H be a connected graph. If S is a central set of $C_d \lozenge H$ then*

$$S = \begin{cases} V(C_d \lozenge H), for\ d\ is\ an\ odd \\ V(C_{d0})\ for\ d\ is\ an\ even \end{cases}$$

Proof: There are two cases for d,

a. For d is an odd

The eccentricity of every vertex on $C_d \lozenge H$ is presented as follows. The eccentricity of every vertex on C_{d0} is $e(v_{01}) = e(v_{02}) = e(v_{03}) = \cdots = e(v_{0d}) = \left\lfloor \frac{d}{2} \right\rfloor + 1$, and the eccentricity of every vertex on H^{e_j} is $e(u_i^{e_1}) = e(u_i^{e_2}) = e(u_i^{e_3}) = \cdots = e(u_i^{e_d}) = \left\lfloor \frac{d}{2} \right\rfloor + 1$ with $i = 1,2,3, \dots, n$. Hence, for every vertex $V(C_d \lozenge H)$ has the same eccentricity, then $rad(C_d \lozenge H) = diam(C_d \lozenge H) = \left\lfloor \frac{d}{2} \right\rfloor + 1$. According to **Lemma 2.2,** the **c**entral set of $C_d \lozenge H$ is $V(C_d \lozenge H)$.

b. For d is an even

The eccentricity every vertex on $C_d \lozenge H$ is presented as follows. The eccentricity of every vertex on C_{d0} is $e(v_{01}) = e(v_{02}) = e(v_{03}) = \cdots = e(v_{0d}) = \left\lfloor \frac{d}{2} \right\rfloor$ and the eccentricity of every vertex on H^{e_j} is $e(u_i^{e_1}) = e(u_i^{e_2}) = e(u_i^{e_3}) = \cdots = e(u_i^{e_d}) = \left\lfloor \frac{d}{2} \right\rfloor + 1$ with $i = 1,2,3, \dots, n$. Hence, vertices $v_{01}, v_{02}, \dots, v_{0d}$ are vertices which have minimum eccentricity, then $\{v_{01}, v_{02}, v_{03}, \dots, v_{0d}\}$ is a central set of $C_d \lozenge H$. Thus, central set of $C_d \lozenge H$ is $V(C_{d0})$.

Lemma 2.16: *Let K_s be a complete graph with $s \geq 4$ and H be a connected graph. If S is a central set of $K_s \lozenge H$ then $S = V(K_{s0})$*

Proof: The eccentricity of every vertex on K_{e0} is $e(v_{01}) = e(v_{02}) = e(v_{03}) = \cdots = e(v_{0e}) = 2$, and the eccentricity of every vertex on H^{e_j} is $e(u_i^{e_1}) = e(u_i^{e_2}) = e(u_i^{e_3}) = \cdots = e\left(u_i^{e_{\frac{e(e-1)}{2}}}\right) = 3$ with $i = 1,2,3, \dots, s$. Hence, $v_{01}, v_{02}, \dots, v_{0e}$ are vertices which have the minimum eccentricity, then $\{v_{01}, v_{02}, \dots, v_{0s}\}$ is a central set of $K_s \lozenge H$. Thus, the central set of $K_s \lozenge H$ is $V(K_{s0})$.

Corollary 2.17: *Let C_d be a cycle with $d \geq 3$, if d is odd and H is a connected graph then $dim_{cen}(C_d \lozenge H) = |V(C_d \lozenge H)|$*

Proof: According to **Lemma 2.2**, $dim_{cen}(C_d \lozenge H) = |V(C_d \lozenge H)|$.

Theorem 2.18: *Let C_d be a cycle with $d > 3$, d even and graph K_e is a complete graph with $e \geq 2$. Then $dim_{cen}(C_d \lozenge K_e) = |V(C_d)| + |E(C_d)|\big(dim(K_e)\big)$.*

Proof: According to **Lemma 2.15**, the central set of $C_d \lozenge K_e$ is $V(C_{d0})$. Let $W = V(C_d \lozenge K_e) \backslash \{u_1^{e_1}, u_1^{e_2}, u_1^{e_3}, \dots, u_1^{e_d}\}$ so $|W| = |V(C_{d0})| + |E(C_{d0})|\big(\dim(K_e)\big)$ and $V(C_{d0}) \subseteq W$. Take any two distinct vertices $x, y \in V(C_d \lozenge K_e) \backslash W$, then x, y are the vertices on the different branch. Let $x = u_1^{e_p}, y = u_1^{e_q}$. Hence, $d(x, v_{0k}) = 1$, $d(v_{0k}, v_{0l}) = z, 1 \leq z \leq \left\lfloor \frac{d}{2} \right\rfloor$ with $k = p, p+1; l = q, q+1$ and

$d(v_{0l}, y) = 1$ then $d(x, v_{0l}) = d(x, v_{0k}) + d(v_{0k,}v_{0l}) = 1 + z$ and $d(x, y) = d(x, v_{0l}) + d(v_{0l}, y) = 2 + z$. So $d(x, v_{0k}) < d(x, v_{0l}) < d(x, y)$. Then for $x, y \in V(C_d \between K_e)\backslash W$ imply $r(x|W) \neq r(y|W)$.

Take any $T \subseteq V(C_d \between K_e)$ with $|T| < |W|$ and T contain a central set $V(C_{d0})$. Let $|T| = |W| - 1$, then there is j such that maximum $e - 2$ vertices on $K_e^{e_j}$ are contained in T. Let $u_a^{e_j}$ and $u_b^{e_j}$ with $a \neq b$ be two vertices that are not contained in T. Then $r(u_a^{e_j}|T) = r(u_b^{e_j}|T)$. So W is a central basis. Because $|V(C_{d0})| = |V(C_d)|$ and $|E(C_{d0})| = |E(C_d)|$, then $dim_{cen}(C_d \between K_e) = |V(C_d)| + |E(C_d)|(\dim(K_e))$.

Theorem 2.19: *Let K_e and K_b be two complete graphs with $e \geq 4$ and $b \geq 2$. Then $dim_{cen}(K_e \between K_b) = |V(K_e)| + |E(K_e)|(dim(K_b))$.*

Proof: According to **Lemma 2.16,** the central set of $K_e \between K_b$ is $V(K_{e0})$. Let $W = V(K_e \between K_b)\backslash\left\{u_1^{e_1}, u_1^{e_2}, u_1^{e_3}, \dots, u_1^{e_{\frac{e(e-1)}{2}}}\right\}$ so $|W| = |V(K_{e0})| + |E(K_{e0})|(dim(K_b))$ and $V(K_{e0}) \subseteq W$. Take any two distinct vertices $x, y \in V(K_e \between K_b)\backslash W$, then x and y are in the different branch. Let $x = u_1^{e_p}, y = u_1^{e_q}$. Hence, $d(x, v_{0k}) = 1$, $d(v_{0k,}v_{0l}) = 1$ for $l \neq k$ and $d(v_{0l}, y) = 1$. Then $d(x, v_{0l}) = d(x, v_{0k}) + d(v_{0k,}v_{0l}) = 2$ and $d(x, y) = d(x, v_{0l}) + d(v_{0l}, y) = 3$. So $d(x, v_{0k}) < d(x, v_{0l}) < d(x, y)$ and $r(x|W) \neq r(y|W)$. With the same step in **Theorem 2.18**, it can be proven that $dim_{cen}(K_e \between K_b) = |V(K_e)| + |E(K_e)|(dim(K_b))$.

Theorem 2.20: *Let $K_{1,f}$ be star with $f \geq 3$ and K_e be complete graph with $e \geq 2$. Then $dim_{cen}(K_{1,f} \between K_e) = |V(K_{1,f})| + |E(K_{1,f})|(dim(K_e))$.*

Proof: Let $E(K_{1,f}) = \{v_0v_i | i = 1,2,3, \dots, f\}$. It is easy to see that the central set of $K_{1,f} \between K_e$ is $\{v_{00}\}$. Choose $W = V(K_{1,f} \between K_e)\backslash\{u_1^{e_1}, u_1^{e_2}, u_1^{e_3}, \dots, u_1^{e_f}\}$, so

$|W| = |V(K_{(1,f)0})| + |E(K_{(1,f)0})|(dim(K_e))$. Take any two distinct vertices $x, y \in V(C_d \between K_e)\backslash W$, then x and y are in the different branch. Let $x = u_1^{e_l}$ and $y = u_1^{e_j}$. Then

$$r(u_1^{e_j}|W) = \begin{cases} (1, a, b, \dots, b), j = 1 \\ (1, b, a, b, \dots, b), j = 2 \\ \left(0, b, \dots, b, \underbrace{a}_{(i+1)-th}, b, \dots, b\right), j = 3,4, \dots, f - 1 \\ (1, b, \dots, b, a), j = f \end{cases}$$

So, W is a central resolver set of $K_{1,f} \between K_e$. For any $T \subseteq V(K_{1,f} \between K_e)$ with $|T| < |W|$ and T contains a central set. Let $|T| = |W| - 1$, then there is j such that there are maximum $e - 2$ vertices in $K_e^{e_j}$ contained in T or there are $f - 1$ vertices in $K_{(1,f)0}$ contained in T. Other vertices that are not contained in T are

$u_a^{e_j}$ and $u_b^{e_j}$ with $a \neq b$ and $a, b = 1,2,3, \dots, e$ or v_{0i} with $i \neq 0$. As consequently $r(u_a^{e_j}|T) = r(u_b^{e_j}|T)$ or $r(v_{0i}|T) = r(u_k^{e_j}|T)$. So W is a basis central of $K_{1,f} \lozenge K_e$. Because $|V(K_{(1,f)0})| = |V(K_{1,f})|$ and $|E(K_{(1,f)0})| = |E(K_{1,f})|$, then $dim_{cen}(K_{1,f} \lozenge K_e) = |V(K_{1,f})| + |E(K_{1,f})|(dim(K_e))$.

For the coronation graphs below, they have the same metric dimension pattern as **Theorem 2.18 - 2.20**.

Theorem 2.21: *Let $K_{1,f}$ be star with $f \geq 2$ and K_e be complete graph with $e \geq 4$. Then $dim_{cen}(K_e \lozenge K_{1,f}) = |V(K_e)| + |E(K_e)|\left(dim(K_{1,f})\right)$.*

Theorem 2.22: *Let $K_{1,f}$ and $K_{1,z}$ be star graphs with $f \geq 3$ and $z \geq 2$. Then $dim_{cen} = (K_{1,f} \lozenge K_{1,z}) = |V(K_{1,f})| + |E(K_{1,f})|\left(dim(K_{1,z})\right)$.*

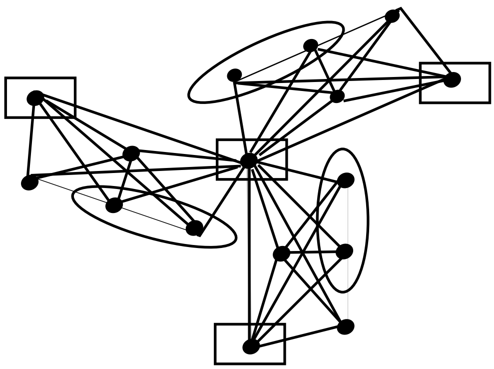

Figure 2
$\boldsymbol{K_{1,3} \lozenge K_{1,3}}$

For example, given the graph $K_{1,3} \lozenge K_{1,3}$ in **Figure 2.** which has a central set consisting of vertices inside the square and has a central basis consisting of vertices inside the square and circle.

The following results, it shows that when a path as the main of the edge coronation graph, then the metric dimension has a different pattern from the results obtained previously.

Theorem 2.25: *Let P_c be path with $c \geq 3$ and K_e be complete graph with $e \geq 2$. Then* $dim_{cen}(P_c \lozenge K_e) = \begin{cases} 3 + |E(P_c)|(dim(K_e)), odd\ c \\ 4 + |V(K_e)| + (|E(P_c)| - 1)(dim(K_e)), even\ c \end{cases}$.

Proof: For c is an odd, the central set of $P_c \lozenge K_e$ is $\left\{v_{0\left\lceil\frac{c}{2}\right\rceil}\right\}$. Choose $W = \left\{v_{01}, v_{0\left\lceil\frac{c}{2}\right\rceil}, v_{0c}, u_i^{e_j} | i = 2,3,4, \dots, e; j = 1,2, \dots, n\right\}$, so that $|W| = 3 + E(P_c)(\dim(K_e))$.

For c is an even, the central set of $P_c \lozenge K_e$ is $\left\{v_{0\frac{c}{2}}, v_{0\left(\frac{c}{2}+1\right)}\right\} \cup \left\{u_i^{e_{\frac{c}{2}}} | \, i = 1,2,3, \dots, e\right\}$. Choose $W = \left\{v_{01}, v_{0\frac{c}{2}}, v_{0\left(\frac{c}{2}+1\right)}, v_{0c}, u_i^{e_j} | i = 2,3,4, \dots, e; j = 1,2,3, \dots, n\right\} \cup \left\{u_1^{e_{\frac{c}{2}}}\right\}$, then $|W| = 4 + |V(K_e)| + |(E(P_c) - 1)|\left(\dim(K_e)\right)$. Take any two distinct vertices $x, y \in V(P_c \lozenge K_e)\backslash W$, then x and y are in the different branch, let $x = u_1^{e_p}, y = u_1^{e_q}$. Hence, $d(x, v_{0k}) = 1$, $d(v_{0k,} v_{0l}) = z, 1 \le z \le c - 2$ with $k = p, p + 1; l = q, q + 1; k, p \ne 1$ and $d(v_{0l}, y) = 1$, so that $d(x, v_{0l}) = d(x, v_{0k}) + d(v_{0k,} v_{0l}) = 1 + z$ and $d(x, y) = d(x, v_{0l}) + d(v_{0l}, y) = 2 + z$. Therefore $d(x, v_{0k}) < d(x, v_{0l}) < d(x, y)$, consequently $r(x|W) \ne r(y|W)$. Take any $T \subseteq V(P_c \lozenge K_e)$ with $|T| < |W|$ and T contain a central set. Let $|T| = |W| - 1$, then first, there is j such that, there are maximum $e - 2$ vertices on $K_e^{e_j}$ contained in T and second, there are maximum $c - 1$ vertices in P_{c0}. First, let $u_a^{e_j}$, $u_b^{e_j} \notin T$, with $a \ne b$. Then $r\left(u_a^{e_j}|T\right) = r\left(u_b^{e_j}|T\right)$. Second, let $v_{0k} \notin T$, $k \ne \frac{c}{2}, \frac{c}{2} + 1$. Then $r(v_{01}|T) = r\left(u_i^{e_1}|T\right), r(v_{02}|T) = r\left(u_i^{e_2}|T\right)$, $\dots, r(v_{0c}|T) = r\left(u_i^{e_{c-1}}|T\right)$ with $i = 1,2,3, \dots, e$. Thus, $dim_{cen}(P_c \lozenge K_e) = 3 + |E(P_c)|\left(dim(K_e)\right)$, for an odd c and $dim_{cen}(P_c \lozenge K_e) = 4 + |V(K_e)| + (|E(P_c)| - 1)\left(dim(K_e)\right)$, for an even c.

Theorem 2.26: *Let P_c be path with $c \ge 3$ and $K_{1,f}$ be star with $f \ge 2$. Then*

$$dim_{cen}\left(P_c \lozenge K_{1,f}\right) = \begin{cases} 3 + |E(P_c)|\left(dim\left(K_{1,f}\right)\right), c \text{ odd} \\ 4 + |V\left(K_{1,f}\right)| + (|E(P_c)| - 1)\left(dim\left(K_{1,f}\right)\right), c \text{ even} \end{cases}$$

Proof: For c is an odd, the central set of $P_c \lozenge K_{1,f}$ is $\left\{v_{0\left\lceil\frac{c}{2}\right\rceil}\right\}$. Choose $W = \left\{v_{01}, v_{0\left\lceil\frac{c}{2}\right\rceil}, v_{0c}, u_i^{e_j} | i = 2,3,4, \dots, f; j = 1,2, \dots, n\right\}$, so that $|W| = 3 + |E(P_c)|\left(dim\left(K_{1,f}\right)\right)$. For c is an even, the central set of $P_c \lozenge K_{1,f}$ is $\left\{v_{0\frac{c}{2}}, v_{0\left(\frac{c}{2}+1\right)}\right\} \cup \left\{u_i^{e_{\frac{c}{2}}} | \, i = 0,1,2, \dots, f\right\}$. Choose $W = \{v_{01}, v_{0\frac{c}{2}}, v_{0\left(\frac{c}{2}+1\right)}, v_{0c}, u_i^{e_j} | \, i = 2,3,4, \dots, f; j = 1,2,3, \dots, n\} \cup \left\{u_0^{e_{\frac{c}{2}}}, u_1^{e_{\frac{c}{2}}}\right\}$, so that $|W| = 4 + |V(K_e)| + (|E(P_c)| - 1)\left(dim\left(K_{1,f}\right)\right)$. With the same steps of **Theorem 2.25**, it can be proven that W is a basis central of $P_c \lozenge K_{1,f}$.

Another metric dimension formula pattern of coronation graphs is presented below.

Theorem 2.27: *Let C_d and C_m be cycles with $d \ge 3, d$ is an even and $m \ge 7$. Then $dim_{cen}(C_d \lozenge C_m) = V(C_d) + E(C_d)\left(dim(K_1 + C_m)\right)$*

Proof: The central set of $C_d \lozenge C_m$ for d even is $V(C_{d0})$. There are five cases for value of m.

a. Case 1: $m \equiv 0(mod\ 5)$. Let $m = 5k$, with $k \geq 2$. Then $\left\lfloor \frac{2m+2}{5} \right\rfloor = 2k$. Choose $W = V(C_{d0}) \cup \{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k-1; j = 1,2,\dots,d\}$.

b. Case 2: $m \equiv 1(mod\ 5)$. Let $m = 5k+1$, with $k \geq 2$. Then $\left\lfloor \frac{2m+2}{5} \right\rfloor = 2k$. Choose $W = V(C_{d0}) \cup \{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k-2; j = 1,2,\dots,d\} \cup \{u_{5k-4}^{e_j}, u_{5k}^{e_j}; j = 1,2,\dots,d\}$.

c. Case 3: $m \equiv 2(mod\ 5)$. Let $m = 5k+2$, with $k \geq 1$. Then $\left\lfloor \frac{2m+2}{5} \right\rfloor = 2k+1$. Choose $W = V(C_{d0}) \cup \{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k-1; j = 1,2,\dots,d\} \cup \{u_{5k-4}^{e_j}, u_{5k}^{e_j}; j = 1,2,\dots,d\}$.

d. Case 4: $m \equiv 3(mod\ 5)$. Let $m = 5k+3$, with $k \geq 1$. Then $\left\lfloor \frac{2m+5}{5} \right\rfloor = 2k+1$. Choose $W = V(C_{d0}) \cup \{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k-2; j = 1,2,\dots,d\} \cup \{u_{5k-4}^{e_j}, u_{5k}^{e_j}, u_{5k+2}^{e_j}; j = 1,2,\dots,d\}$.

e. Case 5: $m \equiv 4(mod\ 5)$. Let $m = 5k+4$, with $k \geq 1$. Then $\left\lfloor \frac{2m+5}{5} \right\rfloor = 2k+2$. Choose $W = V(C_{d0}) \cup \{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k; j = 1,2,\dots,d\}$.

For any two vertices $x, y \in V(C_d \lozenge C_m) \backslash W$ with $x \neq y$, then there are two cases:

1. x and y are in the same branch, let $x = u_t^{e_j}$ and $y = u_l^{e_j}, l \neq t$. Because the central set of $C_d \lozenge C_m$ is contained in W, based on **Theorem 1.4,** W is a central resolver set of $C_d \lozenge C_m$.
2. x and y are in the different branch, let $x = u_i^{e_p}, y = u_n^{e_q}$ with $p \neq q$ and $i, n = 1,2,\dots,m$. Hence, $d(x, v_{0k}) = 1$, $d(v_{0k}, v_{0l}) = z, 1 \leq z \leq \left\lfloor \frac{d}{2} \right\rfloor$ with $k = p, p+1; l = q, q+1$ and $d(v_{0l}, y) = 1$, then $d(x, v_{0l}) = d(x, v_{0k}) + d(v_{0k}, v_{0l}) = 1 + z$ and $d(x, y) = d(x, v_{0l}) + d(v_{0l}, y) = 2 + z$. So $d(x, v_{0k}) < d(x, v_{0l}) < d(x, y)$. Consequently $r(x|W) \neq r(y|W)$.

Take any $T \subseteq V(C_d \lozenge C_m)$ with $|T| < |W|$ and T contain a central set, based on **Theorem 1.4**, T is not a central resolver set. Thus, $dim_{cen}(C_d \lozenge C_m) = |V(C_d)| + |E(C_d)|\big(dim(K_1 + C_m)\big)$.

Theorem 2.28: *Let K_e be a complete with $e \geq 4$ and C_d be a cycle with $d \geq 7$. Then $dim_{cen}(K_e \lozenge C_d) = |V(K_e|) + |E(K_e)|\big(dim(K_1 + C_d)\big)$.*

Proof: By dividing the value of d in five cases and using the steps the steps in **Theorem 2.27,** it can be proven that $dim_{cen}(K_e \lozenge C_d) = |V(K_e|) + |E(K_e)|\big(dim(K_1 + C_d)\big)$.

Theorem 2.29: *Let K_e be a complete graph with $e \geq 4$ and P_c be path with $c \geq 7$. Then* $dim_{cen}(K_e \lozenge P_c) = |V(K_e)| + |E(K_e)|\big(dim(K_1 + P_c)\big), c \equiv b(mod\ 5)$, $b = 2,3,4$

Proof: Based on **Theorem 2.16**, the central set of $K_e \lozenge P_c$ is $V(K_{e0})$.

1. For $c \equiv 2(mod\ 5)$ then $c = 5k + 2$, with $k \geq 1$, and $\left\lfloor \frac{2c+5}{5} \right\rfloor = 2k + 1$. Choose $W = V(K_{e0}) \cup \left\{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k-1; j = 1,2, \dots, \frac{e(e-1)}{2}\right\} \cup \left\{u_{5k-4}^{e_j}, u_{5k}^{e_j}; j = 1,2, \dots, d\right\}$.
2. For $c \equiv 3(mod\ 5)$ then $c = 5k + 3$, with $k \geq 1$, and $\left\lfloor \frac{2c+5}{5} \right\rfloor = 2k + 1$. Choose $W = V(K_{e0}) \cup \left\{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k-2; j = 1,2, \dots, \frac{e(e-1)}{2}\right\} \cup \left\{u_{5k-4}^{e_j}, u_{5k}^{e_j}, u_{5k+2}^{e_j}; j = 1,2, \dots, d\right\}$.
3. For $c \equiv 4(mod\ 5)$ then $c = 5k + 4$, with $k \geq 1$, and $\left\lfloor \frac{2c+5}{5} \right\rfloor = 2k + 2$. Choose $W = V(K_{e0}) \cup \left\{u_{5i+1}^{e_j}, u_{5i+4}^{e_j}; 0 \leq i \leq k; j = 1,2, \dots, \frac{e(e-1)}{2}\right\}$.

By using **Theorem 1.5** it can be proven that W is a central resolver set of $K_e \lozenge P_c$. Take any $T \subseteq V(K_e \lozenge P_c)$ with $|T| < |W|$ and T contain a central set. Because $V(K_{e0})$ is a central set of $K_e \lozenge P_c$, based on **Theorem 1.5,** W is a basis central of $K_e \lozenge P_c$. Thus, $dim_{cen}(K_e \lozenge P_c) = |V(K_e)| + |E(K_e)|\big(dim(K_1 + P_c)\big)$.

Conclusion

From the results of the research, it was found that there are several graphs that have the same metric dimension pattern. This research can be continued to find the properties of a family of graphs coronation that have the same metric dimension pattern, so that more general results are obtained.

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