Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 1, pp. 87–100
DOI : 10.47974/JDMSC-1857

Fuzzy extended filters of MS-algebras

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Abstract

In this article, extended fuzzy filter of s fuzzy filter χ is presented, notated by $\Upsilon_{\chi,W}$ with $W \subseteq \mathcal{L}$. The features of $\Upsilon_{\chi,W}$ are investigated. Furthermore, the strong fuzzy filter is introduced, donated by $\Omega_{\chi,W}$. Many properties are studied. Characterisation of both $\Upsilon_{\chi,W}, \Omega_{\chi,W}$ are clarified by using the notion of dense elements with respect to a fuzzy filter. The homomorphisms of both $\Upsilon_{\chi,W}$ and $\Omega_{\chi,W}$ are triggered.

Subject Classification: *(2010) 94D05, 06D99.*

Keywords: *Bounded distributive lattice, MS-algebras, Fuzzy lattice, Fuzzy filter, Homomorphism.*

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1. Introduction

In 1965, the genesis of the fuzzy mathematics was made by Lotfi Asker Zadeh'publications [18] and [19]. Lotfi Asker Zadeh introduced several fuzzy concepts as fuzzy sets, fuzzy algorithms, etc. Many algebraic structures were fuzzified. In 1971, fuzzy subgroups and fuzzy subgroupoids were settled down by A.Rosenfeld [14]. The evolution of fuzzy mathematics took a great part in lattice theory see [2], [3] and [13]. Many algebraic structures were fuzzified, for instance see [1] and [4].

The variety $\mathbf{MS}$ that contains every *MS*-algebra was considered first by Varlet and Blyth to provide a unified approach to handle Stone and de Morgan algebras, see [6], [7], [8] and [9]. The theory of substructures of $\mathcal{L} \in \mathbf{MS}$ has been considered by many researchers as [5], [15], [16] and [17].

In this paper, for $\mathcal{L} \in \mathbf{MS}$ and a fuzzy filter χ together with $W \subseteq \mathcal{L}$, the fuzzification of the extended filter $E_\chi(W)$ is presented and notated by $\Upsilon_{\chi,W}$. We verify that $\Upsilon_{\chi,W}$ is an prime fuzzy extended filter for χ and properties of $\Upsilon_{\chi,W}$ are studied in a great details. The class of all fuzzy extended filters is proved to be closed under the meet operation. The notion of a strong fuzzy extended filter and notated by $\Omega_{\chi,W}$. The values of both $\Upsilon_{\chi,W}$, $\Omega_{\chi,W}$ are studied in terms of dense elements. The features of $\Upsilon_{\chi,W}$ homomorphism are characterised. We study the elements of the kernel and the cokernel.

2. Preliminaries

To make the paper consistent, we mention some results and definitions that will be used throughout the paper. The set of filters of a lattice $\mathcal{L}$ is notated by $\mathcal{F}(\mathcal{L})$.

A sublattice $\chi \subseteq \mathcal{L}$ is a filter if $\theta \in \chi$, $\varepsilon \in \mathcal{L}$ imply that $\theta \vee \varepsilon \in \chi$. A prime filter $\mathfrak{P}$ is a proper filter such that $\theta \vee \varepsilon \in \mathfrak{P}$f imply that $\theta \in \mathfrak{P}$ or $\varepsilon \in \mathfrak{P}$ for all $\theta, \varepsilon \in \mathcal{L}$. Let $\varepsilon \in \mathcal{L}$. The principal filter $[\varepsilon)$ of $\mathcal{L}$ created by ε is defined as $[\varepsilon) = \{\theta \in \mathcal{L} : \theta \geq \varepsilon\}$. For every $W \subseteq \mathcal{L}$, the filter $[\mathfrak{B})$ generated by the set $\mathfrak{B}$ is defined to be

$$[\mathfrak{B}) = \{\theta \in \mathcal{L} : \theta \geq b_1 \wedge b_2 \wedge \ldots \wedge b_n \text{ for } b_1, b_2, \ldots, b_n \in \mathfrak{B}\}.$$

If $\theta \mapsto \theta^\circ$ is a unary operation on a distributive lattice $(\mathcal{L}; \vee, \wedge, 0, 1)$ satisfying,

$$1^\circ = 0,\ (\theta \wedge \xi)^\circ = \theta^\circ \vee \xi^\circ \text{ and } \theta \leq \theta^{\circ\circ}.$$

Then $\mathcal{L}$ is called an *MS*-algebra, see [8]. The elements in an *MS*-algebra satisfy the following equalities. The notation $\mathcal{L}$ is mentioned through the paper as an *MS*-algebra. We refer to [12] for details of lattice theory.

Proposition 2.1: [10] Let $\theta, \xi \in \mathcal{L}$. The following equalities are valid;

(1) $(\theta \vee \xi)^{\circ} = \theta^{\circ} \wedge \xi^{\circ}$,

(2) $(\theta \vee \xi)^{\circ\circ} = \theta^{\circ\circ} \vee \xi^{\circ\circ}$,

(3) $\theta^{\circ} = \theta^{\circ\circ\circ}$,

9 (4) $0^{\circ} = 1$.

Definition 2.2: [11] Let $\chi \in \mathcal{F}(\mathcal{L})$ and $\mathcal{L} \in \mathbf{MS}$. If W is a nonempty subset of $\mathcal{L}$, define

$$E_{\chi}(W) = \{\theta \in \mathcal{L}\,; \theta \vee w^{\circ\circ} \in \chi \text{ for every } w \in W\}.$$

Recall some basic theorems that characterise the concept of $E_{\chi}(W)$.

Theorem 2.3: [11] *Let* $\chi \in \mathcal{F}(\mathcal{L})$, *then* $E_{\chi}(W) \in \mathcal{F}(\mathcal{L})$ *and* $\chi \in E_{\chi}(W)$.

We call $E_{\chi}(W)$ the extended filter of χ. Let $\mathcal{L}$ be any set. Assume that $[0,1]$ is the unit interval. A fuzzy subset of $\mathcal{L}$ is a map $\eth : \mathcal{L} \to ([0,1], \wedge, \vee)$. Let $\theta, \xi \in [0,1]$. Define the operations as follows

$$\xi \wedge \theta = inf\{\xi, \theta\},$$

$$\xi \vee \theta = Sup\{\xi, \theta\}.$$

The notation $\mathrm{Fuzz}(\mathcal{K})$ is used for the collection of all fuzzy subsets of a bounded distributed lattice $\mathcal{K}$.

Definition 2.4: [2]. Let $\Phi, \Psi \in \mathrm{Fuzz}(\mathcal{K})$. Define fuzzy subsets $\Phi \cup \Psi$ and $\Phi \cap \Psi$ of $\mathcal{K}$ as follows: let $\pi \in \mathcal{K}$,

$$(\Phi \cup \Psi)(\pi) = \Phi(\pi) \vee \Psi(\pi),$$

$$(\Phi \cup \Psi)(\pi) = \Phi(\pi) \wedge \Psi(\pi).$$

Definition 2.5: [3] If $\eth \in \mathrm{Fuzz}(\mathcal{K})$. Then $\eth$ is characterised as a fuzzy sublattice of $\mathcal{K}$ provided that for every $\pi, \xi \in \mathcal{K}$ the following satisfied;

(i) $\eth(\xi \wedge \pi) \geq \eth(\xi) \wedge \eth(\pi)$,

(ii) $\eth(\xi \vee \pi) \geq \eth(\xi) \wedge \eth(\pi)$.

Definition 2.6: [17] *If* $\eth \in \mathrm{Fuzz}(\mathcal{K})$, *then* $\eth$ *is a fuzzy ideal if the following are valid for every* $\pi, \theta \in \mathcal{K}$;

(i) $\eth(0) = 1$,

(ii) $\eth(\pi \wedge \theta) \geq \eth(\pi) \vee \eth(\theta)$,

(iii) $\eth(\pi \vee \theta) \geq \eth(\pi) \wedge \eth(\theta)$.

We notate the following;

$$\mathrm{FID}(\mathcal{K}) = \{\eth \in \mathrm{Fuzz}(\mathcal{K}), \eth \text{ is a fuzzy ideal}\}.$$

Theorem 2.7: [17] *If* $\eth \in Fuzz(\mathcal{K})$, *then* $\eth \in FID(\mathcal{K})$ *if and only if* $\eth(0) = 1$ *and for all* $\pi, \theta \in \mathcal{K}$, $\eth(\theta \vee \pi) = \eth(\theta) \wedge \eth(\pi)$.

Definition 2.8: [17] If $\eth \in Fuzz(\mathcal{K})$. Then $\eth$ is a fuzzy filter provided that For every $\theta, \pi \in \mathcal{K}$;

(i) $\eth(1) = 1$,

(ii) $\eth(\pi \wedge \theta) \geq \eth(\pi) \wedge \eth(\theta)$,

(iii) $\eth(\pi \vee \theta) \geq \eth(\pi) \vee \eth(\theta)$.

Consider the following set;

$$\mathrm{FFL}(\mathcal{K}) := \{\eth \in \mathrm{Fuzz}(\mathcal{K}), \eth \text{ is a fuzzy filter}\}.$$

Theorem 2.9: [17] *Let* $\eth \in FFL(\mathcal{K})$, *then* $\eth \in FFL(\mathcal{K})$ *if and only if* $\eth(1) = 1$ *and for every* $\theta, \pi \in \mathcal{K}$, $\eth(\theta \wedge \pi) = \eth(\theta) \wedge \eth(\pi)$.

A non constant $\eth \in \mathrm{FFL}(\mathcal{K})$ is called proper. A proper $\chi \in \mathrm{FID}(\mathcal{K})$ is promoted to be a prime fuzzy ideal if for every $\Phi, \Psi \in FID(\mathcal{K})$ satisfying that $\Psi \cap \Phi \subseteq \chi$, either $\Psi \subseteq \chi$ or $\Phi \subseteq \chi$. A proper $\eth \in \mathrm{FFL}(\mathcal{K})$ is said to be a prime fuzzy filter if for every $\Psi, \Phi \in \mathrm{FFL}(\mathcal{K})$ satisfying that $\Psi \cap \Phi \subseteq \eth$, we have $\Psi \subseteq \chi$ or $\Phi \subseteq \eth$.

3. Fuzzy Extended filters

Let $W \subseteq \mathcal{L}$ be nonempty and $\chi \in \mathrm{FFL}(\mathcal{L})$. Define

$$\Upsilon_{\chi,W}(\pi) = Sup\{\chi(\pi) \vee \chi(w^{\circ\circ}); \text{for every } w \in W\},$$

for every $\pi \in \mathcal{L}$.

Theorem 3.1: *Let $W \subseteq \mathcal{L}$ and $\Gamma \in FFL(\mathcal{L})$. Then $\Upsilon_{\Gamma,W}$ is a prime fuzzy filter containing Γ.*

Proof: Clearly, $\Upsilon_{\Gamma,W}(1)=1$ as $\Gamma(1)=1$. We have that

$$\begin{aligned}\Upsilon_{\Gamma,W}(\xi\vee\pi) &= Sup\{\Gamma(\xi\vee\pi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &\geq Sup\{(\Gamma(\xi)\vee\Gamma(\pi))\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &= Sup\{\Gamma(\xi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &\vee\ Sup\{\Gamma(\pi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &= \Upsilon_{\Gamma,W}(\xi)\vee\Upsilon_{\Gamma,W}(\pi).\end{aligned}$$

And

$$\begin{aligned}\Upsilon_{\Gamma,W}(\xi\wedge\pi) &= Sup\{\Gamma(\xi\wedge\pi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &\geq Sup\{(\Gamma(\xi)\wedge\Gamma(\pi))\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &= Sup\{\Gamma(\xi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &\wedge Sup\{\Gamma(\pi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\ &= \Upsilon_{\Gamma,W}(\xi)\wedge\Upsilon_{\Gamma,W}(\pi).\end{aligned}$$

Obviously, $\Gamma\subseteq\Upsilon_{\Gamma,W}$. Moreover, if $\Phi\cap\Psi\subset\Upsilon_{\Gamma,W}$. Then $(\Phi\cap\Psi)\ (\pi)\leq\Upsilon_{\Gamma,W}(\pi)$. Therefore $\Phi(\pi)\wedge\Psi(\pi)\leq\Upsilon_{\Gamma,W}(\pi)$. Hence, $\Phi(\pi)\leq\Upsilon_{\Gamma,W}(\pi)$ or $\Psi(\pi)\leq\Upsilon_{\Gamma,W}(\pi)$.

We call $\Upsilon_{\chi,W}$ a fuzzy extended filter of χ. The next result encapsulates essential characterisations of $\Upsilon_{\chi,W}$.

Lemma 3.2: *Let χ be a fuzzy filter. For a subset W and a member θ of $\mathcal{L}$, we have*

1. *If $Z\subseteq W$, then $\Upsilon_{\chi,Z}\subseteq\Upsilon_{\chi,W}$,*
2. *If $\chi_1\subseteq\chi_2$, then $\Upsilon_{\chi_1,W}\subseteq\Upsilon_{\chi_2,W}$,*
3. The equality $\Upsilon_{\chi,W}(\theta)=\chi(\theta)$ is valid providing that $\varpi^{\circ\circ}\leq\theta$ for every $\varpi\in W$,
4. *If χ is a one to one function, then $\Upsilon_{\chi,W}(\theta)=\chi(\theta)$ implies that $\varpi^{\circ\circ}\leq\theta$ for every $\varpi\in W$,*
5. *If $\varpi^{\circ\circ}=1$ for some $\varpi\in W$, then $\Upsilon_{\chi,W}(\theta)=1$,*
6. *$\Upsilon_{\chi,\mathcal{L}}(\theta)=1=\Upsilon_{\chi,\{1\}}(\theta)$.*
7. *The equation $\Upsilon_{\chi,W}(\theta)=1$ implies that $\chi(\theta)=1$ or $\chi(\varpi^{\circ\circ})=1$ for some $\varpi\in W$.*

Proof:

(1) If $Z \subseteq W$, then

$Sup\{\chi(\theta) \vee \chi(a^{\circ\circ})$; for every $a \in Z\} \leq Sup\{\chi(\theta) \vee \chi(\varpi^{\circ\circ})$; for every $\varpi \in W\}$.

Hence $\Upsilon_{\chi,Z} \subseteq \Upsilon_{\chi,W}$.

(2) Suppose that $\chi_1 \subseteq \chi_2$. We see that

$Sup\{\chi_1(\theta) \vee \chi_1(\varpi^{\circ\circ})$; for every $\varpi \in W\} \leq Sup\{\chi_2(\theta) \vee \chi_2(\varpi^{\circ\circ})$; for every $\varpi \in W\}$.

Hence $\Upsilon_{\chi_1,W} \subseteq \Upsilon_{\chi_2,W}$.

(3) Let $\theta \leq \varpi^{\circ\circ}$ for every $\varpi \in W$, then $\chi(\varpi^{\circ\circ}) \leq \chi(\theta)$. Since $\chi(\theta) = \chi(\theta) \vee \chi(\varpi^{\circ\circ})$, then

$$\begin{aligned}\Upsilon_{\chi,W}(\theta) &= Sup\{\chi(\theta) \vee \chi(\varpi^{\circ\circ}); \text{for every } \varpi \in W\} \\ &= Sup\{\chi(\theta)\} \\ &= \chi(\theta).\end{aligned}$$

(4) Suppose that $\chi(\theta) = \Upsilon_{\chi,W}(\theta)$. Then $\chi(\theta) \geq \chi(\varpi^{\circ\circ})$ for every $\varpi \in W$. By the definition of fuzzy filters, for every $\varpi \in W$ we see that $\chi(\varpi^{\circ\circ}) = \chi(\theta \wedge \varpi^{\circ\circ})$. For every $\varpi \in W$ and by the one to one property, $\varpi^{\circ\circ} = \theta \wedge \varpi^{\circ\circ}$. Hence we get the required result.

(5) If $\varpi^{\circ\circ} = 1$ for some $\varpi \in W$, then $\chi(\theta) \vee \chi(\varpi^{\circ\circ}) = 1$. Hence $\Upsilon_{\chi,W}(\theta) = 1$.

(6) Follows directly from (5).

(7) Let $\Upsilon_{\chi,W}(\theta) = 1$, then $\chi(\theta) \vee \chi(\varpi^{\circ\circ}) = 1$ for some $\varpi \in W$. As $\chi(\theta)$, $\chi(\varpi^{\circ\circ}) \in [0,1]$, hence either $\chi(\theta) = 1$ or $\chi(\varpi^{\circ\circ}) = 1$ for some $\varpi \in W$.

Proposition 3.3: Let $\mathcal{L} \in \mathbf{MS}$ and $\pi, \xi \in \mathcal{L}$. For nonempty subsets W of $\mathcal{L}$ and $\Gamma \in FFL(\mathcal{L})$, we have that

(1) $\Upsilon_{\Gamma_1,W}(\pi) \vee \Upsilon_{\Gamma_2,W}(\pi) = \Upsilon_{\Gamma_1 \cup \Gamma_2,W}(\pi)$,

(2) $\Upsilon_{\Gamma,W}(\xi) \wedge \Upsilon_{\Gamma,W}(\pi) = \Upsilon_{\Gamma,W}(\xi \wedge \pi)$.

Proof:

(1) We see that

$$\begin{aligned}\Upsilon_{\Gamma_1,W}(\pi)\vee\Upsilon_{\Gamma_2,W}(\pi) &= Sup\{\Gamma_1(\pi)\vee\Gamma_1(\varpi^{\circ\circ});\text{ for every }\varpi\in W\}\\ &\quad\vee Sup\{\Gamma_2(\pi)\vee\Gamma_2(\varpi^{\circ\circ});\text{ for every }\varpi\in W\}\\ &= Sup\{(\Gamma_1(\pi)\vee\Gamma_1(\varpi^{\circ\circ}))\vee(\Gamma_2(\pi)\vee\Gamma_2(\varpi^{\circ\circ}));\\ &\qquad\text{for every }\varpi\in W\}\\ &= Sup\{(\Gamma_1(\pi)\vee\Gamma_2(\pi))\vee(\Gamma_1(\varpi^{\circ\circ})\vee\Gamma_2(\varpi^{\circ\circ}));\\ &\qquad\text{for every }\varpi\in W\}\\ &= Sup\{((\Gamma_1\cup\Gamma_2)(\pi))\vee((\Gamma_1\cup\Gamma_2)(\varpi^{\circ\circ}));\\ &\qquad\text{for every }\varpi\in W\}\\ &= \Upsilon_{\Gamma_1\cup\Gamma_2,W}(\pi).\end{aligned}$$

(2) By Theorem (3.1), we get the required.

Definition 3.4: Let $\chi\in FFL(\mathcal{L})$, we say that χ is a fixed fuzzy filter relative to a non empty set W if for every θ the equality $\Upsilon_{\chi,W}(\theta)=\chi(\theta)$ is satisfied.

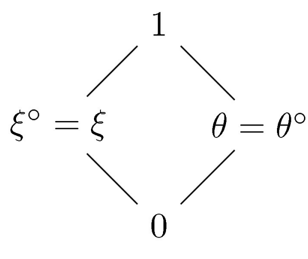
Example 3.5:

(1) In the virtue of Lemma 3.2(3), every $\chi\in \mathrm{FFL}(\mathcal{L})$ is a fixed fuzzy filter relative to $W=\{0\}$. That is $\Upsilon_{\chi,\{0\}}(\theta)=\chi(\theta)$.

(2) Let $A=\{\rho\in\mathcal{L};\rho^{\circ\circ}=0\}$, then $\Upsilon_{\chi,A}(\theta)=\chi(\theta)$. Since $\rho^{\circ\circ}=0\le\theta$ for every $\theta\in\mathcal{L}$. Thus $\chi(\rho^{\circ\circ})\le\chi(\theta)$.

(3) Let $C=\{\rho\in\mathcal{L};\chi(\rho^{\circ\circ})=0\}$, by the definition of $\Upsilon_{\chi,C}$, we get for every $\theta\in\mathcal{L}$ that $\Upsilon_{\chi,C}(\theta)=\chi(\theta)$.

(4) Consider the following Hasse diagram



$\mathcal{L}$

Obviously, $(\mathcal{L},^\circ)\in\mathbf{MS}$. Define χ as $\chi(\theta)=\chi(1)=1$ and $\chi(0)=\chi(\xi)=0.5$. Then $\chi\in\mathrm{FFL}(\mathcal{L})$. Let $W=\{0,\xi\}$. Obviously, χ is a fixed fuzzy filter relative to the set W.

Proposition 3.6: Let $W \subseteq \mathcal{L}$. If a fuzzy filter χ is fixed relative to W, then it is fixed relative to every subset of W.

Proof: Let $Z \subseteq W$ and $\pi \in \mathcal{L}$. Then $\Upsilon_{\chi,Z}(\pi) \leq \Upsilon_{\chi,W}(\pi) = \chi(\pi)$. Therefore $\Upsilon_{\chi,Z}(\pi) = \chi(\pi)$. Hence χ is a fixed fuzzy filter relative to Z.

Proposition 3.7: If χ_1, χ_2 are fixed fuzzy filters relative to a subset $W \subseteq \mathcal{L}$, then $\chi_1 \cup \chi_2$ is a fixed fuzzy filters relative to W.

Proof: This follows easily from Proposition (3.3).

Theorem 3.8: *The image of an element* $\pi, \varpi \in \mathcal{L}$ *under the fuzzy filter* $\Upsilon_{\Gamma,\{\varpi\}}$ *is* $\Gamma(\pi)$ *or* $\Gamma(\varpi^{\circ\circ})$ *for* $\varpi \in \mathcal{L}$.

Proof: One can see that

$$\begin{aligned}\Upsilon_{\Gamma,\{\varpi\}}(\pi) &= Sup\{\Gamma(\pi) \vee \Gamma(\varpi^{\circ\circ}); \text{for every } \varpi \in \{\varpi\}\} \\ &= Sup\{(\Gamma(\pi) \vee \Gamma(\varpi^{\circ\circ})\} \\ &= \Gamma(\pi) \vee \Gamma(\varpi^{\circ\circ}).\end{aligned}$$

This gives immediately the following results.

Corollary 3.9: *For* $\varpi, \theta \in \mathcal{L}$, *if* $\Upsilon_{\chi,\{\varpi\}}(\theta) \neq \chi(\varpi^{\circ\circ})$, *then* χ *is a fixed fuzzy filter with respect to* $\{\varpi\}$.

Corollary 3.10: *If* $\chi \in FFL(\mathcal{L})$ *and* $\varpi \in \mathcal{L}$, *then* $\Upsilon_{\chi,\{\varpi\}}(\varpi) = \chi(\varpi^{\circ\circ})$.

4. Strong Fuzzy Extensions

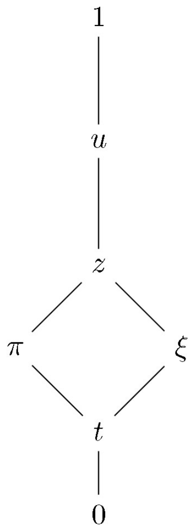
In this section, a more generalized case of fuzzy extended filter called strong fuzzy extension is studied.

Definition 4.1: *Let* $\chi \in \mathrm{FFL}(\mathcal{L})$. *Define the following set*

$$\Omega_{\chi,W}(\theta) = Sup\{\chi(\theta \vee \varpi^{\circ\circ}); \text{for every } \varpi \in W\}.$$

We see that $\Omega_{\chi,W}$ is an extension of $\Upsilon_{\chi,W}(\theta)$. By using the properties of fuzzy filters, we see that for every $\xi, \theta \in \mathcal{L}$, $\chi(\xi \vee \theta) \geq \chi(\xi) \vee \chi(\theta)$. Therefore, $\Omega_{\chi,W}(\theta) \geq \Upsilon_{\chi,W}(\theta)$. Hence $\Upsilon_{\chi,W} \subseteq \Omega_{\chi,W}$. Also, $\Upsilon_{\chi,W} \subseteq \Omega_{\chi,W}$ properly as verified below.

Example 4.2: *On the following lattice* $\mathcal{L}$



$\mathcal{L}$

, define $^\circ$ by $\pi^\circ = t$, $\xi^\circ = t^\circ = z^\circ = u$, $u^\circ = \xi$, $0^\circ = 1$, $1^\circ = 0$. Hence $(\mathcal{L}, ^\circ) \in \mathbf{MS}$. Define the following fuzzy set $\Gamma : \mathcal{L} \to [0,1]$ as the following; $\Gamma(0) = 0$, $\Gamma(u) = 0.8$, $\Gamma(t) = \Gamma(\xi) = \Gamma(\pi) = 0.6$, $\Gamma(z) = 0.7$ and $\Gamma(1) = 0.7$. Obviously, $\Gamma \in FFL(\mathcal{L})$. Then

$$\begin{aligned}\Upsilon_{\Gamma,\{\xi\}}(\pi) &= Sup\{\Gamma(\pi) \vee \Gamma(\xi^{\circ\circ})\} \\ &= \Gamma(\pi) \vee \Gamma(\xi^{\circ\circ}) \\ &= \Gamma(\pi) \vee \Gamma(\xi) = 0.6.\end{aligned}$$

And

$$\begin{aligned}\Omega_{\Gamma,\{\xi\}}(\pi) &= Sup\{\Gamma(\pi \vee \xi^{\circ\circ})\} \\ &= \Gamma(\pi \vee \xi^{\circ\circ}) \\ &= \Gamma(\pi \vee \xi) = 0.7.\end{aligned}$$

We call $\Omega_{\Gamma,W}$ a strong fuzzy extension of Γ.

Theorem 4.3: *Let* $\Gamma \in FFL(\mathcal{L})$. *If* $W \subseteq \mathcal{L}$, *then* $\Omega_{\Gamma,W} \in FFL(\mathcal{L})$.

Proof: Obviously, $\Omega_{\Gamma,W}(1) = 1$. We have that

$$\begin{aligned}\Omega_{\Gamma,W}(\pi \vee \theta) &= Sup\{\Gamma((\pi \vee \theta) \vee \varpi^{\circ\circ})\text{; for every } \varpi \in W\} \\ &= Sup\{\Gamma((\pi \vee \varpi^{\circ\circ}) \vee (\theta \vee \varpi^{\circ\circ}))\text{; for every } \varpi \in W\} \\ &\geq Sup\{\Gamma(\pi \vee \varpi^{\circ\circ}) \vee \Gamma(\theta \vee \varpi^{\circ\circ})\text{; for every } \varpi \in W\}\end{aligned}$$

$$= Sup\{\Gamma(\pi \vee \varpi^{\circ\circ}); \text{for every } \varpi \in W\}$$
$$\vee\, Sup\{\Gamma(\theta \vee \varpi^{\circ\circ}); \text{for every } \varpi \in W\}$$
$$= \Omega_{\Gamma,W}(\pi) \vee \Omega_{\Gamma,W}(\theta).$$

$$\Omega_{\Gamma,W}(\pi \wedge \theta) = Sup\{\Gamma((\pi \wedge \theta) \vee \varpi^{\circ\circ})); \text{for every } \varpi \in W\}$$
$$= Sup\{\Gamma((\pi \vee \varpi^{\circ\circ}) \wedge (\theta \vee \varpi^{\circ\circ})); \text{for every } \varpi \in W\}$$
$$\geq Sup\{\Gamma(\pi \vee \varpi^{\circ\circ}) \wedge \Gamma(\theta \vee \varpi^{\circ\circ}); \text{for every } \varpi \in W\}$$
$$= Sup\{\Gamma(\pi \vee \varpi^{\circ\circ}); \text{for every } \varpi \in W\}$$
$$\wedge\, Sup\{\Gamma(\theta \vee \varpi^{\circ\circ}); \text{for every } \varpi \in W\}$$
$$= \Omega_{\Gamma,W}(\pi) \wedge \Omega_{\Gamma,W}(\theta).$$

Remark 4.4: We see that $\Upsilon_{\Gamma,W}$ is not necessarily equal to $\Omega_{\Gamma,W}$. But the equality occurs if Γ satisfying $\Gamma(\theta \vee \pi) = \Gamma(\theta) \vee \Gamma(\pi)$ for all $\theta, \pi \in \mathcal{L}$, i.e., $\Omega_{\Gamma,W}(\theta) = \Upsilon_{\Gamma,W}(\theta)$ for every $\theta \in \mathcal{L}$.

For $\mu \in \mathrm{Fuzz}(\mathcal{K})$ of S, set $\mu_t := \{\theta \in S; \mu(\theta) \geq t\}$ where $t \in [0,1]$.

Definition 4.5: An element α is said to be a dense element in a set W with respect to fuzzy filter μ if $\alpha \in \mu_t$ such that $t = \max\{\mu(\theta); \theta \in W\}$.

Example 4.6: Let $\mathcal{L} = \aleph$ (Chain of natural numbers), set the1 fuzzy set $\lambda : \mathcal{L} \to [0,1]$ such that $\lambda(n) = \frac{1}{n}$. Obviously, 1 is a dense element with respect to any subset of $\aleph$.

The symbol $W^{\circ\circ}$ stands for the set $\{\varpi^{\circ\circ}; \varpi \in W\}$. The following theorem illustrates the necessity of dense element.

Theorem 4.7: *If $\varpi_\theta^{\circ\circ}$ is a dense element in the set $W^{\circ\circ}$ with respect to a fuzzy filter χ, then $\Upsilon_{\chi,W}(\theta) = \chi(\theta) \vee \chi(\varpi_\theta^{\circ\circ})$.*

Proof: From the concept of dense elements, we get that $\chi(\varpi_\theta^{\circ\circ}) \geq \chi(\varpi^{\circ\circ})$ for every $\varpi^{\circ\circ} \in W^{\circ\circ}$. Hence $\Upsilon_{\chi,W}(\theta) = \chi(\theta) \vee \chi(\varpi_\theta^{\circ\circ})$.

The succeeding theorem characterises the value of the strong fuzzy extended filter in terms of a dense element.

Theorem 4.8: *Let $\varpi \in W$, the equality $\Omega_{\chi,W}(\theta) = \chi(\theta \vee \varpi^{\circ\circ})$ is valid if and only if $\theta \vee \varpi^{\circ\circ}$ is a dense element in the set $\{\theta \vee \varpi^{\circ\circ};\ \varpi \in W\}$ with respect to a fuzzy filter χ.*

Proof: Consider $\Omega_{\chi,W}(\theta)=\chi(\theta\vee\varpi^{\circ\circ})$, thus the element $\theta\vee\varpi^{\circ\circ}$ has the largest value among all such members of the set $\{\theta\vee\varpi^{\circ\circ};\varpi\in W\}$ under the fuzzy set χ. The other direction follows directly.

5. Homomorphisms of fuzzy extensions

Theorem 5.1: *Let $\Gamma\in FFL(\mathcal{L})$ and $W\in\mathcal{L}$. If Γ is a join homomorphism , then $\Upsilon_{\Gamma,W}$ is a lattice homomorphism. Furthermore, if Γ is an MS-homomorphism, then so $\Upsilon_{\Gamma,W}$.*

Proof: Every fuzzy filter μ has the property that $\mu(\pi\wedge\xi) = \mu(\pi)\wedge\mu(\xi)$ for every $\pi,\xi\in\mathcal{L}$. Consequently, $\Upsilon_{\Gamma,W}(\pi\wedge\xi) = \Upsilon_{\Gamma,W}(\pi)\wedge\Upsilon_{\Gamma,W}(\xi)$. Let Γ be a join homomorphism. Then $\Gamma(\pi\vee\xi) = \Gamma(\pi)\vee\Gamma(\xi)$. So,

$$\begin{aligned}
\Upsilon_{\Gamma,W}(\xi\vee\pi) &= Sup\{\Gamma((\xi\vee\pi)\vee\varpi^{\circ\circ})); \text{for every } \varpi\in W\}\\
&= Sup\{\Gamma(\xi)\vee\Gamma(\pi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\
&= Sup\{(\Gamma(\xi)\vee\Gamma(\varpi^{\circ\circ}))\vee(\Gamma(\pi)\vee\Gamma(\varpi^{\circ\circ})); \text{for every } \varpi\in W\}\\
&= Sup\{\Gamma(\xi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\
&\vee Sup\{\Gamma(\pi)\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\
&= \Upsilon_{\Gamma,W}(\xi)\vee\Upsilon_{\Gamma,W}(\pi).
\end{aligned}$$

Furthermore, for every $\pi\in\mathcal{L}$ suppose that $\Gamma(\pi^{\circ\circ})=(\Gamma(\pi))^{\circ\circ}$. Then;

$$\begin{aligned}
\Upsilon_{\Gamma,W}(\pi^{\circ\circ}) &= Sup\{(\Gamma(\pi^{\circ\circ})\vee\Gamma(\varpi^{\circ\circ}); \text{for every } \varpi\in W\}\\
&= Sup\{(\Gamma(\pi))^{\circ\circ}\vee(\Gamma(\varpi^{\circ\circ}))^{\circ\circ}; \text{for every } \varpi\in W\}\\
&= Sup\{(\Gamma(\pi)\vee\Gamma(\varpi^{\circ\circ}))^{\circ\circ}; \text{for every } \varpi\in W\}\\
&= (\Upsilon_{\Gamma,W}(\pi))^{\circ\circ}.
\end{aligned}$$

Proposition 5.2: *Let $\chi\in FFL(\mathcal{L})$ and $W\subseteq\mathcal{L}$. Every element in the kernel of $\Upsilon_{\chi,W}$ is an element of the kernel of χ and $W^{\circ\circ}$ is a subset of the kernel of χ. Furthermore, the converse is true.*

Proof: Assume that θ is an element of the kernel of $\Upsilon_{\chi,W}(\theta)$, then $\chi(\theta)\vee\chi(\varpi^{\circ\circ})=0$ for every $\varpi\in W$. Thus $\chi(\theta)=0$ and $\chi(\varpi^{\circ\circ})=0$ for every $\varpi\in W$. Hence θ and $\varpi^{\circ\circ}$ are elements of the kernel of χ for every $\varpi\in W$. Conversely, let θ be an element of the kernel of χ. For a subset $W^{\circ\circ}$ of the kernel of χ, we have $\chi(\theta)=0$ and $\chi(w^{\circ\circ})=0$ for every $\varpi\in W$. Hence θ is an element of the kernel of $\Upsilon_{\chi,W}$.

Proposition 5.3: An element is in the cokernel of $\Upsilon_{\chi,W}$ if and only if it is an element of the cokernel of χ or the intersection between $\varpi^{\circ\circ}$ and the cokernel of χ is not empty.

Proof: Suppose that θ is an element in the cokernel of $\Upsilon_{\chi,W}$. Then $\Upsilon_{\chi,W}(\theta)=1$. Thus $\chi(\theta) \vee \chi(\varpi^{\circ\circ}) = 1$ for some $\varpi \in W$. As $\chi(\theta), \chi(\varpi^{\circ\circ}) \in [0,1]$, hence either $\chi(\theta)=1$ or $\chi(\varpi^{\circ\circ})=1$ for some $\varpi \in W$. The reverse implication follows directly from the definition of $\Upsilon_{\chi,W}$.

Let $\pi \in \mathcal{L}$, define the following set

$$\underline{\Upsilon}(\Upsilon_{\chi,W}(\pi)) = \{a \in \mathcal{L}; \Upsilon_{\chi,W}(a) = \Upsilon_{\chi,W}(\pi)\}.$$

The set $\underline{\Upsilon}(\Upsilon_{\chi,W}(\pi))$ is the inverse (as a relation) of $\Upsilon_{\chi,W}(\pi)$.

Lemma 5.4: *For every $\pi \in \mathcal{L}$, the set $\underline{\Upsilon}(\Upsilon_{\chi,W}(\pi))$ is closed under the meet operation. Furthermore, If χ is a join homomorphism, then $\underline{\Upsilon}(\Upsilon_{\chi,W}(\pi))$ is a subalgebra of $\mathcal{L}$.*

Proof: Assume that $\xi, \rho \in \underline{\Upsilon}(\Upsilon_{\chi,W}(\pi))$, therefore

$$\begin{aligned}\Upsilon_{\chi,W}(\xi \wedge \rho) &= \Upsilon_{\chi,W}(\xi) \wedge \Upsilon_{\chi,W}(\rho)\\ &= \Upsilon_{\chi,W}(\pi) \wedge \Upsilon_{\chi,W}(\pi)\\ &= \Upsilon_{\chi,W}(\pi).\end{aligned}$$

Hence $\xi \wedge \rho \in \Upsilon_{\chi,W}(\pi)$. Let χ be a $\vee$-homomorphism, by Theorem (5.1), $\Upsilon_{\chi,W}$ is a lattice homomorphism. Let $\xi, \rho \in \underline{\Upsilon}(\Upsilon_{\chi,W}(\pi))$, then

$$\begin{aligned}\Upsilon_{\chi,W}(\xi \vee \rho) &= \Upsilon_{\chi,W}(\xi) \vee \Upsilon_{\chi,W}(\rho)\\ &= \Upsilon_{\chi,W}(\pi) \vee \Upsilon_{\chi,W}(\pi)\\ &= \Upsilon_{\chi,W}(\pi).\end{aligned}$$

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Received September, 2023