

Inverse n-dominating sets in vague graphs with application in decision making

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Abstract

Compared to other graph models, vague graph models are more compatible and flexible in modeling situations with imprecise information. The goal of this study is to introduce the notions of inverse vague-dominating set, inverse vague-domination numbers and some of their properties in vague graphs. Eventually, the researchers have been provided an application that can be used to locate aid centers.

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1. Introduction

Today, graph models as mathematical simulation models are widely used in solving complex problems in various fields. Zadeh [23] first suggested the theory of fuzzy sets. Then, Rosenfeld [21] investigated the concept of fuzzy graph theory as an extension of Euler's graph. Somasundaram and Somasundaram [22] investigated domination in fuzzy graphs. In addition, Gau and Buehrer [11] suggested the nation of the vague set with alternating the value of an element in a set with a subinterval of $[0, 1]$. According to this, Ramakrishna [18] investigated the notion of vague graphs and some of its properties. Borzooei and Rashmanlou [8] investigated different types of dominating set in vague graphs. Vague-dominating sets are studied and investigated as one of the widely used notions in vague graphs. Banitalebi, Borzooei and other researchers had also worked on related studies [1,2,3,4,5,6,7,9,10,12,13,14,15,16,17,19,20]

The goal of this study is to investigate the nations and properties related to n -vague-domination parametr in vague graphs. Eventually, a technique to optimize the vague-dominating parameter in vague graph models will be offered.

2. Preliminaries

A *fuzzy graph* $\mathbb{G}_{\mathbb{F}\mathbb{U}} = (\xi, \chi)$ on crisp graph $\mathbb{G}_{\mathbb{R}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ is a pair of functions $\xi: \mathbb{A}_{\mathbb{V}} \rightarrow [0, 1]$ and $\chi: \mathbb{B}_{\mathbb{E}} \rightarrow [0, 1]$ wherever, for every $cd \in \mathbb{B}_{\mathbb{E}}$, $\chi(cd) \leq \min\{\xi(c), \xi(d)\}$.

A *vague graph* (VG) [18] on crisp graph (CG) $\mathbb{G}_{\mathbb{R}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$ is interpreted to be a pair $\mathbb{G}_{\mathbb{V}\mathbb{G}} = (\kappa, \tau)$, wherever $\kappa = (t_{\kappa}, f_{\kappa})$ is a vague set on $\mathbb{A}_{\mathbb{V}}$ and $\tau = (t_{\tau}, f_{\tau})$ is a vague set on $\mathbb{B}_{\mathbb{E}}$ wherever for every arc $qp \in \mathbb{B}_{\mathbb{E}}$,

$$t_{\tau}(qp) \leq \min\{t_{\kappa}(q), t_{\kappa}(p)\} \text{ and } f_{\tau}(qp) \geq \max\{f_{\kappa}(q), f_{\kappa}(p)\}.$$

Definition 2.1: [8] Put $\mathbb{G}_{\mathbb{V}\mathbb{G}} = (\kappa, \tau)$ be a VG on CG $\mathbb{G}_{\mathbb{R}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$. Then:

- (i) The *order* of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is explained by,

$$|\mathbb{A}_{\mathbb{V}}| = \sum_{q_i \in \mathbb{A}_{\mathbb{V}}} \left(\frac{t_{\kappa}(q_i) + (1 - f_{\kappa}(q_i))}{2} \right).$$

- (ii) The *degree* of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is interpreted with,

$$|\mathbb{B}_{\mathbb{E}}| = \sum_{q_i, q_j \in \mathbb{B}_{\mathbb{E}}} \left(\frac{t_{\tau}(q_i q_j) + (1 - f_{\tau}(q_i q_j))}{2} \right).$$

(iii) The *size* of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is explained as follows,

$$|\mathbb{G}_{\mathbb{V}}| = |\mathbb{A}_{\mathbb{V}}| + |\mathbb{B}_{\mathbb{E}}|.$$

(iv) For any $\mathcal{Y} \subseteq \mathbb{A}_{\mathbb{V}}$, the order of $\mathcal{Y}$ is demonstrated by $P(\mathcal{Y})$ and explained as follows,

$$P(\mathcal{Y}) = \sum_{y_i \in \mathcal{Y}} \left(\frac{t_{\kappa}(y_i) + (1 - f_{\kappa}(y_i))}{2} \right).$$

(v) For any $\mathcal{P} \subseteq \mathbb{B}_{\mathbb{E}}$, the *degree* of $\mathcal{P}$ is demonstrated by $M(\mathcal{P})$ and explained as follows,

$$M(\mathcal{P}) = \sum_{q_i q_j \in \mathcal{P}} \left(\frac{t_{\tau}(q_i q_j) + (1 - f_{\tau}(q_i q_j))}{2} \right).$$

Definition 2.2: [4, 8] Put $\mathbb{G}_{\mathbb{V}\mathbb{G}} = (\kappa, \tau)$ be a VG on CG $\mathbb{G}_{\mathbb{R}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$. Then:

(i) An arc $a = pq \in \mathbb{B}_{\mathbb{E}}$ is denominated a *strong arc* in $\mathbb{G}_{\mathbb{V}}$ if $t_{\tau}(pq) \geq t_{\tau}^{\infty}(pq)$ and $f_{\tau}(pq) \leq f_{\tau}^{\infty}(pq)$, wherever

$$t_{\tau}^{\infty}(pq) = \max\{t_{\tau}^k(pq) \mid k = 1, 2, \dots, l\}, \quad f_{\tau}^{\infty}(pq) = \min\{f_{\tau}^k(pq) \mid k = 1, 2, \dots, l\},$$

and

$$t_{\tau}^k(pq) = \min\{t_{\tau}(pm_1), t_{\tau}(m_1 m_2), \dots, t_{\tau}(m_{k-1} q) \mid p, m_1, \dots, m_{k-1}, q \in \mathbb{A}_{\mathbb{V}}, k = 1, 2, \dots, l\},$$

$$f_{\tau}^k(pq) = \max\{f_{\tau}(pm_1), f_{\tau}(m_1 m_2), \dots, f_{\tau}(m_{k-1} q) \mid p, m_1, \dots, m_{k-1}, q \in \mathbb{A}_{\mathbb{V}}, k = 1, 2, \dots, l\}.$$

(ii) The *neighborhood* of $j \in \mathbb{A}_{\mathbb{V}}$ is demonstrated with $N(j)$ and explained with:

$$N(j) = \{i \in \mathbb{A}_{\mathbb{V}} \mid ji \text{ is a strong arc in } \mathbb{G}_{\mathbb{V}\mathbb{G}}\}.$$

(iii) We say that j *dominate* i in $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ if there is a strong arc between j and i .

(iv) $S_{\mathbb{V}\mathbb{G}} \subset \mathbb{A}_{\mathbb{V}}$ is denominated a *vague-dominating set* (Vds) in $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ if for every $i \in \mathbb{A}_{\mathbb{V}} \setminus S_{\mathbb{V}\mathbb{G}}$, there is $j \in S_{\mathbb{V}\mathbb{G}}$ wherever j dominates i .

(v) A Vds $S_{\mathbb{V}\mathbb{G}}$ in $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is denominated a *minimal vague-dominating set* if no proper subset of $S_{\mathbb{V}\mathbb{G}}$ is a vague-dominating set.

(vi) Minimum order amidst all minimal VDSs of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is denominated *lower vague-domination number* of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ and is indicated with $d_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}\mathbb{G}})$.

- (vii) Maximum order amidst all minimal VDSs of $\mathbb{G}_{\text{vg}}$ is denominated *upper vague-domination number* of $\mathbb{G}_{\text{vg}}$ and is demonstrated by $D_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}})$.
- (viii) The *vague-domination number* of $\mathbb{G}_{\text{vg}}$ is demonstrated with $\Delta_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}})$ and is explained with,

$$\Delta_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}}) = \frac{d_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}}) + D_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}})}{2}.$$

Definition 2.3: [4] Put $\mathbb{G}_{\text{vg}} = (\kappa, \tau)$ be a VG on CG $\mathbb{G}_\text{gr} = (\mathbb{A}_\text{v}, \mathbb{B}_\text{e})$. Afterward:

- (i) The *vague-cobondage set*(Vbs) of a VG $\mathbb{G}_{\text{vg}}$ is the set $\mathbb{W}$ of additional strong arcs of $\mathbb{G}_{\text{vg}}$, that reduces the vague-domination number of $\mathbb{G}_{\text{vg}}$. i.e.

$$\Delta_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}\mathbb{W}}) < \Delta_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}}).$$

- (ii) A Vbs $\mathbb{W}$ of $\mathbb{G}_{\text{vg}}$ is denominated a *minimal vague-cobondage set* if no proper subset of $\mathbb{W}$ is a vague-cobondage set.
- (iii) Minimum order amidst all minimal VBSs of $\mathbb{G}_{\text{vg}}$ is denominated *lower vague-cobondage number* of $\mathbb{G}_{\text{vg}}$ and is demonstrated with $b_{\mathbb{B}_\text{e}}(\mathbb{G}_{\text{vg}})$.
- (iv) Maximum order amidst all minimal VBSs of $\mathbb{G}_{\text{vg}}$ is denominated *upper vague-cobondage number* of $\mathbb{G}_{\text{vg}}$ and is demonstrated with $B_{\mathbb{B}_\text{e}}(\mathbb{G}_{\text{vg}})$.

Definition 2.4: [4, 8] Put $\mathbb{G}_{\text{vg}} = (\kappa, \tau)$ be a VG on CG $\mathbb{G}_\text{gr} = (\mathbb{A}_\text{v}, \mathbb{B}_\text{e})$. Then:

- (i) Two nodes $j, i \in \mathbb{A}_\text{v}$ are denominated *independent* if there is not any strong arc betwixt them.
- (ii) $S_{\text{vg}} \subset \mathbb{A}_\text{v}$ is denominated an *vague-independent set* (VIS) in $\mathbb{G}_{\text{vg}}$ if for any $q, p \in S_{\text{vg}}$, $t_\tau(qp) < t_\tau^\infty(qp)$ and $f_\tau(qp) > f_\tau^\infty(qp)$.
- (iii) A VIS S_{vg} in $\mathbb{G}_{\text{vg}}$ is denominated a *maximal vague-independent set* if for any node $p \in \mathbb{A}_\text{v} \setminus S_{\text{vg}}$, the set $S_{\text{vg}} \cup \{p\}$ is not independent.
- (iv) Minimum order amidst all maximal VISs of $\mathbb{G}_{\text{vg}}$ is denominated *lower vague-independent number* of $\mathbb{G}_{\text{vg}}$ and is demonstrated with $i_{\mathbb{A}_\text{v}}(\mathbb{G}_{\text{vg}})$.

- (v) Maximum order amidst all maximal VISs of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is denominated *upper vague-independent number* of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ and is demonstrated with $I_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}\mathbb{G}})$.
- (vi) The *vague-independent number* of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is demonstrated by $I(\mathbb{G}_{\mathbb{V}\mathbb{G}})$ and explained with,

$$I(\mathbb{G}_{\mathbb{V}\mathbb{G}}) = \frac{i_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}\mathbb{G}}) + I_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}\mathbb{G}})}{2}.$$

Notation: Hereinafter, in this study, we put $G = (\kappa, \tau)$ be a connected VG on CG $\mathbb{G}_{\mathbb{R}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{E}})$.

3. Inverse vague-dominating set of a vague graph

In this part, we discuss the concept of the inverse vague-dominating set of the VGs.

Definition 3.1: Put $\mathbb{D}_{\mathbb{V}\mathbb{G}} \subset \mathbb{A}_{\mathbb{V}}$ be a minimal Vds of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$. Then:

- (i) $\mathbb{D}_{\mathbb{V}\mathbb{G}}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}\mathbb{G}}$ is denominated an inverse vague-dominating set $((Vds)^{-1})$ of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ with respect to $\mathbb{D}_{\mathbb{V}\mathbb{G}}$ if $\mathbb{D}_{\mathbb{V}\mathbb{G}}^{-1}$ is a vague-dominating set of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$.
- (ii) An inverse vague-dominating set $\mathbb{D}_{\mathbb{V}\mathbb{G}}^{-1} \subseteq \mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}\mathbb{G}}$ is denominated minimal inverse vague-dominating set of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ related to $\mathbb{D}_{\mathbb{V}\mathbb{G}}$ if no proper subset of $\mathbb{D}_{\mathbb{V}\mathbb{G}}^{-1}$ is a Vds.
- (iii) Minimum order amidst all minimal $(Vds)^{-1}$ s of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is denominated lower inverse vague-domination number of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ and is demonstrated by $d_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}\mathbb{G}})$.
- (iv) Maximum order amidst all minimal $(Vds)^{-1}$ s of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is denominated upper inverse vague-domination number of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ and is demonstrated by $D_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}\mathbb{G}})$.
- (v) The inverse vague-domination number of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ is demonstrated by $\Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}\mathbb{G}})$ and explained as follows,

$$\Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}\mathbb{G}}) = \frac{d_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}\mathbb{G}}) + D_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}\mathbb{G}})}{2}.$$

Example 3.2: Investigate a vague graph $\mathbb{G}_{\mathbb{V}\mathbb{G}}$ as Figure 1. Clearly, $\mathbb{D}_{\mathbb{V}\mathbb{G}} = \{u_1, u_6\}$ is a minimal Vds in $\mathbb{G}_{\mathbb{V}\mathbb{G}}$. Also, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}\mathbb{G}} = \{u_2, u_3, u_4, u_5\}$ is an independent Vds of $\mathbb{G}_{\mathbb{V}\mathbb{G}}$. Hence, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}\mathbb{G}}$ is a minimal Vds of

$\mathbb{G}_{\mathbb{V}G}$. Therefore, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ of $\mathbb{G}_{\mathbb{V}G}$ related to $\mathbb{D}_{\mathbb{V}G}$. By routine calculations,

$$d_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}) = d_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}G}) = 0.7, \quad D_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}) = D_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}G}) = 1.5,$$

and so

$$\Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}) = \Delta_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}G}) = 1.1.$$

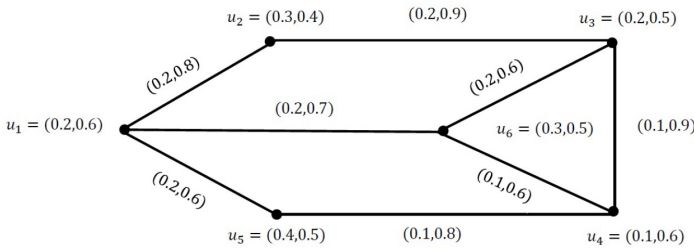


Figure 1
Vague graph $\mathbb{G}_{\mathbb{V}G}$.

Theorem 3.3: A minimal Vds $\mathbb{D}_{\mathbb{V}G}$ of $\mathbb{G}_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ if and only if $\mathbb{G}_{\mathbb{V}G}$ has not any isolated nodes.

Proof: Suppose that $\mathbb{D}_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ and $\mathbb{G}_{\mathbb{V}G}$ has an isolated node $q \in \mathbb{A}_{\mathbb{V}}$, with the contrary. Then q belongs to any Vds of $\mathbb{G}_{\mathbb{V}G}$. Thus, $q \in \mathbb{D}_{\mathbb{V}G}$, and so q does not belong to $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}G}$. Hence, $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}G}$ is not contain Vds of $\mathbb{G}_{\mathbb{V}G}$, which is a contradiction. Therefore, $\mathbb{G}_{\mathbb{V}G}$ has not every isolated node.

Conversely, presume that $\mathbb{D}_{\mathbb{V}G}$ is a minimal Vds of $\mathbb{G}_{\mathbb{V}G}$, and $\mathbb{G}_{\mathbb{V}G}$ has not every isolated node. Then every node in $\mathbb{D}_{\mathbb{V}G}$ is dominated by at least one node in $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}G}$, and $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}G}$ is a Vds and; hence $\mathbb{A}_{\mathbb{V}} \setminus \mathbb{D}_{\mathbb{V}G}$ contains a minimal Vds $\mathbb{D}_{\mathbb{V}G}^{-1}$ of $\mathbb{G}_{\mathbb{V}G}$. Therefore, $\mathbb{D}_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ of $\mathbb{G}_{\mathbb{V}G}$ related to $\mathbb{D}_{\mathbb{V}G}^{-1}$.

Corollary 3.5: Any minimal Vds $\mathbb{D}_{\mathbb{V}G}$ of a connected VG $\mathbb{G}_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$.

Theorem 3.5: If $\mathbb{G}_{\mathbb{V}G}$ has an isolated node, $q \in \mathbb{A}_{\mathbb{V}}$, then $\Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}) = 0$.

Proof: Suppose that $q \in \mathbb{A}_v$ wherever $|N(q)| = 0$. Then u belong to any Vds $\mathbb{D}_{vg}$ of $\mathbb{G}_{vg}$. Thus q does not belong to $\mathbb{A}_v \setminus \mathbb{D}_{vg}$. Hence, $\mathbb{A}_v \setminus \mathbb{D}_{vg}$ does not contain any Vds of $\mathbb{G}_{vg}$. Therefore, $\Delta_{\mathbb{A}_v}^{-1}(\mathbb{G}_{vg}) = 0$.

Note: For any VG $\mathbb{G}_{vg}$ having not an isolated node,

$$d_{\mathbb{A}_v}(\mathbb{G}_{vg}) \leq d_{\mathbb{A}_v}^{-1}(\mathbb{G}_{vg}).$$

Theorem 3.6: If $\mathbb{G}_{vg}$ is a VG, and $\mathbb{D}_{vg}$ is a minimal Vds of $\mathbb{G}_{vg}$, then

$$\mathbb{P}(\mathbb{D}_{vg}) + \mathbb{P}(\mathbb{D}_{vg}^{-1}) \leq |\mathbb{A}_v|.$$

Moreover, equality holds if $\mathbb{G}_{vg}$ has not every isolated node and $\mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a VIS.

Proof: Assume that $\mathbb{G}_{vg}$ be a VG, $\mathbb{D}_{vg}$ be a minimal Vds and $\mathbb{D}_{vg}^{-1}$ be a minimal $(Vds)^{-1}$ related to $\mathbb{D}_{vg}$ of $\mathbb{G}_{vg}$. Then $\mathbb{D}_{vg}^{-1} \subseteq \mathbb{A}_v \setminus \mathbb{D}_{vg}$. Thus $\mathbb{P}(\mathbb{D}_{vg}^{-1}) \leq \mathbb{P}(\mathbb{A}_v \setminus \mathbb{D}_{vg})$, and so $\mathbb{P}(\mathbb{D}_{vg}^{-1}) \leq |\mathbb{A}_v| - \mathbb{P}(\mathbb{D}_{vg})$. Hence,

$$\mathbb{P}(\mathbb{D}_{vg}^{-1}) + \mathbb{P}(\mathbb{D}_{vg}) \leq |\mathbb{A}_v|.$$

If $\mathbb{G}_{vg}$ has not every isolated node and $\mathbb{D}_{vg}$ is a minimal Vds, then $\mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a Vds and so $\mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a $(Vds)^{-1}$ related to $\mathbb{D}_{vg}$. Since $\mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a VIS of $\mathbb{G}_{vg}$, we obtain $\mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a minimal $(Vds)^{-1}$ of $\mathbb{G}_{vg}$ related to $\mathbb{D}_{vg}$.

Corollary 3.7: If $\mathbb{G}_{vg}$ has a $(Vds)^{-1}$, then

$$d_{\mathbb{A}_v}(\mathbb{G}_{vg}) + d_{\mathbb{A}_v}^{-1}(\mathbb{G}_{vg}) \leq |\mathbb{A}_v|.$$

Note: If $\mathbb{D}_{vg} \subset \mathbb{A}_v$ is a minimal Vds of $\mathbb{G}_{vg}$, then Vds $\mathbb{D}_{vg}^{-1} \subseteq \mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a minimal $(Vds)^{-1}$ of $\mathbb{G}_{vg}$ related to $\mathbb{D}_{vg}$ if and only if for every node $j \in \mathbb{D}_{vg}^{-1}$, one of the following conditions holds.

- (i) $IN(j) \subseteq \mathbb{A}_v \setminus \mathbb{D}_{vg}^{-1}$,
- (ii) there is a node $i \in \mathbb{A}_v \setminus \mathbb{D}_{vg}^{-1}$ wherever $IN(j) \cap \mathbb{D}_{vg}^{-1} = \{i\}$.

Theorem 3.8: If $d_{\mathbb{A}_v}(\mathbb{G}_{vg}) > \frac{|\mathbb{A}_v|}{2}$, then $\mathbb{G}_{vg}$ has no $(Vds)^{-1}$.

Proof: Suppose that $\mathbb{D}_{vg}$ is a minimal Vds of $\mathbb{G}_{vg}$ and $\mathbb{P}(\mathbb{D}_{vg}) = d_{\mathbb{A}_v}(\mathbb{G}_{vg}) \geq \frac{|\mathbb{A}_v|}{2}$. If $\mathbb{D}_{vg}^{-1} \subseteq \mathbb{A}_v \setminus \mathbb{D}_{vg}$ is a minimal $(Vds)^{-1}$ related to $\mathbb{D}_{vg}$ and $d_{\mathbb{A}_v}^{-1}(\mathbb{G}_{vg}) \leq \mathbb{P}(\mathbb{D}_{vg}^{-1})$, with the contrary. Then with Theorem 3.6,

$$\mathbb{P}(\mathbb{D}_{\mathbb{V}G}) + \mathbb{P}(\mathbb{D}_{\mathbb{V}G}^{-1}) \leq |\mathbb{A}_{\mathbb{V}}|,$$

and so,

$$\mathbb{P}(\mathbb{D}_{\mathbb{V}G}^{-1}) \leq |\mathbb{A}_{\mathbb{V}}| - \mathbb{P}(\mathbb{D}_{\mathbb{V}G}).$$

Thus,

$$\mathbb{P}(\mathbb{D}_{\mathbb{V}G}^{-1}) < \frac{|\mathbb{A}_{\mathbb{V}}|}{2},$$

hence,

$$d_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}) < d_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}G}),$$

which is a contradiction. Therefore, $\mathbb{G}_{\mathbb{V}G}$ has not any $(Vds)^{-1}$.

Corollary 3.9: *If $\mathbb{G}_{\mathbb{V}G}$ has an $(Vds)^{-1}$, then $d_{\mathbb{A}_{\mathbb{V}}}(\mathbb{G}_{\mathbb{V}G}) \leq \frac{|\mathbb{A}_{\mathbb{V}}|}{2}$.*

Notation: If in a graph $\mathbb{G}_{\mathbb{V}} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{V}})$ we add an arc e to $\mathbb{B}_{\mathbb{V}}$, then we demonstrate it with $\mathbb{B}_{\mathbb{V}e} = \mathbb{B}_{\mathbb{V}} \cup \{e\}$ and $\mathbb{G}_{\mathbb{V}e} = (\mathbb{A}_{\mathbb{V}}, \mathbb{B}_{\mathbb{V}e})$. Moreover, if a VG $\mathbb{G}_{\mathbb{V}G} = (\kappa, \tau)$ on $\mathbb{G}_{\mathbb{V}}$ extends on $\mathbb{G}_{\mathbb{V}e}$, then we denote it with $\mathbb{G}_{\mathbb{V}Ge} = (\kappa_e, \tau_e)$.

Notation: If arc e in VG $\mathbb{G}_{\mathbb{V}Ge}$ is a strong arc, then we indicate $\mathbb{G}_{\mathbb{V}Ge}^s = (\kappa_e^s, \tau_e^s)$ instead of $\mathbb{G}_{\mathbb{V}Ge} = (\kappa_e, \tau_e)$.

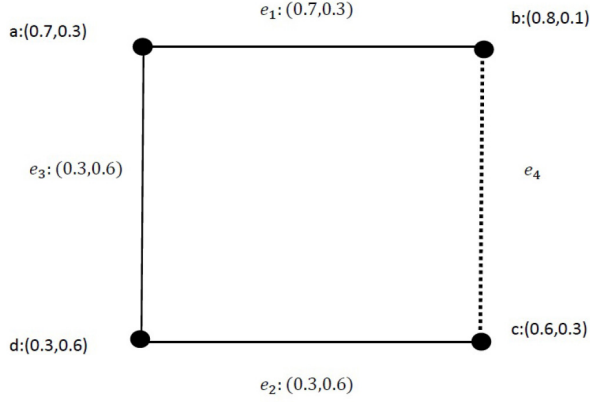
Theorem 3.10: *Put $\mathbb{G}_{\mathbb{V}G}$ has not every isolated node, and $a = zt$ be an additional strong arc in $\mathbb{G}_{\mathbb{V}Ga}^*$. Then,*

$$\Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}Ga}^s) \leq \Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}).$$

Proof: If $S_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ of $\mathbb{G}_{\mathbb{V}G}$ and $z, t \in S_{\mathbb{V}G}$ wherever $N(z) \subseteq \mathbb{A}_{\mathbb{V}} \setminus S_{\mathbb{V}G}$, then with adding $a = zt$, $S_{\mathbb{V}G} \setminus \{z\}$ is a minimal $(Vds)^{-1}$ in $\mathbb{G}_{\mathbb{V}Ga}^s$. Otherwise, $S_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ in $\mathbb{G}_{\mathbb{V}Ga}^s$. Therefore,

$$\Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}Ga}^s) \leq \Delta_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}).$$

Example 3.11: Investigate a VG $\mathbb{G}_{\mathbb{V}G}$ as Figure 2. Obviously, two sets $\mathbb{D}_{\mathbb{V}G1} = \{a, d\}$ and $\mathbb{D}_{\mathbb{V}G1}^{-1} = \{b, c\}$ are the minimal Vdss of $\mathbb{G}_{\mathbb{V}G}$ ($d_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}G}) = 1.05$). In this case, with adding $e_4 = (0.5, 0.5)$ to $\mathbb{G}_{\mathbb{V}G}$, $\mathbb{D}_{\mathbb{V}G2} = \{c, d\}$ and $\mathbb{D}_{\mathbb{V}G2}^{-1} = \{a, b\}$ are minimal Vdss of $\mathbb{G}_{\mathbb{V}Ge_4}$ ($d_{\mathbb{A}_{\mathbb{V}}}^{-1}(\mathbb{G}_{\mathbb{V}Ge_4}) = 1.0$).

**Figure 2****Vague graph $\mathbb{G}_{\mathbb{V}G}$.**

Theorem 3.12: For every vague graph $\mathbb{G}_{\mathbb{V}G}$, $d_{\mathbb{A}_V}^{-1}(\mathbb{G}_{\mathbb{V}G}) \leq i_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G})$.

Proof: We consider the following cases:

Case 1: If $\mathbb{G}_{\mathbb{V}G}$ has an isolated node, then the proof is obvious and $d_{\mathbb{A}_V}^{-1}(\mathbb{G}_{\mathbb{V}G}) \leq i_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G})$.

Case 2: If $\mathbb{G}_{\mathbb{V}G}$ has not every isolated node, and $S_{\mathbb{V}G}$ is a maximal VIS of $\mathbb{G}_{\mathbb{V}G}$, where $\mathbb{P}(S_{\mathbb{V}G}) = i_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G})$, $S_{\mathbb{V}G}$ is a minimal Vds. Also, $\mathbb{A}_V \setminus S_{\mathbb{V}G}$ is a Vds, and so $\mathbb{A}_V \setminus S_{\mathbb{V}G}$ contains a minimal Vds $\mathbb{D}_{\mathbb{V}G}$ of $\mathbb{G}_{\mathbb{V}G}$. Hence $S_{\mathbb{V}G}$ is a minimal $(Vds)^{-1}$ of $\mathbb{G}_{\mathbb{V}G}$ for $\mathbb{D}_{\mathbb{V}G}$. Therefore $d_{\mathbb{A}_V}^{-1}(\mathbb{G}_{\mathbb{V}G}) \leq i_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G})$.

Corollary 3.13: For every VG $\mathbb{G}_{\mathbb{V}G}$,

$$d_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G}) \leq d_{\mathbb{A}_V}^{-1}(\mathbb{G}_{\mathbb{V}G}) \leq i_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G}).$$

Theorem 3.14: For any VG $\mathbb{G}_{\mathbb{V}G}$ having not an isolated node,

$$d_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G}) \leq \frac{|\mathbb{A}_V| + d_{\mathbb{A}_V}^{-1}(\mathbb{G}_{\mathbb{V}G})}{3}.$$

Proof: Suppose that $\mathbb{G}_{\mathbb{V}G} = (\kappa, \tau)$ be a the VG. Then with Theorem 3.8, $d_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G}) \leq \frac{|\mathbb{A}_V|}{2}$, also $d_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G}) \leq d_{\mathbb{A}_V}^{-1}(\mathbb{G}_{\mathbb{V}G})$. Hence the result holds.

4. Inverse n-vague-dominating set of a vague graph

In this part, we discuss about the notion of inverse n-vague-dominating set of VG.

Definition 4.1: $(\mathbb{D}_{\text{vg}})_n \subset \mathbb{A}_{\text{v}}$ is denominated n-vague-dominating set $(Vds)^n$ of $\mathbb{G}_{\text{vg}}$ if for every node $i \in \mathbb{A}_{\text{v}} \setminus (\mathbb{D}_{\text{vg}})_n$ we get i dominate with at least n nodes in $(\mathbb{D}_{\text{vg}})_n$.

Definition 4.2:

- (i) A n-vague-dominating set $(\mathbb{D}_{\text{vg}})_n$ of $\mathbb{G}_{\text{vg}}$ is denominated a minimal $(Vds)^n$ if no proper subset of $(\mathbb{D}_{\text{vg}})_n$ is a $(Vds)^n$ of $\mathbb{G}_{\text{vg}}$.
- (ii) Minimum order amidst all minimal $(Vds)^n$ s of $\mathbb{G}_{\text{vg}}$ is denominated lower n-vague-domination number of $\mathbb{G}_{\text{vg}}$ and is demonstrated with $d_{\mathbb{A}_{\text{v}}}^n(G)$.
- (iii) Maximum order amidst all minimal $(Vds)^n$ s of $\mathbb{G}_{\text{vg}}$ is denominated upper n-vague-domination number of $\mathbb{G}_{\text{vg}}$ and is demonstrated with $D_{\mathbb{A}_{\text{v}}}^n(G)$.
- (iv) The n-vague-domination number of $\mathbb{G}_{\text{vg}}$ is demonstrated with $\Delta_{\mathbb{A}_{\text{v}}}^n(\mathbb{G}_{\text{vg}})$ and is explained with,

$$\Delta_{\mathbb{A}_{\text{v}}}^n(\mathbb{G}_{\text{vg}}) = \frac{d_{\mathbb{A}_{\text{v}}}^n(\mathbb{G}_{\text{vg}}) + D_{\mathbb{A}_{\text{v}}}^n(\mathbb{G}_{\text{vg}})}{2}.$$

Definition 4.3: Put $(\mathbb{D}_{\text{vg}})_n \subset \mathbb{A}_{\text{v}}$ be a minimal $(Vds)^n$ of $\mathbb{G}_{\text{vg}}$. Then:

- (i) $(\mathbb{D}_{\text{vg}})_n^{-1} \subseteq \mathbb{A}_{\text{v}} \setminus (\mathbb{D}_{\text{vg}})_n$ is denominated an inverse n-vague-dominating set $((Vds)^{-n})$ of $\mathbb{G}_{\text{vg}}$ to $(\mathbb{D}_{\text{vg}})_n$ if $(\mathbb{D}_{\text{vg}})_n^{-1}$ is a $(Vds)^n$ of $\mathbb{G}_{\text{vg}}$.
- (ii) An inverse n-vague-dominating set $(\mathbb{D}_{\text{vg}})_n^{-1} \subseteq \mathbb{A}_{\text{v}} \setminus (\mathbb{D}_{\text{vg}})_n$ is denominated minimal $(Vds)^{-n}$ of $\mathbb{G}_{\text{vg}}$ for $(\mathbb{D}_{\text{vg}})_n$ if no proper subset of $(\mathbb{D}_{\text{vg}})_n^{-1}$ is a $(Vds)^n$.
- (iii) Minimum order amidst all minimal $(Vds)^{-n}$ s of $\mathbb{G}_{\text{vg}}$ is denominated lower inverse n-vague-domination number of $\mathbb{G}_{\text{vg}}$ is demonstrated with, $(d_{\mathbb{A}_{\text{v}}}^n)^{-1}(\mathbb{G}_{\text{vg}})$.
- (iv) Maximum order amidst all minimal $(Vds)^{-n}$ s of $\mathbb{G}_{\text{vg}}$ is denominated upper inverse n-vague-domination number of $\mathbb{G}_{\text{vg}}$ is demonstrated with, $(D_{\mathbb{A}_{\text{v}}}^n)^{-1}(\mathbb{G}_{\text{vg}})$.
- (v) The inverse n-vague-domination number of $\mathbb{G}_{\text{vg}}$ is demonstrated by $(\Delta_{\mathbb{A}_{\text{v}}}^n)^{-1}(\mathbb{G}_{\text{vg}})$ and explained as follows,

$$(\Delta_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg}) = \frac{(d_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg}) + (D_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg})}{2}.$$

Theorem 4.4: If $\mathbb{G}_{vg}$ has a node, $j \in \mathbb{A}_v$ wherever $|\text{IN}(j)| \leq n-1$, then $(\Delta_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg}) = 0$.

Proof: Assume that $j \in \mathbb{A}_v$ wherever $|\text{IN}(j)| = n-1$. Then q belongs to any $(Vds)^n$ of $\mathbb{G}_{vg}$. Thus, $j \in (\mathbb{D}_{vg})_n$, and so j does not belong to $\mathbb{A}_v \setminus (\mathbb{D}_{vg})_n$. Hence, $\mathbb{A}_v \setminus (\mathbb{D}_{vg})_n$ does not contain any $(Vds)^n$ of $\mathbb{G}_{vg}$. Therefore, $(\Delta_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg}) = 0$.

Note:

(i) If $\mathbb{G}_{vg}$ has not any node $j \in \mathbb{A}_v$ wherever $|\text{IN}(j)| \leq n-1$, we have

$$d_{\mathbb{A}_v}^n(\mathbb{G}_{vg}) \leq (d_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg}).$$

(ii) If $\mathbb{G}_{vg}$ is a vague graph, and $(\mathbb{D}_{vg})_n$ is a minimal $(Vds)^n$ of $\mathbb{G}_{vg}$, then

$$\mathbb{P}((\mathbb{D}_{vg})_n) + \mathbb{P}((\mathbb{D}_{vg})_n^{-1}) \leq |\mathbb{A}_v|.$$

(iii) If $\mathbb{G}_{vg}$ has a $(Vds)^{-n}$, then

$$d_{\mathbb{A}_v}^n(\mathbb{G}_{vg}) + (d_{\mathbb{A}_v}^n)^{-1}(\mathbb{G}_{vg}) \leq |\mathbb{A}_v|.$$

(iv) If $d_{\mathbb{A}_v}^n(\mathbb{G}_{vg}) > \frac{|\mathbb{A}_v|}{2}$, then $\mathbb{G}_{vg}$ has no $(Vds)^{-n}$.

Theorem 4.5: Put $\mathbb{G}_{vg}$ be a VG and $(\mathbb{D}_{vg})_n$ is an independent $(Vds)^n$ of $\mathbb{G}_{vg}$. Then $(\mathbb{D}_{vg})_n$ is a minimal Vds of $\mathbb{G}_{vg}$.

Proof: If $(\mathbb{D}_{vg})_n$ is an independent $(Vds)^n$ of VG $\mathbb{G}_{vg}$, then $(\mathbb{D}_{vg})_n$ is a Vds of $\mathbb{G}_{vg}$, and $(\mathbb{D}_{vg})_n - \{q\}$ is not a Vds for any $q \in (\mathbb{D}_{vg})_n$. Hence $(\mathbb{D}_{vg})_n$ is a minimal Vds of $\mathbb{G}_{vg}$.

Theorem 4.6: A minimal n -vague-dominating set $(\mathbb{D}_{vg})_n$ of $\mathbb{G}_{vg}$ is a minimal $(Vds)^{-n}$ if $\mathbb{G}_{vg}$ has not any node $j \in \mathbb{A}_v$ wherever $|N(j)| \leq n-1$.

Proof: Suppose that $(\mathbb{D}_{vg})_n$ is a minimal inverse $(Vds)^{-n}$ and $\mathbb{G}_{vg}$ has a node $j \in \mathbb{A}_v$ wherever $|N(j)| \leq n-1$, with the contrary. Then j belongs to any $(Vds)^n$ of $\mathbb{G}_{vg}$. Thus, $j \in (\mathbb{D}_{vg})_n$, and so j does not belong to $\mathbb{A}_v \setminus (\mathbb{D}_{vg})_n$. Hence, $\mathbb{A}_v \setminus (\mathbb{D}_{vg})_n$ does not contain a $(Vds)^n$ of $\mathbb{G}_{vg}$, which is a contradiction. Therefore, $\mathbb{G}_{vg}$ has no node $j \in \mathbb{A}_v$ wherever $|N(j)| \leq n-1$.

Theorem 4.7: $\mathbb{G}_{\mathbb{V}G}$ has at least two distinct $(Vds)^n$'s if and only if $(\Delta_{\mathbb{A}_V}^n)^{-1}(\mathbb{G}_{\mathbb{V}G}) > 0$.

Proof: Suppose that $\mathbb{D}_{\mathbb{V}G1}$ and $\mathbb{D}_{\mathbb{V}G2}$ are two distinct $(Vds)^n$'s of $\mathbb{G}_{\mathbb{V}G}$. Then $\mathbb{D}_{\mathbb{V}G1}$ and $\mathbb{D}_{\mathbb{V}G2}$ contain minimal n -vague-dominating sets D'_1 and D'_2 , respectively. Hence, $D'_1 \subseteq \mathbb{A}_V \setminus D'_2$ is a minimal $(Vds)^{-n}$ of $\mathbb{G}_{\mathbb{V}G}$ for D'_2 and $D'_2 \subseteq \mathbb{A}_V \setminus D'_1$ is a minimal $(Vds)^{-n}$ of $\mathbb{G}_{\mathbb{V}G}$ for D'_1 . Therefore, $(\Delta_{\mathbb{A}_V}^n)^{-1}(\mathbb{G}_{\mathbb{V}G}) > 0$.

Conversely, assume that $(\Delta_{\mathbb{A}_V}^n)^{-1}(\mathbb{G}_{\mathbb{V}G}) > 0$, $(\mathbb{D}_{\mathbb{V}G})_n$ is a minimal $(Vds)^n$, and $(\mathbb{D}_{\mathbb{V}G})_n^{-1}$ is a minimal $(Vds)^{-n}$ for $(\mathbb{D}_{\mathbb{V}G})_n$ of $\mathbb{G}_{\mathbb{V}G}$. Then, clearly $(\mathbb{D}_{\mathbb{V}G})_n$ and $(\mathbb{D}_{\mathbb{V}G})_n^{-1}$ are distinct.

In the following example, with the help of the Petersen graph, we show that if $(\mathbb{D}_{\mathbb{V}G})_n$ is an independent $(Vds)^n$ of vague graph $\mathbb{G}_{\mathbb{V}G}$, then $(\mathbb{D}_{\mathbb{V}G})_n^{-1}$ is not necessarily independent.

Example 4.8: Investigate the vague graph $\mathbb{G}_{\mathbb{V}G}$ as Figure 3. Obviously, $\mathbb{D}_{\mathbb{V}G2} = \{u_1, u_4, u_7, u_8\}$ is an independent $(Vds)^2$ of $\mathbb{G}_{\mathbb{V}G}$ but $\mathbb{D}_{\mathbb{V}G2}^{-1} = \{u_2, u_3, u_5, u_6, u_9, u_{10}\}$ is not a VIS.

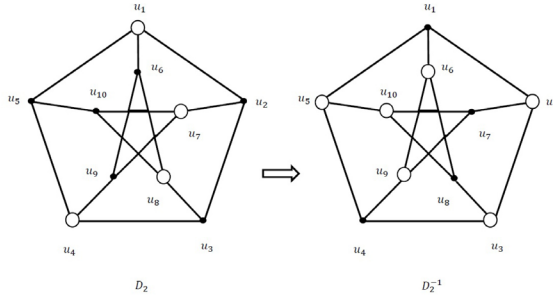


Figure 3
Vague graph $\mathbb{G}_{\mathbb{V}G}$.

Theorem 4.9: Put G^* be a Petersen graph and $\mathbb{G}_{\mathbb{V}G}$ be a strong VG on G^* . Then every minimal $(Vds)^2$ $\mathbb{D}_{\mathbb{V}G2}$ of $\mathbb{G}_{\mathbb{V}G}$ is a VIS of $\mathbb{G}_{\mathbb{V}G}$.

Theorem 4.10: Put G^* be a Petersen graph and $\mathbb{G}_{\mathbb{V}G}$ be a strong VG on G^* . Then

$$D_{\mathbb{A}_V}^2(\mathbb{G}_{\mathbb{V}G}) = (D_{\mathbb{A}_V}^2)^{-1}(\mathbb{G}_{\mathbb{V}G}) = D_{\mathbb{A}_V}(\mathbb{G}_{\mathbb{V}G}).$$

Proof: Assume that G^* is a Petersen graph, $\mathbb{G}_{\text{vg}}$ be a strong VG on G^* and $\mathbb{D}_{\text{vg}2}$ is a minimal $(Vds)^2$ of $\mathbb{G}_{\text{vg}}$. Then, with Theorem 4.9, $\mathbb{D}_{\text{vg}2}$ is a VIS of $\mathbb{G}_{\text{vg}}$, and with Theorem 4.5, $\mathbb{D}_{\text{vg}2}$ is a minimal Vds of $\mathbb{G}_{\text{vg}}$. Hence, we have

$$D_{\mathbb{A}_v}^2(\mathbb{G}_{\text{vg}}) \leq (D_{\mathbb{A}_v}^2)^{-1}(\mathbb{G}_{\text{vg}}) \leq D_{\mathbb{A}_v}(\mathbb{G}_{\text{vg}}) = D_{\mathbb{A}_v}^2(\mathbb{G}_{\text{vg}}),$$

Therefore,

$$D_{\mathbb{A}_v}^2(\mathbb{G}_{\text{vg}}) = (D_{\mathbb{A}_v}^2)^{-1}(\mathbb{G}_{\text{vg}}) = D_{\mathbb{A}_v}(\mathbb{G}_{\text{vg}}).$$

5. Application

Vague-dominating sets are studied and investigated as one of the widely used notions in VGs. Dominating sets play more of these roles as guiding and controlling sections. Based on this, it can be said that inverse dominating sets also play a significant role in guiding and controlling.

Location of relief centers

Investigate the VG in Figure 4. Assume that graph $\mathbb{G}_{\text{vg}}$ shows the map of a city. In these graphs, the vertices indicate that the areas and the arcs indicate its communication routes in this city. In this case, $\mathbb{T}_{\text{vg}} = \{A, E, F\}$ shows the fire stations, and $\mathbb{T}_{\text{vg}}^{-1} = \{B, C, D\}$ represents the emergency medical centers settled throughout the city. it is necessary to mention $\mathbb{T}_{\text{vg}}$ is the minimal Vds of $\mathbb{G}_{\text{vg}}$ with order of 0.8, and $\mathbb{T}_{\text{vg}}^{-1}$ is a $(Vds)^{-1}$ of $\mathbb{G}_{\text{vg}}$ for $\mathbb{T}_{\text{vg}}$ with order of 1.2. We describe the f -strength, and t -strength values in any node and arc (path) as follows:

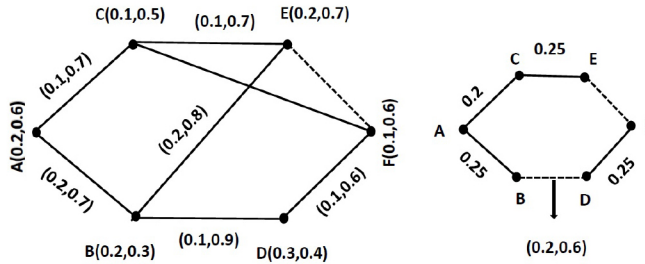


Figure 4
Vague graph $\mathbb{G}_{\text{vg}}$.

For any $z \in \mathbb{A}_v$ and $a \in \mathbb{B}_E$, we have

$t_\kappa(z)$: The minimum certitude of non-incident in mission scope of z .

$f_\kappa(z)$: The minimum certitude of incident in mission scope of z .

$t_\tau(e)$: The minimum certitude of well timed attendance at the incident scene through the a path.

$f_\tau(e)$: The minimum certitude of not being on time at the incident scene through the a path.

Thus, the order of any node is $|z| = \frac{1+t_\kappa(z)-f_\kappa(z)}{2}$, for any $z \in \mathbb{A}_v$, which indicated the optimal level of certitude of the non-incident in mission scope of that region. It is noteworthy that the optimal value of confidence in the mission scope of nodes A , E and F has the lowest value, which justifies the location of fire stations in these areas. Also, for any $e \in \mathbb{B}_E$, the degree of any arc is $|a| = \frac{1+t_\tau(a)-f_\tau(a)}{2}$ which indicated the optimal certitude of well timed attendance at the incident scene through that arc (path). It is noteworthy that the highest certitude of well timed attendance at the incident scene and also the lowest certitude of well timed attendance at the incident scene are related to these paths. Eventually, considering the t -strength and f -strength, the following relationships are also evident:

$$t_\tau(gh) \leq \min(t_\kappa(g), t_\kappa(h)) \text{ and } f_\tau(gh) \geq \max(f_\kappa(g), f_\kappa(h)).$$

Purpose: How can the number of fire stations and emergency medical centers in the city be reduced at the same time as increasing the optimal reliability of timely presence in any of the incident scenes?

Answer: If a new path with coordinates $EF = (0.1, 0.7)$ betwixt E and F areas is re-opened and a path with coordinates $BD = (0.2, 0.6)$ betwixt B and D areas is corrected (which are additional strong arcs), then the $\mathbb{T}_{vg}$ set of fire stations can be $\mathbb{T}_{vg}^* = \{A, F\}$ set of fire stations with node value of 0.55 and the $(\mathbb{T}_{vg}^{-1})^* = \{C, D\}$ set of medical centers with a node value of 0.75 while increasing the optimum level of certitude in the timely presence in the city with 0.5, simultaneously.

$\mathbb{T}_{vg} - \text{set}$	$\mathbb{P}(\mathbb{T}_{vg})$	Optimum level of certitude
$\mathbb{T}_{vg} = \{A, E, F\}$	0.8	0.95
$\mathbb{T}_{vg}^{-1} = \{B, C, D\}$	1.2	0.95

$\mathbb{T}_{\text{VG}} - \text{set}$	$\mathbb{P}(\mathbb{T}_{\text{VG}})$	Optimum level of certitude
$\mathbb{T}_{\text{VG}}^* = \{A, F\}$	0.55	1.45
$(\mathbb{T}_{\text{VG}}^{-1})^* = \{C, D\}$	0.75	1.45

6. Conclusion

Compared to other graph models, VG models are more compatible and flexible in modeling situations with imprecise information. In this research, considering this point and also the role of dominating sets in controlling and optimizing graph models, we investigated the vague-dominating sets and inverse vague-dominating sets in VGs and obtained useful results in this regard. Inverse vague-dominating sets can play an effective role in model control and optimization when there are restrictions on the vague-dominating set. With the aim of presenting a decision-making model in complex hybrid environments based on cognitive layers, we have focused our recent studies [3, 4, 5, 6, 7] on the investigation of dominating sets in Pythagorean fuzzy graphs and Neutrosophic graphs, which are undoubtedly interesting and useful for researchers in this field.

References

- [1] M. Akram, F. Feng, S. Sawar, and Y. B. Jun, "Certain types of vague graph," Univ. Politehnica of Bucharest Sci. Bull., Ser. A: Appl. Math. Phys., vol. 76, no. 1, pp. 141-154 (2014).
- [2] M. Akram, K. P. Wenjuan Chen, and Shum, "Some properties of vague graphs," *Southeast Asian Bull. Math.*, vol. 37, pp. 307-324 (2013).
- [3] S. Banitalebi, "Irregular vague graphs," *J. Algebraic Hyperstructures Log. Algebras*, vol. 2, no. 2, pp. 73-90 (2021).
- [4] S. Banitalebi and R. A. Borzooei, "Domination of vague graphs by using strong arcs," *J. Math. Extens.*, vol. 16, no. 3, pp. 1-22 (2022).
- [5] S. Banitalebi and R. A. Borzooei, "2-Domination in vague graphs," *J. Algebraic Struct. Their Appl.*, vol. 8, no. 2, pp. 203-222 (2021).
- [6] S. Banitalebi and R. A. Borzooei, "Neutrosophic special dominating set in neutrosophic graphs," *J. Neutrosophic Sets Syst.*, vol. 45, no. 1, pp. 26-39 (2021).
- [7] S. Banitalebi and R. A. Borzooei, "Domination in Pythagorean fuzzy graphs," *J. Granular Comput.*, vol. 8, pp. 959-966 (2023).

- [8] R. A. Borzooei and H. Rashmanlou, "Domination in vague graph and its application," *J. Intell. Fuzzy Syst.*, vol. 29, pp. 1933-1940 (2015).
- [9] R. A. Borzooei and H. Rashmanlou, "Degree of vertices in vague graphs," *J. Appl. Math. Informatics*, vol. 33, pp. 545-557 (2015).
- [10] E. J. Cockayne and S. Hedetniem, "Towards a theory of domination in graphs," *Networks*, vol. 7, pp. 247-261 (1977).
- [11] W. L. Gau and D. J. Buehrer, "Vague sets," *IEEE Trans. Syst. Man Cybern.*, vol. 23, no. 2, pp. 610-614 (1993).
- [12] V. R. Kulli and B. Janakiram, "The cobondage number of a graph," *Discuss. Math. Graph Theory*, vol. 16, no. 2, pp. 111-117 (1996).
- [13] A. Nagoor Gani and V. T. Chandrasekaran, "Domination in fuzzy graph," *Adv. Fuzzy Sets Syst.*, vol. 1, no. 1, pp. 17-26 (2006).
- [14] A. Nagoor Gani and K. Prasanna Devi, "2-domination in fuzzy graphs," *Int. J. Fuzzy Math.*, vol. 1, pp. 119-124 (2015).
- [15] A. Nagoor Gani and K. Prasanna Devi, "Reduction of domination parameter in fuzzy graphs," *Global J. Pure Appl. Math.*, vol. 7, pp. 3307-3315 (2017).
- [16] S. Mirvakili and H. Naraghi, "Connections between reversible regular hypergroups, t-fuzzy subgroups and t-fuzzy graphs," *J. Algebraic Hyperstructures Log. Algebras*, vol. 1, no. 4, pp. 71-82 (2020).
- [17] R. Parvathi and G. Thamizhendhi, "Domination in intuitionistic fuzzy graphs," *Proc. Fourteenth Int. Conf. Intuitionistic Fuzzy Sets, Sofia*, vol. 16, no. 2, pp. 39-49 (2010).
- [18] N. Ramakrishna, "Vague graphs," *Int. J. Comput. Cogn.*, vol. 7, pp. 51-58 (2009).
- [19] H. Rashmanlou and R. A. Borzooei, "Product vague graphs and its applications," *J. Intell. Fuzzy Syst.*, vol. 30, no. 1, pp. 371-382 (2016).
- [20] H. Rashmanlou, S. Samanta, M. Pal, and R. A. Borzooei, "Product of bipolar fuzzy graphs and their degree," *Int. J. Gen. Syst.*, vol. 45, no. 1, pp. 1-14 (2016).
- [21] A. Rosenfeld, "Fuzzy graphs," *Fuzzy Sets and Their Applications*, Academic Press, New York, pp. 77-95 (1975).
- [22] A. Somasundaram and S. Somasundaram, "Domination in fuzzy graphs-I," *Pattern Recognit. Lett.*, vol. 19, pp. 787-791 (1998).
- [23] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, pp. 338-353 (1965).