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On the Cayley type graph construction and characterization associated with ternary semigroups

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Abstract

Cayley graphs of binary semigroup was introduced by Bohdan Zelinka. We introduce two Cayley type graph construction from ternary semigroups. The graph of first kind $\mathcal{C}_h(T,B)$ is a oriented hypergraph and the graph of second kind $\mathcal{C}_d(T,B)$ is a oriented multigraph for a semigroup T with ternary operation and a connection set $B \subseteq T$. We study subgraphs of these graphs and investigate some properties of these graphs. We give characterization theorems for graphs which are first kind and second kind for a ternary semigroup T.

Subject Classification (2020): *20M75, 05C25, 05C15, 05E16, 20M14.*

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1. Introduction

Cayley graphs of semigroups are a way to visualize the structure and behavior of semigroups using graph theory. A vital study on the conditions regarding the isomorphism for Cayley oriented-graphs of Brandt semigroups is explored by Meksawang, J., & Wongpinit, W [5]. The key research on the finite Cayley graphs' spectrum and the Edge coloring of cyclic groups' unitary endo-Cayley graphs are efficiently carried out by Ghorbani, M., & Larki, F. N and Promsakon, C respectively in 2018 [7, 6]. A semigroup is a collection of elements with a binary operation that obeys the associativity law, which is usually denoted by #, which means that for a randomly selected three elements *a, b* and *c* we have $(a\#b)\#c = a\#(b\#c)$ [12, 8]. A Cayley graph of a semigroup S associated to a connection set A is graph with node collection S and an oriented arc (x, y) is in the edge set implies and implied by when there exists a $t \in A$ such that $y = tx$ [2]. Cayley graphs provide insights into the algebraic properties of semigroups, such as their symmetry, connectivity, and other structural properties. They are widely used in various fields, including combinatorics, group theory, and computer science, to study and analyze the properties and behaviors of semigroups. Additionally, they can be useful in designing algorithms and solving problems related to semigroups [11, 13, 15, 16, 17, 18, 21, 22, 23]. A study on connected cayley graphs carried out by Sukumaran, Teerapong and Sayan Panma[20]. A ternary semigroup is a mathematical structure that generalizes the concept of binary operations to ternary operations. In a ternary semigroup, instead of just one operation combining two elements, there are three elements involved in each operation[19]. Formally, a semigroup T with ternary operation is a set with $\# : T \times T \times T \rightarrow T$ that satisfy the associative property meaning that for any elements $a, c, d, e \in T$ we have $[ab[cde]] = [a[bcd]e] = [[abc]de]$ where $\#(a,b,c) = [abc]$. Ideal theory and congruences on ternary semigroups are explored by Dutta, Tapan, S. Kar, B. Maity[9, 14]. Ternary semigroups have applications in various areas of mathematics and theoretical computer science. They provide a framework for studying systems where interactions involve three elements or entities. However, compared to binary semigroups, the theory of ternary semigroups is less developed, and fewer properties and results are known about them. Nevertheless, ternary semigroups are of interest in mathematical research and can offer insights into more complex algebraic structures. This paper introduces the Cayley type construction of oriented graphs and hypergraphs and characterizes ternary semigroups for given graph.

2. Preliminaries

2.1 *Ternary Semigroups*

Definition 1: [19] A semigroup with a ternary map is a non-void set T with $(q,r,j) \rightarrow [qrj]$ meeting the first-kind law of associativity $[qr[jkl]] = [q[rjk]l] = [[qrj]kl]$ for all $q,r,j,k,l \in T$.

Example 1: Here we have given three examples.

1. Any binary semigroup with operation $*$ becomes a ternary semigroup by defining $[abc] = a*b*c$.
2. The collection of all odd permutations under mapping composition.

Definition 2: [19] Let T be ternary semigroup.

1. An element $g \in T$ becomes selfpotent if $g^3 = g$. $A \subseteq T$ is said to be selfpotent if each element of A is selfpotent.
2. An element $h \in T$ is said to be biunital if $[hhx] = x = [xhh]$ for all $x \in T$.
3. T satisfies commutativity if for a randomly selected $a,b,c \in T$, $[abc] = [acb] = [bca] = [cba] = [bac] = [cab]$.

Lemma 1: [1] *Every regular element has an inverse.*

Proposition 1: [1] Let q be a member of a semigroup T with a ternary process characterized by an index m and a period r. Then,

1. for every u,v of even integers, $q^{m+u} = q^{m+v}$ implies and implied by $u = v\ (mod\ r)$, ,
2. $<q>=\{q,q^3,\ldots,q^m,q^{m+2},\ldots,q^{m+r-2}\}$ and $|q| = \frac{m+r-1}{2}$

2.2 *Graphs and Hypergraphs*

Definition 3: [4, 3] A multi-oriented graph is an unordered-pair $X=(V,E)$ with V is the node collection and E is the arc collection of X and there is map $\phi_X : E \rightarrow V \times V$. A simple oriented graph X_0 is the one with ϕ_{X_0} is an injection(one-to-one map). For a simple oriented graph the pair (x,y) uniquely represents an edge. A multigraph is given in the figure 1 and a simple oriented graph is given in the figure 2.

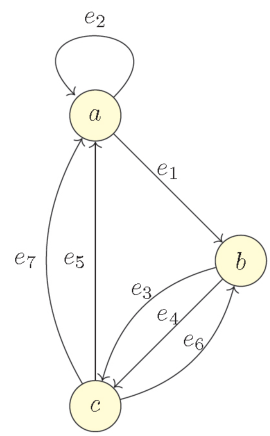

Figure 1

A multi-oriented graph

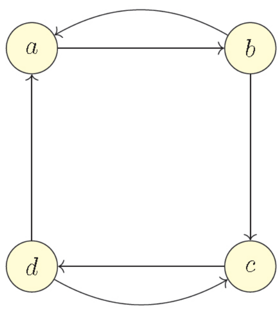

Figure 2

A simple oriented graph

Definition 4: [4, 3] A simple oriented graph X with node set V and edge E is said to be k-out-regular if for each $v \in V$ the count of outgoing arcs from v is equal to k.

Definition 5: [4, 3] A hyper-graph is represented by $\mathcal{H} = \langle \mathcal{W}, \mathcal{E} \rangle$, where $\mathcal{W} = \{q_1, q_2, \ldots, q_n\}$ is the vertices and $\mathcal{E} = \{R_1, R_2, \ldots, R_m\}$, $E_i \subseteq \mathcal{W}$ $(i = 1, 2, \ldots, m)$, is the collection of hyper edges. If each R_i contains t elements from $\mathcal{V}$, then $\mathcal{H}$ is called t-uniform hyper-graph. And if $\mathcal{E}$ contains all t-member subsets of $\mathcal{V}$, then $\mathcal{H}$ is called t-uniform complete hyper-graph. An example is given in the figure 3.

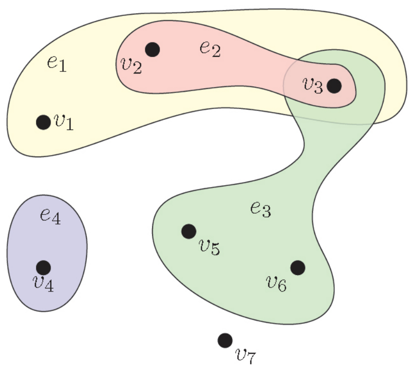

Figure 3

A hypergraph

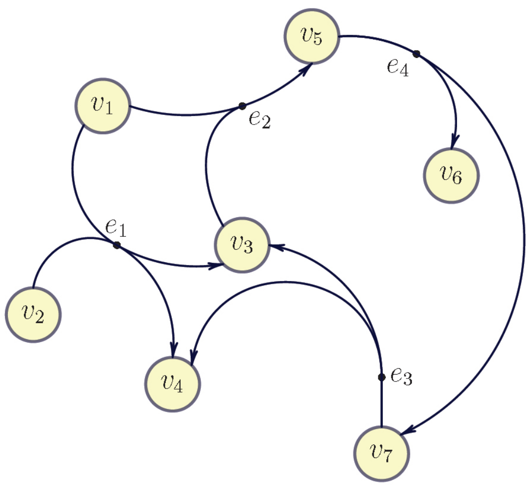

Figure 4

An oriented hypergraph

Definition 6: [10] A oriented hyper edge $Q = (Q_1, Q_2)$, where Q_1 and Q_2 are the tail and head of Q respectively. We identify this by $T(Q) = Q_1$ and $H(Q) = Q_2$. A oriented hypergraph is given in the figure 4.

Definition 7: [10] A **B-arc** is hyper arc $Q = (T(Q), H(Q))$ with $H(Q)$ singleton. A **F-arc** is with $T(Q)$ containing one member. A **B-graph** is a hyper-graph whose hyper arcs are B-arcs. An **F-graph** is a hyper-graph whose hyper arcs are F-arcs.

Let $FS(v) = \{Q \in \mathcal{Q} : v \in T(Q)\}$, $BS(v) = \{Q \in \mathcal{Q} : v \in H(Q)\}$.

3. Main Definitions and Results

Definition 8: Let T be a semigroup with a ternary operation and M be a non-empty subset of T. The hyper-graph of first kind is denoted by $\mathcal{C}_h(T,M)$ and its subgraph $\hat{\mathcal{C}}_h(T,M)$, with respect to a connection set M is defined by the following way,

1. The node set of $\mathcal{C}_h(T,M)$ is equal to T.
2. The arc set of $\mathcal{C}_h(T,M)$ is denoted by,
   $E(\mathcal{C}_h(T,M)) = \{(\{x,y\}, \{z,w\}) \mid$ There exists an $m \in M$ such that $z=[mxy]$ & $w=[myx]\}$. Also for any two edges E_i and E_j, either $T(E_i) \neq T(E_j)$ or $H(E_i) \neq H(E_j)$.
3. The graph $\hat{\mathcal{C}}_h(T,M)$ is the subgraph of $\mathcal{C}_h(T,M)$ with $T(E) \cap H(E) = \phi$ for all $E \in E\ (\hat{\mathcal{C}}_h(T,M))$.

Definition 9: The **incidence matrix** corresponds to the hypergraph $\mathcal{C}_h(T,M)$ is defined as follows

$$\mathcal{M}(\mathcal{C}_h(T,M)) = m_{ij} = \begin{cases} 1 & \text{if } v_i \in H(E_j) - T(E_j) \\ -1 & \text{if } v_i \in T(E_j) - H(E_j) \\ 2 & \text{if } v_i \in H(E_j) \cap T(E_j) \\ 0 & \text{Otherwise.} \end{cases}$$

Hence there is a bijection between Cayley hypergraphs of first kind and $(-1,0,1,2)$ matrices.

Let us look at a group with induced ternary operation. $T = \mathbb{Z}_3 = \{0,1,2\}$ and $M = \{1\}$.

Define the ternary operation $[abc] = a + b + c(mod 3)$. Here $[100]=1$, $[111]=0$, $[122]=2$, $[112]=1$, $[121]=1$, $[101]=2$, $[110]=2$, $[120]=0$, $[102]=0$. The graph $\mathcal{C}_h(\mathbb{Z}_3,\{1\})$ is illustrated in the figure 5.

The hyperarcs are the following,

$$\hat{D}_1 = (0,1), \hat{D}_2 = (1,0), \hat{D}_3 = (\{1,2\},1), \hat{D}_4 = (\{0,2\},0), \hat{D}_5 = (\{0,1\},2), \hat{D}_6 = (2,2).$$

The incidence matrix $\mathcal{M}(\mathcal{C}_h(\mathbb{Z}_3,\{1\}))$ is given by,

	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$
0	–1	1	0	1	0	0
1	1	–1	2	0	–1	0
2	0	0	–1	2	1	2

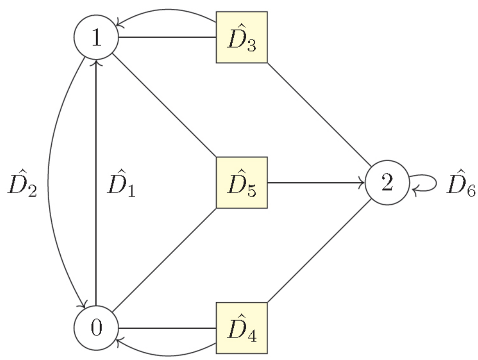

Figure 5

$\mathcal{C}_h(\mathbb{Z}_3, \{1\})$

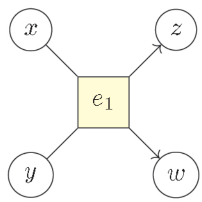

Figure 6

$e_1 = (\{x, y\}, \{z, w\}) x \neq y \neq z \neq w$

Definition 10: For a subset s of T with $|s| \leq 2$, $O_s = \{E \mid T(E) = s\}$ and $I_s = \{E \mid H(E) = s\}$. The numbers $|O_s|$ and $|I_s|$ are defined to be **out-degree(*s*)** and **in-degree(*s*)** respectively.

Definition 11: For an edge E of a oriented hyper graph, $T(E)$ and $H(E)$ will be called **hyper node**. Let s and t hyper vertices. Then a **hyper path** of length l is sequence $sE_1s_1...E_lt$ of $s_i \neq s_j$ for $i \neq j$. If $s = t$ then the sequence will be called an **hyper circuit**. The graph $\mathcal{C}_h(T, M)$ is said to be **hyper connected** if for any two s and t, either there is a hyper path from s to t or from t to s. The graph is **strongly hyper connected** if there are paths from s to t and t to s. An identification of edge connecting hypervertices is given in the figure 13.

All possible hyperedges of the graph $\mathcal{C}_h(T, M)$ are illustrated in the figures 6, 7, 8, 9, 10, 11, 12. For $M = \{a\}$, we shall write simply $\mathcal{C}_h(T, a)$ and $\hat{\mathcal{C}}_h(T, a)$ instead of $\mathcal{C}_h(T, \{a\})$ and $\hat{\mathcal{C}}_h(T, \{a\})$. Here we illustrate seven type of hyper edges denoted by e which is possible in a $\mathcal{C}_h(T, M)$,

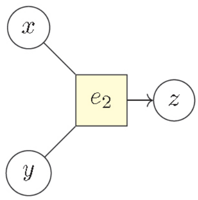

Figure 7

$e_2 = (\{x, y\}, \{z\}), z = w$

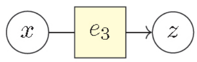

Figure 8

$e_3 = (\{x, y\}, \{z\}), z = w, x = y$

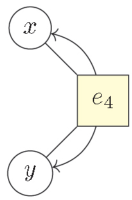

Figure 9

$e_4 = (\{x, y\}, \{x, y\}), (x = z, y = w)$ or $(x = w, y = z), z \neq w$

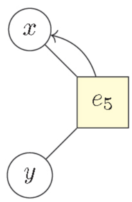

Figure 10

$e_5 = (\{x, y\}, \{x\}), z = w = x, x \neq y$

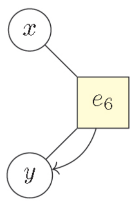

Figure 11

$e_6 = (\{x, y\}, \{x\}), z = w = y, x \neq y$

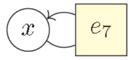

Figure 12

$e_7 = (\{x\}, \{x\}), x = y = z = w$

Proposition 2: Let T be a semigroup with a ternary operation and $M \neq \phi$ be a right ideal of T. Then for $\hat{\mathcal{C}}_h(T, M)$,

1. For each hyper edge $E \in E(\hat{\mathcal{C}}_h(T, M)), H(E) \subseteq M..$
2. $x \notin M$ implies $BS(x) = \phi$ and the converse is true if T has a biunital element which is not in M or M is a complete right ideal.
3. Let $x \in M$. For each $m \in M$ and $y \in T$, $[mxy] = x = [myx]$ implies $FS(x) = \phi$.
4. If $x \notin M$ then $FS(x) \neq \phi$.

Proof:

1. If $E \in E$ $(\hat{\mathcal{C}}_h(T, M))$ then $E = (\{x, y\}, \{z, w\})$ such that $z = [mxy]$ and $w = [myx]$ for some $m \in M$. Since M is right ideal of T, we have $[MTT] \subseteq M$. Then we must have $z = [mxy] \in M$ and $w = [myx] \in M$. That is $\{z, w\} = H(E) \subseteq M$.
2. Let $x \notin M$. Assume that $BS(x) \neq \phi$. Then there exists an hyper edge $E \in E$ $(\hat{\mathcal{C}}_h(T, M))$ such that $E \in BS(x)$. That is $x \in H(E) \subseteq M$, which is not possible. Hence $BS(x) = \phi$. For the partial converse, suppose that T has a biunital element $f \notin M$. let $x \in M$. Then $x = [xff]$ by the property of the biunital element. This implies that the edge $E = (\{f\}, \{x\})$ is belongs to $E \in E$ $(\hat{\mathcal{C}}_h(T, M))$. Clearly $x \in H(E) \subseteq M$. this shows that $E \in BS(x)$, which is not possible. Hence $x \notin M$.
3. Let $x \in M$. Assume that $FS(x) \neq \phi$. Then there obtains an edge $E \in E$ $(\hat{\mathcal{C}}_h(T, M))$ with $x \in T(E)$. That is there exists an $m \in M$ and $y \in T$ such that $[mxy] = z$ or $[myx] = w$. But by assumption $z = w = x$. That is $E \in E$ $(\hat{\mathcal{C}}_h(T, M))$. Hence $FS(x) = \phi$.
4. If $x \notin M$ choose $m \in M$, $(M = \phi)$ then $[mxx] \in M$. That is the edge $E = (\{x\}, \{[mxx]\})$ is in $E \in E$ $(\hat{\mathcal{C}}_h(T, M))$ and $x \in T(E)$. This implies $E \in FS(x)$. Hence $FS(x) \neq \phi$.

Proposition 3: Let T be a semigroup with ternary operation and $a \in T$. The for a finite graph $\mathcal{C}_h(T, \alpha)$ and $s \subseteq T$ with $|s| \leq 2$. Then for $E \in O_s$ we have $|H(E)| \leq 2$, out-degree $(s) = 1$ and Each hyper connected component of $\mathcal{C}_h(T,\alpha)$ contains exactly one oriented hyper circuit and for each s there is a oriented hyper path from s to this unique oriented hyper circuit.

Proof: Suppose that $s=\{x,y\}$ and $E \in O_s$. By the definition of $\mathcal{C}_h(T,a)$ we have $H(E)$ = {[axy], [ayx]}. This implies $|H(E)| \leq 2$ and out-degree(s) $=|O_s|=1$. Let $\tilde{C}$ be a hyper connected component of $\mathcal{C}_h(T,a)$. Let P_s := $s_1E_1s_2$ ... $s_lE_ls_1$ and P_t := $t_1\tilde{E}_1t_2$ ... $t_r\tilde{E}_rt_1$ be two two hyper circuits in the component $\tilde{C}$ of length l and r respectively. Since $\tilde{C}$ is connected there is a path from s_1 to t_1 (say). Let this path be $s_1K_1h1K_2h_2$ ... K_dt_1. But $|O_{s_1}|=1$ implies $K_1=E_1$ and $h_1=s_2$. Then we have only two possibilities. Either P_s is a sub hyper circuit of P_t or P_t is sub hyper circuit of P_s. In the former case $|O_{s_i}|=2>1$ and in the latter case $|O_{t_i}|=2>1$ which are contradictions to the fact that the vertices of the hyper circuit are distinct and $|O_s|=1$ for each s. This scenario is well illustrated in the figures 14 and 15. That is there is a unique hyper circuit corresponds to each hyper connected component of $\mathcal{C}_h(T,a)$ (figure 16). Following the removal of all hyper arcs of this circuit a hyper forest is obtained. Each hyper tree of this hyper forest has a characteristic is that every node has a hyperpath connecting it to another node in the circuit. This can be seen by looking at a minimal oriented hyper path from a hyper node out side the hyper circuit to a hyper node of the unique hyper circuit. oriented Hyper path exist because both hyper vertices are in the same hyper connected component. Let s be a hyper node and P_s be a path follows from s. Then P_s will eventually contain a hyper circuit due to finiteness and this hyper circuit is the unique hyper circuit that we proved have formerly. Now if you delete this circuit, you have the desired hyper path.

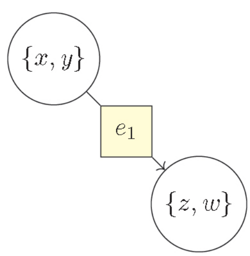

Figure 13

Representation of $e_1=(\{x,y\},\{z,w\})\ x \neq y \neq z \neq w$ **using hyper node.**

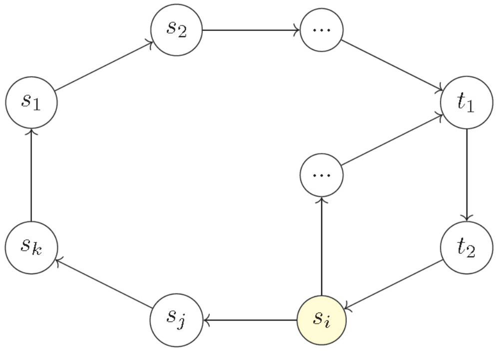

Figure 14

P_t is a sub hyper circuit of P_s.

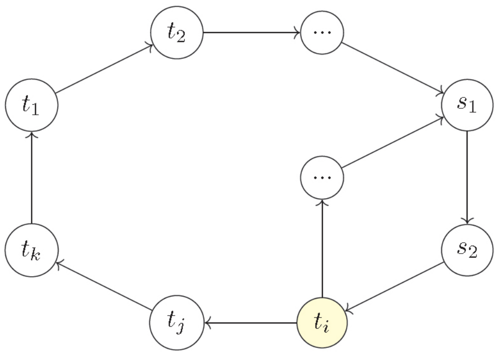

Figure 15

P_s is a sub hypercircuit of P_t.

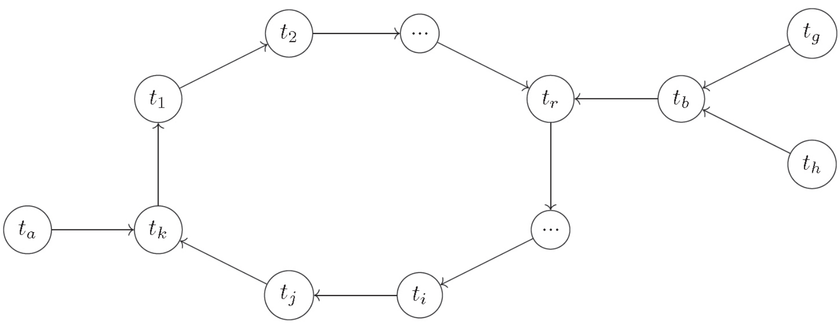

Figure 16

A hyper connected component of $\mathcal{C}_h(T, a)$.

Definition 12: For a semigroup T with ternary operation and an element a, let $\tilde{\chi}(C)$ represent the length of the unique hyper circuit included in the hyper connected component C of $\mathcal{C}_h(T,a)$ and by $\tilde{\lambda}(C)$ we provide the maximum length-measure of an oriented walk that does not contain an edge of the circuit.

Note that $\mathcal{C}_h(T,a)$ is a B-graph if T is commutative.

Theorem 1: *Let $\mathcal{G}$ be a oriented graph with out-degree one for each node. The graph $\mathcal{G}$ is isomorphically equal to a sub-oriented hypergraph $\mathcal{H}$ of the graph $C_h(T,\alpha)$ for a commutative ternary semigroup T and its elements α such that $[\alpha y^m] = [\alpha y^2]$ for each $y \neq \alpha$ and for each even positive integer m and for each edge E of $\mathcal{H}, |T(E)|=1$ implies and implied by when it contains a connected part R with every connected part J of $\mathcal{G}$ the number $\tilde{\chi}(R)$ is multiple of $\tilde{\chi}(J)$ and $\tilde{\lambda}(J) \leq \tilde{\lambda}(R)+1$.*

Proof: Assume that $\mathcal{G}$ is isomorphically equal to a sub-oriented hypergraph $\mathcal{H}$ of the graph $C_h(T,\alpha)$ for some commutative ternary semigroup T and $\alpha \in T$ with the property mentioned in the theorem. Consider the oriented path starting from α in $C_h(T,\alpha)$. Then we will get a sequence of vertices $\alpha,\ \alpha^3,\ \alpha^7,\ \alpha^{15},\ \ldots$. The sequence b_n in the powers of α can be recursively defined by,

$$b_0 = 1, b_n = 1+2b_{n-1}, n \geq 1.$$

Now by applying the method of generating functions we have,

$$\begin{aligned}
\sum_{n\geq 1} b_n x^n &= \sum_{n\geq 1}(1+2b_{n-1})x^n \\
&= \sum_{n\geq 1} x^n + 2\sum_{n\geq 1} b_{n-1}x^n \\
\text{Let}\qquad M &= \sum_{n\geq 0} b_n x^n \\
&= 1+\sum_{n\geq 1} x^n + 2x\sum_{n\geq 1} b_{n-1}x^{n-1} \\
M &= \frac{1}{1-x} + 2xM \\
&= \frac{1}{(1-2x)(1-x)} \\
&= \frac{2}{1-2x} - \frac{1}{1-x} \\
&= 2\sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} x^n
\end{aligned}$$

That is $b_n = 2^{n+1} - 1,\ n \geq 0$ or $b_n = 2^n - 1,\ n \geq 1$. Since T is finite, the sequence α^{b_n} must repeat. Then the set $\{x \in \mathbb{Z}_+ :$ There exists $y \in \mathbb{Z}_+$ such that $[\alpha^{2^x-1}] = [\alpha^{2^y-1}],\ x \neq y\}$ is non-empty and has a least element say k. Then the set $\{x \in \mathbb{Z}_+ : [\alpha^{2^{k+x}-1}] = [\alpha^{2^k-1}]\}$ is non-empty and the least element say h. This means that the elements $\alpha, [\alpha^3], [\alpha^7], \ldots, [\alpha^{2^k-1}], [\alpha^{2^{k+1}-1}], \ldots,$

$[\alpha^{2^{k+h}-1}]$ are pairwise different and $[\alpha^{2^{k+h}-1}] = [\alpha^{2^{k}-1}]$. Hence in $\mathcal{G}$ there is an h-length circuit in existence and a oriented k-length path whose end node falls into this hyper circuit; the start node of this walk is the member α. Now let $x \neq \alpha$ be any node of $\mathcal{G}$. Let J be the linked portion of $\mathcal{G}$ including x. Let p represents the length of the oriented walk starting from x, arriving into a node of a circuit and including no arc of this circuit; evidently $p \leq \tilde{\lambda}(J)$. Let $q = \tilde{\chi}(J)$. Then we get members

$$x,[\alpha x^2],[\alpha[\alpha x^2][\alpha x^2]],[\alpha[\alpha[\alpha x^2][\alpha x^2]][\alpha[\alpha x^2][\alpha x^2]]],\dots,w(\alpha,x).$$

Where $w(\alpha,x)$ is product of powers of α and x such that sum of powers of x is equal to 2^{p+q-1} and sum of powers of α is equal to $2^{p+q-1}-1$. Since T is commutative; we can rewrite the sequence in the following manner

$$x,[\alpha x^2],[\alpha^3 x^4],[\alpha^7 x^8],\dots,[\alpha^{2^{p+q-1}-1}x^{2^{p+q-1}}].$$

Since $[\alpha y^m]=[\alpha y^2]$ for each $y \neq \alpha$ and for each even positive integer m; the sequence will be

$$x,[\alpha x^2],[\alpha^3 x^2],[\alpha^7 x^2],\dots,[\alpha^{2^{p+q-1}-1}x^2].$$

Here in the sequence the elements are pairwise different and $[\alpha^{2^{p+q}-1}x^2] = [\alpha^{2^{p}-1}x^2]$. If $q \nmid h$, then $k \not\equiv h+k \pmod{q}$ and thus $[\alpha^{2^{k+h}-1}x^2] \neq [\alpha^{2^{k}-1}x^2]$, which is a contradiction to the premise $[\alpha^{2^{k+h}-1}] = [\alpha^{2^{k}-1}]$. hence q should divide h and h is $\tilde{\chi}(R)$, where R is a linked part of $\mathcal{G}$ including α. Now suppose $p \geq k+2$. Then $[\alpha^{2^{k+1}-1}x^2]$ is distinct from $[\alpha^{2^{l}-1}x^2]$ for each $l \neq k+1$. But, as $[\alpha^{2^{k+h}-1}] = [\alpha^{2^{k}-1}]$, we must have $[\alpha^{2^{k+1}-1}x^2] = [\alpha^{2^{h+k+1}-1}x^2]$, which is a contradiction. Hence $p \leq k+1 \leq \tilde{\lambda}(R)+1$.

Suppose, for the moment, that $\mathcal{G}$ meets the requirement. In R take a $\tilde{\lambda}(R)$-length oriented hyper path including no arc of a circuit; its start node will be α. The connected component containing α is given in the figure 17. Take all vertices of $\mathcal{G}$ having no incoming edges and if $\mathcal{G}$ includes connected parts different from R which are happens to be circuits, select one node in each of those parts. The collection constructed is identified by $\tilde{\mathbf{B}}$. The node α and the nodes of $\tilde{\mathbf{B}}$ identified as the elements of a ternary semigroup T. Each leftover node will be represented as b_n^{th} power of α or a ternary multiple of a member of $\tilde{\mathbf{B}}$ with the power of α corresponds to the definition of $C_h(T,\alpha)$. Now the node set of $\mathcal{G}$ along with the set $\{[\alpha xy]: x,y \in \tilde{\mathbf{B}}\}$ will form the node set of $C_h(T,\alpha)$. We got a semigroup T with ternary process such that $\mathcal{G}$ is isomorphic to a sub-oriented hypergraph $\mathcal{H}$ of the graph $C_h(T,\alpha)$ such that for each edge E of $\mathcal{H}, |T(E)|=1.$ Hence the proof.

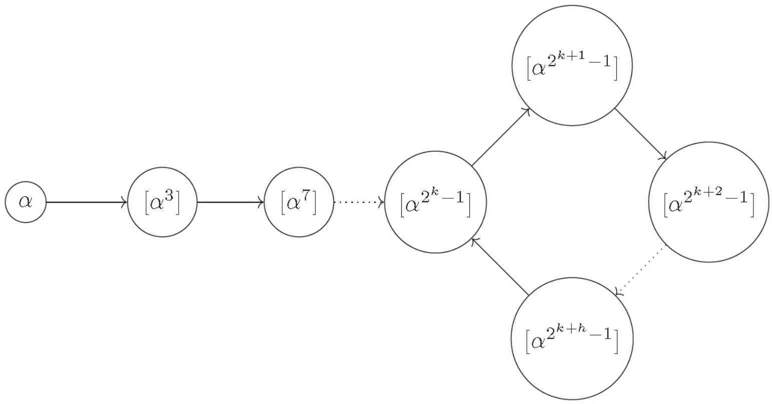

Figure 17

Connected component of $\mathcal{G}$ containing α.

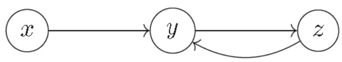

Figure 18

The oriented graph $\mathcal{G}$

Corollary 1: *Given a connected oriented graph $\mathcal{G}$ with a unique source α having each node has out-degree* 1, *there exists a monogenic ternary semigroup M such that $\mathcal{G}$ is a sub-oriented graph of $C_h(M,\alpha)$. If the minimal length of the path from α to the unique circuit is λ and $\tilde{\chi}(\mathcal{G})=\chi$ then T has index λ and period χ. Further more the count of arcs in the graph $C_h(M,\alpha)$ is equal to* $\binom{\frac{\lambda+\chi-1}{2}}{2}+\frac{\lambda+\chi-1}{2}$.

Here we are going to give a procedure to construct the ternary semigroup from the given oriented graph $\mathcal{G}$ (figure 18) with out-degree 1. Let $\mathcal{G}$ be the graph given below. first let us label $x=\alpha,\ y=[\alpha^3],\ z=[\alpha^7]$.

Then from the graph we set $[\alpha^{15}]=[\alpha^3]$ in order to balance with the definition of Cayley hypergraph of first kind. Now construct a monogenic ternary semigroup with these information. Let $T=\{[\alpha], [\alpha^3], [\alpha^5], ..., [\alpha^{13}]\}$. Here note that $[\alpha^{15}] = [\alpha^{3+12-2}] = [\alpha^3]$. That is index $m=3$ and period $r=12$. Now add vertices $\{[\alpha^5], [\alpha^9], [\alpha^{11}], [\alpha^{13}]\}$ to the graph $\mathcal{G}$ appropriately according to the definition of the hypergraph. In the figure 19 we modified $\mathcal{G}$ so that it contains all vertices are elements of the monogenic ternary semigroup T and all the edges of $C_h(T,\alpha)$ such that $\mathcal{G}$ is a sub-oriented graph of $C_h(T,\alpha)$. In the figure 19 orange vertices are the vertices of the oriented graph $\mathcal{G}$. green vertices are added to complete the vertices of $C_h(T,\alpha)$. Given below is the corresponding calculations using $[\alpha^{15}]=[\alpha^3]$ for the construction of edges. There are $28=\binom{7}{2}+7$ edges.

$$[\alpha\alpha^3\alpha^3]=[\alpha^7], [\alpha\alpha^9\alpha^9]=[\alpha^7], [\alpha\alpha^5\alpha^5]=[\alpha^{11}], [\alpha\alpha^{11}\alpha^{11}]=[\alpha^{11}],$$
$$[\alpha\alpha^{13}\alpha^{13}]=[\alpha^3], [\alpha\alpha\alpha]=[\alpha^3].,[\alpha\alpha^7\alpha^7]=[\alpha^3], [\alpha\alpha^5\alpha^{11}]=[\alpha^5],$$
$$[\alpha\alpha^{11}\alpha^{13}]=[\alpha^{13}], [\alpha\alpha^5\alpha^{13}]=[\alpha^7], [\alpha\alpha^9\alpha^5]=[\alpha^3]., [\alpha\alpha^9\alpha^{11}]=[\alpha^9],$$
$$[\alpha\alpha\alpha^9]=[\alpha^{11}], [\alpha\alpha\alpha^5]=[\alpha^7], [\alpha\alpha\alpha^{11}]=[\alpha^{13}], [\alpha\alpha\alpha^{13}]=[\alpha^3].,$$
$$[\alpha\alpha\alpha^7]=[\alpha^9], [\alpha\alpha^3\alpha^{11}]=[\alpha^3], [\alpha\alpha^3\alpha^5]=[\alpha^9], [\alpha\alpha^3\alpha^9]=[\alpha^{13}],$$
$$[\alpha\alpha^3\alpha^{13}]=[\alpha^5]., [\alpha\alpha^{13}\alpha^9]=[\alpha^{11}], [\alpha\alpha^5\alpha^7=[\alpha^{13}], [\alpha\alpha^7\alpha^{13}]=[\alpha^9],$$
$$[\alpha\alpha^7\alpha^{11}]=[\alpha^7].$$

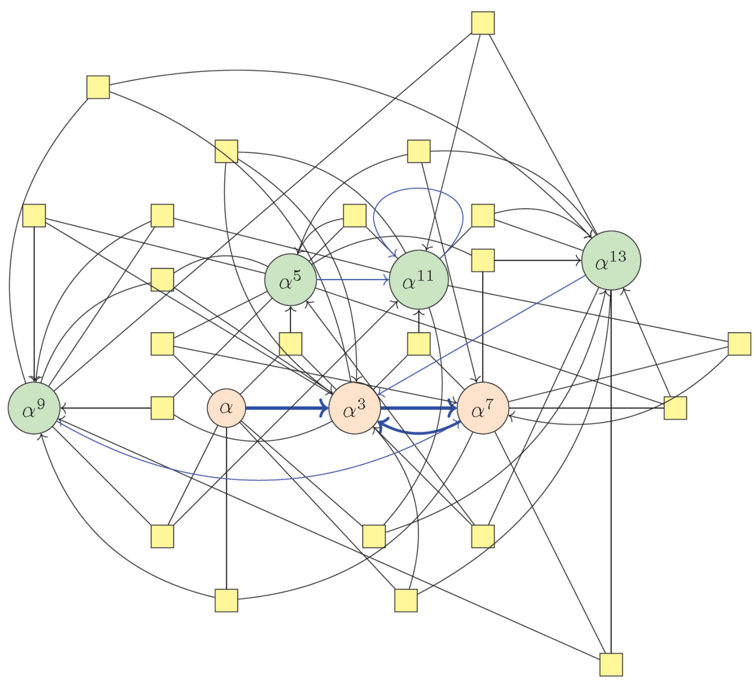

Figure 19

The oriented graph $\mathcal{G}$ modified to the oriented hypergraph $C_h(T,\alpha)$.

Definition 13: Let T be a semigroup with ternary function and B be a non-void subset of T. The oriented multigraph of second kind is denoted by $\mathcal{C}_d(T,B)$ with respect to a connection set B is defined by the following way,

1. The node set of $\mathcal{C}_d(T,B)$ coincides with T.
2. The edge $(x,y) = E(\mathcal{C}_d(T,B))$ if and only if $y = [abx]$ for some $a,b \in B$.
3. The simple oriented graph $\hat{\mathcal{C}}_d(T,B)$ is a subgraph of $\mathcal{C}_d(T,B)$ in which there is exactly one oriented edge (q,r) for $q,r \in T$ if there exists $l,m \in T, r = [lmq]$.

Definition 14: Let T be a semigroup with ternary function and B be a subset of T. For each $t \in T$, define a binary relation R_t^B on $B \times B$ with (a,b) R_t^B (c,d) implies and implied by $[abt] = [cdt]$.

Proposition 4: Let T be a semigroup with ternary map and B be a non-void subset of T.

1. The relation R_t^B is an equivalence on $B \times B$.
2. The number of distinct classes of equivalence of R_t^B is equal to the number of out-going edges from t in $\hat{\mathcal{C}}_d(T,B)$. Moreover, each equivalence class represents a distinct edge from t.
3. The oriented graph $\hat{\mathcal{C}}_d(T,B)$ is k-out-regular implies and implied by for each $t \in T$, Either
   $|(B \times B) \ / \ R_t^B| = k$ or $(B \times B) \ / \ R_t^B = \phi$. Where $(B \times B) \ / \ R_t^B$ is the collection of distinct equivalence classes under the relation R_t^B.

Proof:

(1) Since $[abt] = [abt]$, (a,b) R_t^B (a,b) for all $(a,b) \in B \times B$. That is R_t^B is reflexive. Let (a,b) R_t^B (c,d). Since $[cdt] = [abt]$, (c,d) R_t^B (a,b). That is R_t^B is symmetric. Let (a,b) R_t^B (c,d) and (c,d) R_t^B (e,f) then $[abt] = [cdt]$ and $[cdt] = [eft]$ this implies $[abt] = [eft]$. That is (a,b) R_t^B (e,f). The relation is transitive.

(2) Let $E_t = \{(t,y_1), (t,y_2), ..., (t,y_k)\}$ be the set of all distinct outgoing edges from t in $\hat{\mathcal{C}}_d(T,B)$. Then there exists (a_i,b_i), $1 \le i \le k$ where $(a_i,b_i) \neq (a_j,b_j)$, $i \neq j$ such that $y_i = [a_i b_i t]$. This implies that (a_i,b_i)'s are not R_t^B related. That is (a_i,b_i) R_t^B, $1 \le i \le k$ are distinct equivalence classes of R_t^B. Now define a map Λ from $(B \times B) \ / \ R_t^b$ to E_t such that the element $(a,b)R_t^B$ maps to $(t,[abt])$. This map is well-defined. For if $(a,b)R_t^B = (c,d)R_t^B$ then (a,b) R_t^B (c,d). That is

$(a,b)R_t^B = (c,d)R_t^B$ implies $(t,[abt]) = (t,[cdt])$. Observe that Λ is a bijection. If $(t,[abt]) = (t,[cdt])$ then $[abt] = [cdt]$ implies that (a,b) R_t^b (c,d). That is $(a,b)R_t^B = (c,d)R_t^B$. This shows that Λ is injective. Now let $(x,y) \in E_t$ then $y = [abt]$ for some $a,b \in B$. That is $(a,b) \in (a,b)R_t^B$. That is Λ maps $(a,b)R_t^B$ to (x,y). This proves Λ is bijection and each class $(a,b)R_t^B$ corresponds to an edge $(t,[abt])$ in $\hat{\mathcal{C}}_d(T,B)$.

(3) For the necessary part, assume that $\hat{\mathcal{C}}_d(T,B)$ is *k*-out regular. If $t \in T$ is with out-degree 0 then clearly $(B \times B) \ / \ R_t^B = \phi$. Let $t \in T$ be a node with non-zero out-degree. Since the graph is *k*-out-regular, there exists (g_1,h_1), (g_2,h_2), ..., $(g_k,h_k) \in B \times B$ such that $(t,[g_ih_it])$ is an edge for $1 \le i \le k$ in $\hat{\mathcal{C}}_d(T,B)$. Thus for each node *t* with non-zero out-degree, we have $[g_ih_it]$ are all distinct for $1 \le i \le k$ and (g_i,h_i)'s represents distinct equivalence classes under the relation R_t^B. That is equivalence classes $(g_i,h_i)R_t^B$'s are all distinct and $|(B \times B) \ / \ R_t^B|$ $= k$. For sufficiency part, assume that for each $t \in T$, Either $|(B \times B)$ $/ \ R_t^B| = k$ or $(B \times B) \ / \ R_t^B = \phi$. Let $t \in T$ be an node with non-zero out-degree and $(B \times B) \ / \ R_t^B = \{(g_i,h_i)R_t^B \ : \ 1 \le i \le k\}$. Then by (2) each $(g_i,h_i)R_t^B$ is an edge outgoing from *t*. That is *t* has *k* distinct outgoing edges. This proves that $\hat{\mathcal{C}}_d(T,B)$ is *k*-out-regular.

We will now describe finite graphs $\mathcal{C}_d(T,B)$, where *B* is a singleton set. Then we have $B = \{b\}$ and in the place of $\mathcal{C}_d(T,\{b\})$ we denote $\mathcal{C}_d(T,b)$. We allow loops and accepts them as 1-length circuits.

Each graph $\mathcal{C}_d(T,b)$ has the characteristic that the out-degree of each of its vertices is 1. Every connected portion of such a graph that is finite will have precisely one oriented circuit in it.

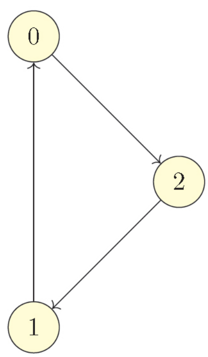

Figure 20

The oriented graph $\mathcal{C}_d(\mathbb{Z}_3,1)$

After removing all oriented arcs of this oriented circuits a forest is formed. Every tree in this forest has the feature that every node has a oriented path connecting it to another node in the oriented circuit. If R is a linked part of such a graph, then by $\chi(R)$ we identify the length-measure of the circuit included in R and by $\lambda(R)$ we represent the largest length of an oriented path in this connected portion which contains no arc of the circuit. The oriented graph $\mathcal{C}_d(\mathbb{Z}_3,1)$ is illustrated in the figure 20.

Theorem 2: *Let X be a finite oriented graph with out-degree 1 for every node. X is isomorphic to $\mathcal{C}_d(T,\beta)$ for ternary semigroup T and its member β implies and implied by when it includes a linked part R with the characteristic that for each linked part J of X the count $\chi(J)$ is factor of $\chi(R)$ and $\lambda(J)\leq\lambda(R)+1$.*

Proof: Assume the candidate X is isomorphically equal to $\mathcal{C}_d(T,\beta)$ for some T and β. Let β have a period h and index k. This implies that the members,

$$[\beta],[\beta^3],[\beta^5],\ldots,[\beta^{h+k-2}].$$

are pairwise different and $[\beta^{h+k-2}]=[\beta^k]$. Hence in X there is a circuit of length $\frac{h}{2}$ and oriented walk of the length $\frac{k-1}{2}$ whose end node is in this circuit; the start node corresponds to the element β. The connected component containing β is given in the figure 21. Now let x be any node of X. Let J be the connected portion of X including x. Let $\hat{p}$ identified as the length-measure of the oriented walk propagating from x, arrives at a node of a circuit and including no arc of this circuit; clearly $\hat{p}\leq\lambda(J)$. Let $\hat{q}=\chi(J)$. Then the members,

$$x,[\beta(\beta x)],[\beta^3(\beta x)],[\beta^5(\beta x)],\ldots,[\beta^{\hat{p}+\hat{q}-2}(\beta x)].$$

are pairwise disjoint and $[\beta^{\hat{p}+\hat{q}}(\beta x)] = [\beta^{\hat{p}}(\beta x)]$. If $\hat{q}\nmid\frac{h}{2}$, then $k\not\equiv h+k$ congruent modulo q and thus $[\beta^k(\beta x)]\neq[\beta^{h+k}(\beta x)]$, which runs counter to the premise $[\beta^{h+k}]=[\beta^k]$. Hence $\hat{q}$ must divide $\frac{h}{2}$ and $\frac{h}{2}=\chi(R)$, where R is the connected part of X including β. Now suppose $\hat{p}\geq\frac{k+3}{2}$. Then $[\beta^{k+2}(\beta x)]$ is distinct from $[\beta^l(\beta x)]$ for each $l\neq k+2$. But, as $[\beta^{h+k}]=[\beta^k]$, we must have $[\beta^{k+2}(\beta x)]=[\beta^{h+k+2}(\beta x)]$, which is a contradiction. Hence $\hat{p}\leq\frac{k+1}{2}=\frac{k-1}{2}+1\leq\lambda(R)+1$.

Now assume that the criterion is achieved. In R take a oriented walk that does not include any edges from a circuit with the length $\lambda(R)$; its start node will be β. Take all vertices of X with no incoming edges and if X includes connected parts different from R which are circuits, select one node in each of those linked portions. The resulting set will be designated as **B**. The node β and the nodes of **B** together forms the ternary semigroup T. Each leftover node will be identified as a power of β or a multiple of an element of **B** with a power of β matches the definition of $\mathcal{C}_d(T,\beta)$. Further, we define the equality $[xyz]=z$ for each $x\neq\beta,\ y\neq\beta,\ z\in T$. Associativity can be seen by observing the fact that

$$[xy[zhk]]=[xyk]=k, [x[yzh]k]=[xhk]=k, [[xyz]hk]=[zhk]=k.$$

Where $\beta\notin\{x,y,z,h\}\subseteq T$. Therefore, we have established a precise definition for a semigroup T such that X is isomorphic to $\mathcal{C}_d(T,\beta)$.

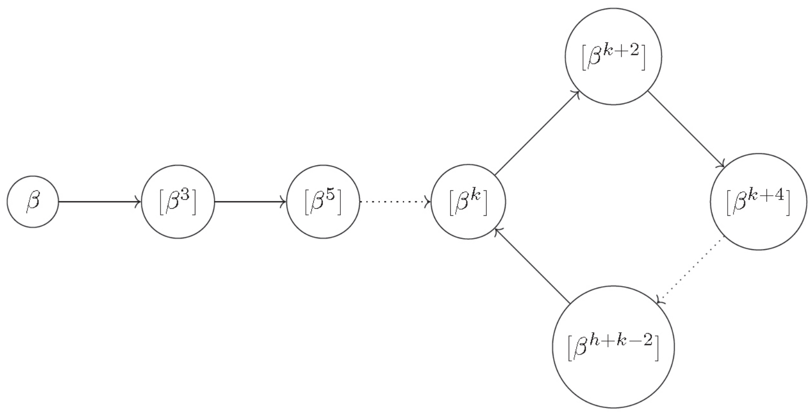

Figure 21

Connected component of X containing β.

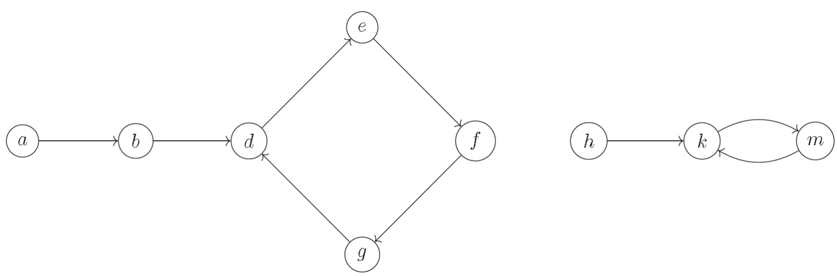

Figure 22

The oriented graph X_0.

Let us look at an example of constructing the ternary semigroup from the given oriented graph X_0 with out-degree 1 for each node (figure 22).

Given X_0 below, Here in X_0 there is a connected component R with $\chi(R)=4$ and $\lambda(R)=2$. Also note that for the component J with node set $\{h,k,m\}$ we have $\chi(J)$ divides $\chi(R)$ and $\lambda(J)\leq\lambda(R)+1$. So the condition for the converse of the Theorem 2 is fulfilled.

From the Theorem 2 the set $B=\{h\}$ and set $b=[a^3]$, $d=[a^5]$, $e=[a^7]$, $f=[a^9]$, $g=[a^{11}]$ and $[a^5]=[a^{11}]$. Also $k=[a^2h]$, $m=[a^4h]$ and $[a^6h]=k$. So the corresponding ternary semigroup $T(X_0)=\{a, [a^3], [a^5], [a^7], [a^9], h, [a^2h], [a^4h]\}$. We have modified the figure 22 into figure 23.

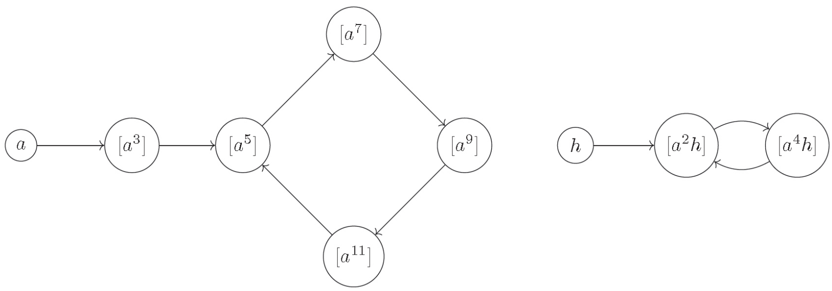

Figure 23

The oriented graph $\mathcal{C}_d(T(X_0),a)$.

4. Conclusion

We have implemented two important graph construction arises from ternary semigroups. The first one is oriented hypergraph and the second one is a oriented graph. We have characterized ternary semigroups in which the first and second kind of graphs arises. These graphs play vital role in the structure and properties of the respective ternary semigroups and thus opens a new route to study combinatorial properties of finite ternary semigroups. The research helps in computational algorithms associated with finite ternary semigroups. One can extend the investigation on the properties of $\mathcal{C}_d(T,B)$ and $\mathcal{C}_h(T,B)$ such as Hamiltonicity, regularity and unorientedness of edges. This paper throws light on new methods related combinatorial ternary semigroup theory via hypergraphs.

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