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Note on the size of standard two-character sets in PG(d,q)

Daniela Tondini
Faculty of Political Sciences
University of Teramo
Campus Universitario di Coste Sant'Agostino
Via R. Balzarini, 1
I-64100 Teramo
Italy

Abstract

This note concerns skills and research areas relating to the foundations of mathematics and the aspects of complementary and elementary mathematics from a higher point of view necessary for their treatment. In particular, we prove that, in PG(d,q), with d odd or q a square, any c-set of type (a,b) with $c \in \left\{ \frac{a\left[\vartheta_{d-1} + q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}}, \frac{b\left[\vartheta_{d-1} - q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}} \right\}$ has standard parameters.

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1. Introduction and Motivation.

Foundations of mathematics deal with what objects are studied and what are basic properties of the studied objects. In this note the problem of what are two-character sets with standard parameters is considered because it is related to the problem of knowledge unification, see [1], [3], [4] and [7]. Let PG(d,q) denote a finite d–dimensional projective space over the field $\mathbb{F}_q$ of order q, $q=p^e$, p a prime. Let $\vartheta_r \coloneqq \sum_{i=0}^{r} q^i$ denote the size of a subspace of dimension r. An c–set S of type (a,b) in PG(d,q) is a set of c points with the intersection size of S with any hyperplane only takes two values, a and b, and both values occur, see [8], [10] and [11]. The integers a and b are said to be the *intersection numbers*. Purely combinatorial argument shows that b–a is a divisor of q^{d-1}. When d is odd or q is a square, if $b - a = q^{\frac{d-1}{2}}$, the intersection numbers are said to be

E-mail: **dtondini@unite.it**

standard, see [5], [6] and [9]. In [2], De Finis showed that the size of a standard two-character set is $c \in \left\{ \frac{a\left[\vartheta_{d-1}+q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}}, \frac{b\left[\vartheta_{d-1}-q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}} \right\}$.

In this paper, we prove the following converse

Theorem: *In* PG(d,q), *d odd or q square, any c–set of type* (a,b), *with size* $c \in \left\{ \frac{a\left[\vartheta_{d-1}+q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}}, \frac{b\left[\vartheta_{d-1}-q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}} \right\}$ *has standard intersection numbers.*

2. The proof of the theorem.

Let i be an integer such that $0 \le i \le \vartheta_{d-1}$. Let us denote by $t_i = t_i(S)$ the size of the hyperplane set of PG(d,q) meeting S in exactly i points. The integers t_i are said to be the *characters* of S with respect to the hyperplanes, cfr. [11]. Throughout this section, let S denote a c–set of PG(d,q), $q>2$, $q=p^e$, p prime, of type (a,b), with size $c \in \left\{ \frac{a\left[\vartheta_{d-1}+q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}}, \frac{b\left[\vartheta_{d-1}-q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}} \right\}$. Double counting the hyperplanes, the pairs (P,H), with $P \in S$ and H is a hyperplane containing P, and the pairs ($\{P, Q\}$,H), with $\{P,Q\} \subset S$ and H is a hyperplane containing $\{P,Q\}$, we have:

$$t_a + t_b = \vartheta_d, \tag{2.1.1}$$

$$at_a + bt_b = c\vartheta_{d-1}, \tag{2.1.2}$$

$$a(a-1)t_a + b(b-1)t_b = c\,(c-1)\vartheta_{d-2}. \tag{2.1.3}$$

Firstly, we show that $a \ge 1$. Indeed, let us suppose, on the contrary, that $a = 0$. From (2.1.2) and (2.1.3) we immediately have that $b-1 = \frac{(c-1)\vartheta_{d-2}}{\vartheta_{d-1}} \in \left\{ a - \frac{\vartheta_{d-2}-q^{\frac{d-1}{2}}}{\vartheta_{d-1}}, b - \frac{\vartheta_{d-2}+q^{\frac{d-1}{2}}}{\vartheta_{d-1}} \right\}$, where $0 < \frac{\vartheta_{d-2} \mp q^{\frac{d-1}{2}}}{\vartheta_{d-1}} < 1$, for any $q \ge 2$. So, we have that b is not an integer, a contradiction.

From (2.1.1) and (2.1.2) we get

$$(b-a)t_a = b\vartheta_d - c\vartheta_{d-1}, \tag{2.1.4}$$

$$(b-a)t_b = c\vartheta_{d-1} - a\vartheta_d. \tag{2.1.5}$$

Taking into account (2.1.3), (2.1.4), and (2.1.5), we get

$$ab\vartheta_d - [(a+b)\vartheta_{d-1} - q^{d-1}]c + c^2\,\vartheta_{d-2} = 0. \tag{2.1.6}$$

In order to prove that, if $c \in \left\{\frac{a\left[\vartheta_{d-1}+q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}}, \frac{b\left[\vartheta_{d-1}-q^{\frac{d-1}{2}}\right]}{\vartheta_{d-2}}\right\}$, then the pair $\left(a, a+q^{\frac{d-1}{2}}\right)$ is the only solution of (2.1.6), put $b=a+x$. We get

$$aq^{\frac{d-1}{2}}\left(q^{\frac{d-1}{2}}+\vartheta_{d-1}\right)\left(q^{\frac{d-1}{2}}-x\right)=0 \Rightarrow x=q^{\frac{d-1}{2}}$$

This concludes the proof of the theorem.

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