

Left generalized homoderivations in 3-prime near-rings

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Abstract

In this paper, the structure of 3-prime near-rings with left generalized homoderivation satisfying certain conditions on a semigroup ideal will be investigated. After that, we'll provide examples to explain why the hypotheses used in the various results are necessary.

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1. Introduction

Throughout this paper, $\mathcal{N}$ stands for a left near-ring with center $Z(\mathcal{N})$. Usually $\mathcal{N}$ will be 3-prime, if for $a, b \in \mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies that $a = 0$ or $b = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $x0 = 0$). Recalling that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. A non-empty subset U of $\mathcal{N}$ is called a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$), and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$.

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In (2019), A. Boua [3] has introduced the notion of homoderivation of a 3-prime near-ring $\mathcal{N}$ in the following way. An additive mapping $h: \mathcal{R} \rightarrow \mathcal{R}$ is homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in \mathcal{N}$. In (2021) A. Boua and E. Farhan [4] have defined the generalized homoderivation of a near-ring $\mathcal{N}$ as follows. An additive mapping $G: \mathcal{N} \rightarrow \mathcal{N}$ is called left (resp. right) generalized homoderivation associated with homoderivation h on $\mathcal{N}$ if $G(xy) = h(x)G(y) + h(x)y + xG(y)$ (resp. $G(xy) = G(x)h(y) + G(x)y + xh(y)$) for all $x, y \in \mathcal{N}$. An additive mapping $G: \mathcal{N} \rightarrow \mathcal{N}$ is called a generalized homoderivation associated with homoderivation h on $\mathcal{N}$ if it is both a left and a right generalized homoderivation with associated homoderivation h . A homoderivation h is of course a generalized homoderivation.

For all $x, y \in \mathcal{N}$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator $xy + yx$. Let $E \subseteq \mathcal{N}$ and f a map from $\mathcal{N}$ to $\mathcal{N}$. If $f(E) \subseteq E$, then f is said to preserve E , and if for each $x \in E$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$, the map f is said zero-power valued in E .

Many authors have recently investigated the commutativity of prime and semiprime near-rings (resp. gamma near-rings) admitting suitably constrained additive mappings, such as generalized derivations, generalized semi-derivations, and generalized higher derivations acting on appropriate subsets of the near-rings (resp. gamma near-rings) (see [[2], [5], [6]], ..., etc. for reference, where further references can be found). In this work, we'll look at the structure of a 3-prime near-ring with generalized homoderivations that satisfy suitable algebraic conditions on the semigroup ideals of $\mathcal{N}$, and we generalize a few results from [7].

2. Some preliminaries

The literature contains numerous near-ring outcomes that are well-known, and we list a few of them below. These results will be heavily cited to support our results.

Lemma 1 : [1, Lemmas 1.2 (i), 1.2 (iii), and 1.3 (iii)] *Let $\mathcal{N}$ be 3-prime near-ring.*

- (i) *If $a \in Z(\mathcal{N}) \setminus \{0\}$, then a is not a zero divisor.*
- (ii) *If $Z(\mathcal{N})$ contains a nonzero element a for which $a + a \in Z(\mathcal{N})$, then $\mathcal{N}$ is abelian.*

- (iii) If $a \in Z(\mathcal{N}) \setminus \{0\}$ and $t \in \mathcal{N}$ such that $at \in Z(\mathcal{N})$ or $ta \in Z(\mathcal{N})$, then $t \in Z(\mathcal{N})$.

Lemma 2 : [1, Lemmas 1.3 (i), 1.4 (i), and 1.3 (iii)] Let $\mathcal{N}$ be a 3-prime near-ring.

- (i) If I is a nonzero semigroup right (resp. semigroup left) ideal of $\mathcal{N}$ and $a \in \mathcal{N}$ such that $Ia = \{0\}$ (resp. $aI = \{0\}$), then $a = 0$.
- (ii) If I is a nonzero semigroup ideal of $\mathcal{N}$ and $a, b \in \mathcal{N}$ such that $aIb = \{0\}$, then $a = 0$ or $b = 0$.
- (iii) If I is a nonzero semigroup right ideal of $\mathcal{N}$ and $a \in \mathcal{N}$ which centralizes I , then $a \in Z(\mathcal{N})$.

Lemma 3 : [7, Lemma 1.5] If $\mathcal{N}$ is a 3-prime near-ring and $Z(\mathcal{N})$ contains a nonzero semigroup left ideal or semigroup right ideal, then $\mathcal{N}$ is a commutative ring.

Lemma 4 : [7, Lemma 6] Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero additive map f on $\mathcal{N}$ which is zero-power valued on $\mathcal{N}$. Then the following assertions are equivalent:

- (i) $f(t) + t \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$.
- (ii) $t + f(t) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$.
- (iii) $\mathcal{N}$ is a commutative ring.

Lemma 5 : Let f be a nonzero additive map which is zero-power valued on a near-ring $\mathcal{N}$.

- (i) If $t \in \mathcal{N}$ such that $f(t) + t = 0$, then $t = 0$.
- (ii) If $t \in \mathcal{N}$ such that $t + f(t) = 0$, then $t = 0$.

Proof: (i) Let $t \in \mathcal{N}$. Suppose that $f(t) + t = 0$. By recurrence, we can easily arrive at $f^n(t) + (-1)^{n+1}t = 0$. Given that f is zero-power value on $\mathcal{N}$, there is an integer $k(t) > 1$ such that $f^{k(t)}(t) = 0$. In the above equation, we obtain $t = 0$ by substituting $k(t)$ for n .

(ii) We arrive at the proof by employing reasoning similar to (i). $\square$
The result that follows is a generalization of [3, Lemma 2.5].

Lemma 6 : Let $\mathcal{N}$ be a 3-prime near-ring. If G is a left generalized homoderivation associated with a nonzero homoderivation h on $\mathcal{N}$, then for all $a, b, c \in \mathcal{N}$ we have

$$h(ab)(G(c)+c) = h(a)h(b)(G(c)+c) + h(a)b(G(c)+c) + ah(b)(G(c)+c).$$

Proof: We calculate $G(abc)$ in two different ways, we have

$$G(abc) = h(ab)G(c) + h(ab)c + abG(c) = h(ab)(G(c)+c) + abG(c).$$

On the other hand

$$\begin{aligned} G(abc) &= h(a)G(bc) + h(a)bc + aG(bc) \\ &= h(a)h(b)G(c) + h(a)h(b)c + h(a)bG(c) + h(a)bc + ah(b)G(c) \\ &\quad + ah(b)c + abG(c). \end{aligned}$$

We arrive at the needed result by comparing the two final expressions. $\square$

The following result is a generalization of [7, Lemma 3.2].

Lemma 7 : *Let $\mathcal{N}$ be near-ring. If $\mathcal{N}$ is 2-torsion free and admits a left generalized homoderivation G which is zero-power valued on $\mathcal{N}$ associated with a nonzero homoderivation h , then $\mathcal{N}$ is zero-symmetric near-ring.*

Proof: By using the definition of G , we get

$$\begin{aligned} G(0t) &= h(0)G(t) + h(0)t + 0G(t) \\ &= 0G(t) + 0n + 0G(t) \text{ for all } t \in \mathcal{N}. \end{aligned}$$

On the other hand

$$\begin{aligned} G(0t) &= G(0(0t)) \\ &= 0G(t) + 0t + 0G(t) + 0t + 0G(t) + 0t + 0G(t) \text{ for all } t \in \mathcal{N}. \end{aligned}$$

Thus, for all $t \in \mathcal{N}$, we find that $2(0G(t) + 0t) = 0$ by comparing the last two expressions. So, we obtain $0G(t) + 0t = 0$ for all $t \in \mathcal{N}$ using the 2-torsion freeness of $\mathcal{N}$. By recurrence, we obtain $0G^p(t) + (-1)^{p+1}0t = 0$ for all $t \in \mathcal{N}, p \in \mathbb{N}^*$. Using the fact that G is zero-power valued on $\mathcal{N}$, there exists an integer $k(t) > 1$ such that $G^{k(t)}(t) = 0$. Taking $k(t)$ in place of p in last equation, we obtain $0t = 0$ for all $t \in \mathcal{N}$. Which means that $\mathcal{N}$ is zero-symmetric near-ring. $\square$

3. Results for semigroup ideal and generalized homoderivation

The following result is a generalization of [7, Lemma 3.3].

Lemma 8 : Let $\mathcal{N}$ be a 3-prime near-ring and G be a left generalized homoderivation associated with a nonzero homoderivation h on $\mathcal{N}$.

- (i) If I is a nonzero semigroup left ideal of $\mathcal{N}$, then $G(I) \neq \{0\}$.
- (ii) If $\mathcal{N}$ is zero-symmetric and I is a nonzero semigroup right ideal of $\mathcal{N}$, then $G(I) \neq \{0\}$.

Proof: (i) Assume that $G(I) = \{0\}$, where I is a nonzero semigroup left ideal on $\mathcal{N}$. Then, we have $0 = G(tu) = h(t)G(u) + h(t)u + tG(u) = h(t)u$ for all $u \in I, t \in \mathcal{N}$. This implies that $h(t)I = \{0\}$ for all $t \in \mathcal{N}$. Hence $h = 0$ by Lemma 2 (i); which gives a contradiction with our hypothesis.

(ii) Suppose that I be a nonzero semigroup right ideal on $\mathcal{N}$ and $G(I) = \{0\}$. It follows that $0 = G(uvt) = h(u)G(vt) + h(u)vt + uG(vt) = h(u)vt$ for all $u, v \in I, t \in \mathcal{N}$. So, $h(u)It = \{0\}$ for all $u \in I, t \in \mathcal{N}$. By using lemma 2 (i), we get a contradiction. $\square$

We now give a generalization of [7, Lemma 3.5].

Lemma 9 : Let I be a nonzero semigroup ideal of a 3-prime near-ring $\mathcal{N}$ and G be a left generalized homoderivation associated with a nonzero homoderivation h on $\mathcal{N}$ such that h preserves I . If $t \in \mathcal{N}$ and $[G(t) + t, h(I)] = \{0\}$, then $G(t) + t \in Z(\mathcal{N})$.

Proof: Let $t \in \mathcal{N}$ such that $[G(t) + t, h(I)] = \{0\}$. We set

$C(t) = \{x \in \mathcal{N} \mid (G(t) + t)x = x(G(t) + t)\}$. Since $h(I) \subset C(t) \cap I$, it follows that both $h(yu)$, $h(u)$, $h(y)$ and $yh(u)$ are in $C(t)$ for all $u \in I$, $y \in C(t) \cap I$. In view of $(G(t) + t)h(yu) = h(yu)(G(t) + t)$, we arrive at $h(y)u \in C(t)$ for all $u \in I$. So $h(y)uv \in C(t)$ for all $u, v \in I$ and $0 = [G(t) + t, h(y)uv] = h(y)u[(G(t) + t), v]$. Consequently $h(y)I[(G(t) + t), v] = \{0\}$ for all $v \in I$. Using the fact that $h(I) \neq \{0\}$ and Lemma 2 (ii), we get $[G(t) + t, I] = \{0\}$. Thus, $G(t) + t \in Z(\mathcal{N})$ according to Lemma 2 (iii). $\square$

The following result extends [7, Theorem 3.6].

Theorem 1 : Let I be a nonzero semigroup left ideal of a 3-prime near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a left generalized homoderivation G associated with a nonzero homoderivation h such that h and G are zero-power valued on $\mathcal{N}$. Then the following assertions are equivalent:

- (i) $G(i) + i \in Z(\mathcal{N})$ for all $i \in I$.
- (ii) $-i + G(-i) \in Z(\mathcal{N})$ for all $i \in I$.
- (iii) $\mathcal{N}$ is a commutative ring.

Proof: (i) $\Rightarrow$ (iii) Assume that $G(i) + i \in Z(\mathcal{N})$ for all $i \in I$. Thus, $G(ti) + ti \in Z(\mathcal{N})$ for all $i \in I, t \in \mathcal{N}$. Which gives

$$\begin{aligned} G(ti) + ti &= h(t)G(i) + h(t)i + tG(i) + ti \\ &= (G(i) + i)(h(t) + t) \in Z(\mathcal{N}) \text{ for all } i \in I, t \in \mathcal{N}. \end{aligned}$$

Since $G(i) + i \in Z(\mathcal{N})$, by Lemma 1 (iii), we obtain $G(i) + i = 0$ for all $i \in I$ or $h(t) + t \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$.

According to Lemma 5, if $G(i) + i = 0$ for every $i \in I$, we arrive at the paradoxical result $I = \{0\}$. Therefore, $h(t) + t \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$. By Lemma 4, it follows that $\mathcal{N}$ is a commutative ring.

(ii) $\Rightarrow$ (iii) Suppose that $-i + G(-i) \in Z(\mathcal{N})$ for all $i \in I$. Then

$$\begin{aligned} -(G(ti) + ti) &= -(h(t)G(i) + h(t)i + tG(i) + ti) \\ &= (-i - G(i))(t + h(t)) \in Z(\mathcal{N}) \text{ for all } i \in I, t \in \mathcal{N}. \end{aligned}$$

As $-i - G(i) \in Z(\mathcal{N})$, using Lemma 1 (iii), we find $-i - G(i) = 0$ for all $i \in I$ or $t + h(t) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$.

First assume that $-i - G(i) = 0$ for all $i \in I$. Using Lemma 5, we obtain $I = \{0\}$; a contradiction. Hence, $t + h(t) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$. Consequently $\mathcal{N}$ is a commutative ring by Lemma 4.

The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are clear. $\square$

The following Theorem is a generalization of [7, Theorem 3.8].

Theorem 2 : Let I be a nonzero semigroup ideal of a 3-prime near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a left generalized homoderivation G associated with a nonzero homoderivation h , which satisfies the following conditions:

- (i) h and G are zero-power valued on $\mathcal{N}$,
- (ii) h preserves I ,
- (iii) $[h(I), G(I) + I] = \{0\}$,

then $\mathcal{N}$ is a commutative ring.

Proof: A direct application of Theorem 1 and Lemma 9 is used to prove this result. $\square$

Theorem 3 : Let I be a nonzero semigroup ideal of a 3-prime near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a left generalized homoderivation G which is zero-power valued on $\mathcal{N}$, and associated with a nonzero homoderivation h . Then the following assertions are equivalent:

- (i) $G([x, y]) + [x, y] = [h(x), y]$ for all $x, y \in I$.

(ii) $\mathcal{N}$ is a commutative ring.

Proof: (i) $\Rightarrow$ (ii) Suppose that

$$G([x, y]) + [x, y] = [h(x), y] \quad \text{for all } x, y \in I. \quad (1)$$

It is clear that $[h(x), x] = 0$ for all $x \in I$. Replacing y by xy in (1), we find

$$h(x)(G([x, y]) + [x, y]) + x(G([x, y]) + [x, y]) = x[h(x), y] \quad \text{for all } x, y \in I.$$

By using (1), we arrive at $h(x)yh(x) = h(x)h(x)y$ for all $x, y \in I$. Taking yt instead of y in last equation, and using it we obtain $h(x)I[t, h(x)] = \{0\}$ for all $x \in I, t \in \mathcal{N}$. Using Lemma 2 (ii), we obtain $h(x) = 0$ or $h(x) \in Z(\mathcal{N})$ for all $x \in I$. Consequently, $h(I) \subseteq Z(\mathcal{N})$. From (1), it follows that $G([x, y]) + [x, y] = 0$ for all $x, y \in I$. Due to Lemma 5, we get $[x, y] = 0$ for all $x, y \in I$, and from Lemma 2 (iii), we conclude that $I \subseteq Z(\mathcal{N})$. Consequently, $\mathcal{N}$ is a commutative ring by Lemma 3.

(ii) $\Rightarrow$ (i) is obvious.

The result that follows is a generalisation of [7, Theorem 3.12]

Theorem 4 : Let I be a nonzero semigroup ideal of a 3-prime near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a left generalized homoderivation G associated with a nonzero homoderivation h such that h and G are zero-power valued on $\mathcal{N}$, then the following assertions are equivalent:

- (i) $G(xy) + xy \in Z(\mathcal{N})$ for all $x, y \in I$.
- (ii) $\mathcal{N}$ is a commutative ring.

Proof: (i) $\Rightarrow$ (ii) Suppose that $G(xy) + xy \in Z(\mathcal{N})$ for all $x, y \in I$. Then $G(txy) + txy = (G(xy) + xy)(h(t) + t) \in Z(\mathcal{N})$ for all $x, y, t \in I$. By Lemma 1 (iii), we get $G(xy) + xy = 0$ for all $x, y \in I$ or $h(t) + t \in Z(\mathcal{N})$ for all $t \in I$.

Assume That $G(xy) + xy = 0$ for all $x, y \in I$, by using Lemma 5, we get $xy = 0$ for all $x, y \in I$. 3-primeness of $\mathcal{N}$ leads to $I = \{0\}$; a contradiction. Consequently, $h(t) + t \in Z(\mathcal{N})$ for all $t \in I$, so $\mathcal{N}$ is a commutative ring by Theorem 1.

(ii) $\Rightarrow$ (i) This implication is trivial. □

The following result is an extension of [7, Theorem 3.13].

Theorem 5 : Let I be a nonzero semigroup ideal of a 2-torsion free 3-prime near-ring $\mathcal{N}$. There is no left generalized homoderivation G which is zero-power valued on $\mathcal{N}$, and associated with a nonzero homoderivation h such that $G(x \circ y) + x \circ y = [x, y]$ for all $x, y \in I$.

Proof: Assume that

$$G(x \circ y) + x \circ y = [x, y] \quad \text{for all } x, y \in I. \quad (2)$$

So,

$$\begin{aligned} y[x, y] &= G(yx \circ y) + yx \circ y \\ &= h(y)G(x \circ y) + h(y)(x \circ y) + yG(x \circ y) + y(x \circ y) \\ &= h(y)[x, y] + y[x, y] \quad \text{for all } x, y \in I. \end{aligned}$$

Which implies that $h(y)xy = h(y)yx$ for all $x, y \in I$. Putting xn in place of x in the last equation, we find $h(y)I[n, y] = \{0\}$ for all $y \in I, n \in \mathcal{N}$. Using Lemma 2 (ii), we get

$$y \in Z(\mathcal{N}) \quad \text{or} \quad h(y) = 0 \quad \text{for all } y \in I. \quad (3)$$

If there is an element y_0 in I such that $y_0 \in Z(\mathcal{N})$, from (2) it follows that $G(2xy_0) + 2xy_0 = 0$, for all $x \in I$. By using Lemma 5, we arrive at $2xy_0 = 0$ for all $x \in I$, and by the 2-torsion freeness of $\mathcal{N}$, we find that $Iy_0 = \{0\}$. Thus, $y_0 = 0$ according to Lemma 2 (i). Therefore, (3) gives $h(I) = \{0\}$ which contradicts the Lemma 8. $\square$

The result that follows is a generalisation of [7, Theorem 3.14].

Theorem 6 : *Let I be a nonzero semigroup ideal of a 2-torsion free 3-prime near-ring $\mathcal{N}$. There is no left generalized homoderivation G associated with a nonzero homoderivation h on $\mathcal{N}$ such that $G([x, y]) + [x, y] = x \circ y$ for all $x, y \in I$.*

Proof: Assume that

$$G([x, y]) + [x, y] = x \circ y \quad \text{for all } x, y \in I. \quad (4)$$

That gives

$$\begin{aligned} y(x \circ y) &= G([yx, y]) + [yx, y] \\ &= h(y)(G([x, y]) + [x, y]) + y(G([x, y]) + [x, y]) \\ &= h(y)(x \circ y) + y(x \circ y) \quad \text{for all } x, y \in I. \end{aligned}$$

Which implies that $h(y)(x \circ y) = 0$ for all $x, y \in I$. Thus, $h(y)xy = -h(y)yx$ for all $x, y \in I$. Taking xm in place of x in the last equation, we arrive at $h(y)I(-m(-y) + (-y)m) = \{0\}$ for all $y \in I, m \in \mathcal{N}$. By Lemma 2 (ii), we have

$$-y \in Z(\mathcal{N}) \quad \text{or} \quad h(y) = 0 \quad \text{for all } y \in I. \quad (5)$$

Let's suppose that I has an element y_0 such that $-y_0 \in Z(\mathcal{N})$. Taking $-y_0$ in place of y in (4) we find $2(-y_0)x = 0$ for all $x \in I$. 2-torsion freeness of $\mathcal{N}$, leads to $(-y_0)I = \{0\}$ and by Lemma 2 (i), it follows that $y_0 = 0$. Hence, (5) yields to $h(I) = \{0\}$, which is a contradiction by Lemma 8. $\square$

A generalisation of [7, Theorem 3.15] is the result that follows:

Theorem 7 : *Let I be a nonzero semigroup ideal of a 3-prime near-ring $\mathcal{N}$. There is no left generalized homoderivation G which is zero-power valued on $\mathcal{N}$, and associated with a nonzero homoderivation h such that $G(xy) + xy = [x, y]$ for all $x, y \in I$.*

Proof: Assume that

$$G(xy) + xy = [x, y] \quad \text{for all } x, y \in I. \quad (8)$$

Therefore,

$$\begin{aligned} y[x, y] &= G(yxy) + yxy \\ &= h(y)(G(xy) + xy) + y(G(xy) + xy) \\ &= h(y)[x, y] + y[x, y] \quad \text{for all } x, y \in I. \end{aligned}$$

Which gives $h(y)[x, y] = 0$ for all $x, y \in I$. In virtue of the Theorem 6 proof's it follows that

$$y \in Z(\mathcal{N}) \quad \text{or} \quad h(y) = 0 \quad \text{for all } y \in I. \quad (7)$$

If I contains an element y_0 such that $y_0 \in Z(\mathcal{N})$, by using (6) it follows that $G(xy_0) + xy_0 = 0$ for all $x \in I$. By Lemma 5, we find that $xy_0 = 0$ for all $x \in I$, so $Iy_0 = \{0\}$. Using Lemma 2 (i), yields to $y_0 = 0$. So, (7) implies that $h(I) = \{0\}$ which gives a contradiction by Lemma 8. $\square$

The following result extends [7, Theorem 3.17].

Theorem 8 : *Let I be a nonzero semigroup ideal of a 2-torsion 3-prime near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a left generalized homoderivation G associated with a nonzero homoderivation h such that h and G are zero-power valued on $\mathcal{N}$, then the following assertions are equivalent:*

- (i) $G(x \circ y) + x \circ y \in Z(\mathcal{N})$ for all $x, y \in I$.
- (ii) $\mathcal{N}$ is a commutative ring.

Proof: It is easy to see the implication (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Assume that

$$G(x \circ y) + x \circ y \in Z(\mathcal{N}) \quad \text{for all } x, y \in I. \quad (8)$$

Replacing y by xy in (8), we obtain $h(x)G(x \circ y) + h(x)(x \circ y) + xG(x \circ y) + x(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in I$. Using (8), leads to $(G(x \circ y) + x \circ y)(h(x) + x) \in Z(\mathcal{N})$ for all $x, y \in I$. So, by Lemma 1 (iii), we obtain

$$G(x \circ y) + x \circ y = 0 \quad \text{or} \quad h(x) + x \in Z(\mathcal{N}) \quad \text{for all } x, y \in I. \quad (9)$$

If there exists an element $x_0 \in I$ such that $G(x_0 \circ y) + x_0 \circ y = 0$ for all $y \in I$, then, by Lemma 5, we get

$$x_0 \circ y = 0 \quad \text{for all } y \in I. \quad (10)$$

Taking x_0 instead of y in (10), we find $2x_0^2 = 0$. Using 2-torsion freeness of $\mathcal{N}$ yields to $x_0^2 = 0$. Putting $x_0 y$ in place of y in (10), we get $x_0 y x_0 = 0$ for all $y \in I$, so $x_0 I x_0 = \{0\}$. Thus, $x_0 = 0$ according to Lemma 2 (ii). Using (9) yields to $h(x) + x \in Z(\mathcal{N})$ for all $x \in I$. Hence, $\mathcal{N}$ is a commutative ring by Theorem 1. $\square$

4. Examples

The following example proves that the 3-primeness of $\mathcal{N}$ in Theorems 2, 3, 4, 5, 6, 7 and 8 can not be omitted.

Example 1 : Let $\mathcal{A}$ be a zero-symmetric 2-torsion free left near-ring and

$$\mathcal{N} = \left\{ \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u, v, 0 \in \mathcal{A} \right\}, \quad I = \left\{ \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u, 0 \in \mathcal{A} \right\}.$$

Then I is a nonzero semigroup ideal of $\mathcal{N}$ and $\mathcal{N}$ is a 2-torsion free left near-ring that is not 3-prime. Defining $h = G : \mathcal{N} \rightarrow \mathcal{N}$ as follow:

$$G \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that G is a left generalized homoderivation associated with a nonzero homoderivation h such that both h and G are zero-power valued on $\mathcal{N}$ which satisfy the following identities:

- (i) $[h(I), G(I) + I] = \{0\},$ (ii) $G([A, B]) + [A, B] = [h(A), B],$
- (iii) $G(AB) + AB \in Z(\mathcal{N}),$ (iv) $G(A \circ B) + A \circ B \in Z(\mathcal{N}),$
- (v) $G(A \circ B) + A \circ B = [A, B],$ (vi) $G([A, B]) + [A, B] = A \circ B,$
- (vii) $G(AB) + AB = [A, B],$ (viii) $G(AB) + AB = A \circ B$

for all $A, B \in I$, but $\mathcal{N}$ is not commutative.

The following example shows that the condition “zero-power valued on $\mathcal{N}$ of G and h ” must be included in the hypothesis of Theorems 1, 2, 3 and 5.

Example 2 : Let $\mathcal{N} = I = M_2(\mathbb{Z})$ be the 3-prime 2-torsion free ring. We consider $G = h = id_{\mathcal{N}}$, then it is clear that G is a left generalized homoderivation G associated with nonzero homoderivation h such that both h and G are not “zero-power valued on $\mathcal{N}$ ” which satisfy the following identities:

- (i) $[h(I), G(I) + I] = \{0\},$ (ii) $G(A) + A \in Z(\mathcal{N}),$
- (iii) $-A + G(-A) \in Z(\mathcal{N}),$ (iv) $G(AB) + AB \in Z(\mathcal{N}),$
- (v) $G(A \circ B) + A \circ B \in Z(\mathcal{N}),$

for all $A, B \in I$, but $\mathcal{N}$ is not commutative.

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